Continuum models: specimen questions

[* means parts which are too much or too hard]

1. A model for fox rabies outbreaks is given by

$$\begin{aligned}\dot{S} &= (a-b)[1-N/K]S - \beta RS,\\ \dot{I} &= \beta SR - \sigma I - [b+(a-b)N/K]I,\\ \dot{R} &= \sigma I - \alpha R - [b+(a-b)N/K]R,\end{aligned}$$

where $N = S + I + R$, and $S, I, R \geq 0$. By choosing suitable scales for the variables, show that a non-dimensional form of the model can be written as

$$\begin{aligned}\dot{s} &= \nu(1-n)s - \rho s,\\ \dot{q} &= \rho s - (1+\delta+\nu n)q,\\ \mu\dot{\rho} &= q - (d+\varepsilon n)\rho,\\ n &= s + q + \mu\rho,\end{aligned}$$

and define the parameters $\nu, \delta, \mu, d, \varepsilon$.

Using typical values of $a = 1\ \mathrm{y}^{-1}$ (per year), $b = 0.5\ \mathrm{y}^{-1}$, $\alpha^{-1} = 5$ days, $\sigma^{-1} = 28$ days, $K = 2$ foxes km^{-2}, $\beta = 80\ \mathrm{km}^2\ \mathrm{fox}^{-1}\ \mathrm{y}^{-1}$, estimate the parameters, and deduce that an approximation to the model is given (if $s, q, \rho \sim 1$) by

$$\begin{aligned}\dot{s} &\approx -sq/d,\\ \dot{q} &\approx q\left(\frac{s}{d} - 1\right).\end{aligned}$$

*By consideration of the behaviour of the solutions of these equations, show that a uniform approximation, valid also for $s \ll 1$, $q \ll 1$, is

$$\begin{aligned}\dot{s} &\approx \nu(1-s)s - \frac{sq}{d},\\ \dot{q} &\approx q\left(\frac{s}{d} - 1\right).\end{aligned}$$

2. Burgers' equation is

$$u_t + uu_x = \varepsilon u_{xx}.$$

Describe, using diagrams, the evolution of the solution if $u = e^{-x^2}$ at $t = 0$, and $\varepsilon \ll 1$.

By analysing the structure of the shock-like transition layer which forms, show that the speed of the front at x_s is given by $\dot{x}_s \approx \frac{1}{2}(u_+ + u_-)$.

*Assuming $x_s \to \infty$ as $t \to \infty$, use the solution of the equation with $\varepsilon = 0$ to show that for large t, $u \approx 2x/t$, and that

$$2\dot{x} = \exp[-(x_s - 2\dot{x}t)^2]\,;$$

hence show that $x_s \approx \pi^{1/4}t^{1/2}$ for large t.

Describe briefly what you expect to happen if u satisfies

$$u_t + uu_x = \varepsilon u_{xx} - u, \quad \varepsilon \ll 1.$$

3. Use the lubrication approximation to derive the model equation

$$\eta_t = \boldsymbol{\nabla}.\left[\frac{\rho g}{3\mu}\eta^3\boldsymbol{\nabla}\eta\right]$$

for the relaxation of a viscous droplet of thickness η, where ρ is the density and μ the viscosity.

A volume of V of fluid is released at the origin at $t = 0$. Show that in the resulting motion the boundary of the drop, $R(t)$, spreads at a finite rate, and find a formula for it. [You may assume $\boldsymbol{\nabla}\phi(r) = \mathbf{e}_r\dfrac{\partial\phi}{\partial r}$ and $\boldsymbol{\nabla}.[\psi(r)\mathbf{e}_r] = \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\psi)$ in cylindrical polar coordinates.]

Discuss briefly what happens if uniform ablation (a term -1) is added to the right hand side.

4. The temperature T of a reacting solid satisfies the heat equation

$$T_t = T_{xx} + \lambda e^T,$$

with $T_x = 0$ on $x = 0$ and $T_x = -\mu T$ on $x = 1$, $\mu > 0$.

*Show that two solutions for the temperature can exist if $\lambda < \lambda_c(\mu)$, and give a recipe for computing λ_c.

Find limiting values for λ_c as $\mu \to \infty$ and $\mu \to 0$, and indicate how you expect λ to vary with μ.

Discuss briefly how time-dependent solutions might be expected to behave (i) for $\lambda < \lambda_c$; (ii) for $\lambda > \lambda_c$.

5. A reaction scheme is given by

$$\begin{aligned} U + Y &\xrightarrow{k_1} X + P, \\ X + Y &\xrightarrow{k_2} 3X, \\ U + X &\xrightarrow{k_3} Z, \\ Z &\xrightarrow{k_4} fY. \end{aligned}$$

Write down a mathematical model based on the law of mass action.

By scaling the model, show that the equations can be written in the form

$$
\begin{aligned}
\dot{u} &= -u(y+x), \\
\delta_1 \dot{x} &= uy + 2xy - ux, \\
\delta_2 \dot{y} &= -uy - xy + fz, \\
\delta_3 \dot{z} &= ux - z.
\end{aligned}
$$

and define the parameters $\delta_1, \delta_2, \delta_3$. [Hint: choose $U = U_0 u$, where U_0 is the initial concentration of U.]

