Continuum models

PROBLEM SHEET 1.

2.1 For the damped pendulum equation

$$\ddot{\theta} + \alpha\dot{\theta} + \beta \sin \gamma\theta = 0, \quad \theta(0) = 0, \ \dot{\theta}(0) = 1,$$

find suitable rescalings for the cases:

$$\begin{array}{rl} \text{(i)} & \alpha, \gamma = O(1), \quad \beta \gg 1; \\ \text{(ii)} & \alpha, \beta = O(1), \quad \gamma \gg 1; \\ \text{(iii)} & \alpha \sim \beta\gamma \sim 1/\gamma \gg 1. \end{array}$$

2.3 A model for earthquakes (Carlson and Langer 1989) is as follows: N blocks of equal mass are situated at positions $x_j(t)$, $j = 1, \ldots N$, on a level plane. They are connected to each other and to a plate above by springs (see figure 1). The plate moves at speed V. The blocks are subject to a friction force $f(\dot{x}_j)$, where f is an odd function, taken as $f(v) = f_0 \,\mathrm{sgn}\,(v)/(1 + |v|/v_0)$. Also, f is indeterminate when $v = 0$, satisfying only $f \in (-1, 1)$.

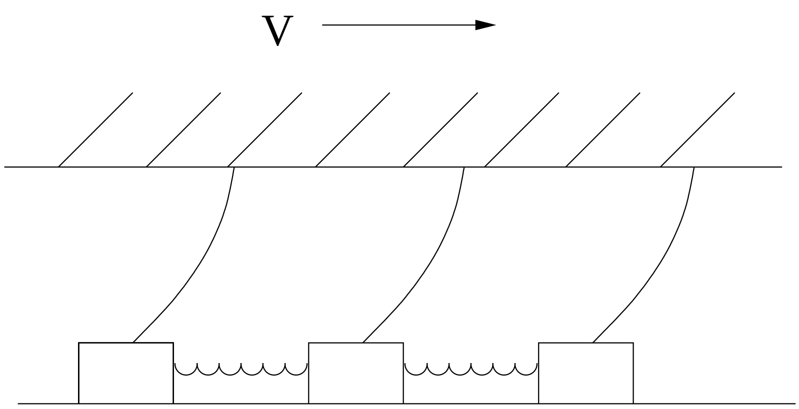


Figure 1: The Carlson-Langer earthquake model

(i) Show that a model for the motion is

$$m\ddot{x}_j = k_c(x_{j+1} - 2x_j + x_{j-1}) - k_p(x_j - Vt) - f(\dot{x}_j),$$

where k_c and k_p are the spring constants of the block-block and block- plate springs.

(ii) If the block-block springs are of natural length a, show that, by putting $x_j = Vt + u$, a continuous approximation to the model is

$$mu_{tt} = k_c a^2 u_{xx} - k_p u - f(V + u_t).$$

(iii) By suitably non-dimensionalising the model, show that a dimensionless version is

$$u_{tt} = u_{xx} - u - f(\varepsilon + u_t),$$

where $\varepsilon = V(mk_p)^{1/2}/f_0$, and $f(w) = \text{sgn}(w)/[1+\alpha|w|]$, $\alpha = f_0/v_0(mk_p)^{1/2}$.

(iv) For values (in consistent units) $f_0 = 50$, $v_0 = 1$, $m = 1$, $k_c = 2500$, $k_p = 40$, $a = 1$, $V = .01$, show that $\varepsilon \approx .001$, $\alpha \approx 9$. Deduce that, for $t \sim 1$, the motion of u is described approximately (if $u_t > 0$) by

$$u_{tt} = u_{xx} - u - \frac{1}{1+\alpha u_t}.$$

(v) If $t = \tau/\varepsilon$, show that an approximate model is

$$0 = u_{xx} - u - \text{sgn}(1 + u_\tau).$$

(vi) Show that the steady state $u = -1/(1+\alpha\varepsilon)$ is unstable on the fast time scale $t = O(1)$, providing $\alpha > 0$.

2.4 The Mackey-Glass model of respiratory ventilation (see Glass and Mackey 1988) is given by

$$\dot{x} = \lambda - \frac{\alpha x V_m x_\tau^n}{\theta^n + x_\tau^n},$$

where $x_\tau = x(t-\tau)$, and x is the arterial CO_2 concentration. The parameters are given by

$$\lambda \sim 6 \text{ mm Hg min}^{-1}, \quad V_m = 80 \text{ liter min}^{-1}, \quad \tau = 0.25 \text{ min};$$

the values of α, θ and n are determined from the observed steady state values of x, $V = V_m x^n/(\theta^n + x^n)$ and $S = dV/dx\,|_{x^*}$. These are $x^* = 40$ (mm Hg), $V^* = 7$ litre min^{-1}, $S^* = 4$ litre min^{-1} (mm Hg)$^{-1}$. Show that if the variables x, V, t are scaled with θ, V_m, τ, then a dimensionless model is

$$\varepsilon\dot{x} = 1 - \beta x v(x_1),$$

where

$$v(x) = \frac{x^n}{1+x^n}, \quad \varepsilon = \frac{\theta}{\lambda\tau}, \quad \beta = \frac{\alpha V_m \theta}{\lambda}.$$