Use the values

$$
\begin{aligned}
k_1 &= 10^6 \text{ M}^{-1} \text{ s}^{-1}, \\
k_2 &= 10^7 \text{ M}^{-1} \text{ s}^{-1}, \\
k_3 &= 10^5 \text{ M}^{-1} \text{ s}^{-1}, \\
k_4 &= 10^3 \text{ s}^{-1}, \\
U_0 &= 10^{-4} \text{ M},
\end{aligned}
$$

to estimate values of δ_1, δ_2 and δ_3, and hence show that

$$
\dot{u} \approx -\frac{-2f(1-f)u^2}{(2f-1)}, \quad \text{with} \quad u(0) = 1,
$$

provided that $\frac{1}{2} < f < 1$.

6. A first order exothermic gas-solid reaction occurs in a spherical porous pellet V of radius a, and the gas concentration c and pellet temperature T are described by the equations

$$
\frac{\partial c}{\partial t} = D\nabla^2 c - r,
$$

$$
\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T + r|\Delta H|,
$$

$$
r = k_0 c \exp\left[-\frac{E}{RT}\right].
$$

Explain the meaning of the terms in these equations.

The temperature and concentration are prescribed as $T = T_0$ and $c = c_0$ on ∂V, the boundary of V. Show that a dimensionless model can be written as

$$
\begin{aligned}
c_t &= \frac{1}{Le}\nabla^2 c - \mu r, \\
\theta_t &= \nabla^2 \theta + \beta r, \\
r &= c \exp\left[\frac{\theta}{1+\varepsilon\theta}\right],
\end{aligned}
$$

with $\theta = 0$, $c = 1$ on ∂V, and define the parameters Le, μ, β and ε.

Suitable values of the constants for the gasification of coal, $C + O_2 \to CO_2$, are $D = 0.1$ cm^2 s^{-1}, $\rho c_p = 2$ J cm^{-3} K^{-1}, $k = 0.2$ W m^{-1} K^{-1}, $\Delta H = -3 \times 10^8$ J mole^{-1}, $k_0 = 10^4$ s^{-1}, $E/R = 7000$ K. Assuming also that the particle radius is $a = 0.1$ mm, the external gas stream temperature is $T_0 = 400$ K, and the external gas (O_2) concentration is $c_0 = 7$ mole m^{-3}, find approximate values of the parameters.

Hence show that $c \approx 1$ in the pellet, and deduce that for these parameter values, thermal runaway will not occur. (You may assume the critical value for thermal runaway of the equation $\nabla^2\theta + \lambda e^\theta = 0$ in $0 < r < 1$, with $\theta = 0$ on $r = 1$, is $\lambda = 2$.)

*Show also that thermal runaway will occur if $T_0 \gtrsim 475$ K.

7. Groundwater flows in a porous layer, bounded by impermeable boundaries at $z = 0$ and $z = d$, and is heated from below by a differential temperature ΔT (between top and bottom). Write down suitable equations governing the motion, and describe what the Boussinesq approximation for the model is.

   Assuming the Boussinesq approximation, derive a non-dimensional form of the model, and show that it can be written in the form

$$\boldsymbol{\nabla}.\mathbf{u} = 0,$$

$$\mathbf{u} = -\boldsymbol{\nabla} p + RT\mathbf{k},$$

$$T_t + \mathbf{u}.\boldsymbol{\nabla} T = \nabla^2 T,$$

   and give an expression for the Rayleigh number R.

   Write down a steady state solution in which $\mathbf{u} = \mathbf{0}$, and show that small perturbations to this uniform solution are unstable if $R > 4\pi^2$.

8. Liquid and gas flow adiabatically as a two-phase flow in a tube, $0 < z < l$. Assuming constant liquid and gas densities ρ_l and ρ_g and equal pressures, write down a two-fluid model for the liquid and gas velocities u and v, assuming no interactive drag terms.

   By writing the equations in the form

$$A\boldsymbol{\psi}_t + B\boldsymbol{\psi}_z = 0,$$

   where $\boldsymbol{\psi} = (\alpha, u, v, p)^T$, α is the void fraction and p is the pressure, show that the characteristics $dz/dt = \lambda$ satisfying $\det(\lambda A - B) = 0$ are given by

$$\lambda = \frac{u \pm isv}{1 \pm is}, \quad s = \left[\frac{\rho_g(1-\alpha)}{\rho_l \alpha}\right]^{1/2},$$

   and comment on the significance of this.

*If, instead, the gas pressure is given by

$$p_g = p_l + C\rho_l v^2,$$

show that the characteristics are given by

$$s^2(\lambda - v)(\lambda - Mv) + (\lambda - u)^2 = 0,$$

where

$$M = 1 + \frac{2C(1-\alpha)}{\alpha s^2},$$

and deduce that if s is small, the characteristics are real if, approximately,

$$C > \frac{2\alpha(v-u)}{(1-\alpha)v}.$$