Show that if the dimensionless observed steady value of x is ξ, then

$$\begin{aligned} \xi &= x^*/\theta, \\ \frac{\xi^n}{1+\xi^n} &= V^*/V_m, \\ S^* &= \frac{nV^*}{x^*}\left(1 - \frac{V^*}{V_m}\right). \end{aligned}$$

Deduce that, roughly

$$n \approx 25, \quad \xi^n \approx 0.1, \quad \text{hence} \quad \xi \approx 0.9,$$

and thus

$$\theta \approx 44 \text{ mm Hg}.$$

Hence find the corresponding values of ε, β and α. Is the model sensibly scaled?

Continuum models

Problem sheet 2.

5.1 (modified) (i) The Korteweg equation is

$$u_t + uu_x = \varepsilon u_{xxx}.$$

Show that if $\varepsilon = 0$, shocks can develop in finite time. Show that if $\varepsilon \gg 1$, then the solutions are *dispersive* waves. Also show that if $\varepsilon \neq 0$, then travelling (*cnoidal*) waves exist, of periodic or *solitary* form. In what way do these solutions represent a balance between nonlinearity and dispersion?

(ii) A thin drop controlled by surface tension has thickness h determined by

$$h_t + hh_x = -\varepsilon h_{xxxx}, \quad \varepsilon \ll 1.$$

By considering the Fourier transform of the linear terms, show that the fourth derivative term is smoothing, and show that it is plausible that a local shock structure exists. (Hint: derive the differential equation in the shock by writing $x = x_s(t) + \delta(\varepsilon)X$ for some suitable δ, and then consider the problem of solving the resulting equation for $h = H(X)$ by shooting from $H = h_-$ as $X \to -\infty$ to $H = h_+$ as $X \to +\infty$ (how many conditions need to be specified for H near $h_\pm$?).)

5.2 Water flow in saturated soil is governed by *Darcy's law*:

$$\mathbf{q} = -\frac{k}{\mu}[\boldsymbol{\nabla} p - \rho_l g \mathbf{k}],$$

where $\mathbf{q}$ is water flux, k is permeability, μ is viscosity, p is pressure, ρ_l is water density, g is gravity, and $\mathbf{k}$ is a unit vector in the vertical direction (downwards). Conservation of mass is described by

$$\phi_t + \boldsymbol{\nabla}.q = 0,$$

where ϕ is the porosity, and the capillary relation gives $p_e = p_e(\phi)$, where p_e is the effective pressure, $p_e = P - p$, P is the overburden pressure (due to weight of soil, etc.), and usually $dp_e/d\phi < 0$.

Assuming $k = k(\phi)$, $p_e = p_e(\phi)$, $P = P_0 + \rho_s g z$, and other properties are constant, derive the equation for one-dimensional flow in the (downwards) z direction:

$$\phi_t - V\phi_z = (D\phi_z)_z,$$

where

$$V = \frac{(\rho_s - \rho_l) g k'(\phi)}{\mu}, \quad D = -\frac{kp_e'}{\mu}.$$

Taking $p_e = p_0(\phi_0 - \phi)/\phi_0$, $k = k_0(\phi/\phi_0)^m$, $m > 0$, derive a dimensionless model suitable to describe infiltration over a length scale l, and show that the advective term can be neglected if the Péclet number

$$Pe = \frac{\Delta\rho g l}{p_0} \ll 1,$$

where $\Delta\rho = \rho_s - \rho_l$. If $p_0 = 1$ bar (10^5 N m^{-2}), $\rho_s = 3$ gm cm^{-3}, $\rho_l = 1$ gm cm^{-3}, $\phi_0 = 0.4$, for what values of l is this satisfied?

If $Pe \ll 1$, show that the resulting degenerate diffusion equation

$$\phi_t = (\phi^m \phi_z)_z$$

has similarity solutions $\phi = t^{-\beta} f(\eta)$, $\eta = z/t^{\alpha}$, where $m\beta + 2\alpha = 1$. Deduce that if $\phi = \delta(z)$ at $t = 0$, so that $\int_{-\infty}^{\infty} \phi dz$ is constant and $\phi \to 0$ as $z \to \pm\,\infty$, then $\alpha = \beta = 1/(m+2)$, and hence find $f(\eta)$ if $m > 0$. (Hint: f is not smooth). If $\int_{-\infty}^{\infty} \phi dz = 1$, show that $f = 0$ for $\eta > \eta_0$, where

$$\eta_0 = \left[\frac{(2\gamma)^{(\gamma-1)/2}}{2I_\gamma}\right]^{1/\gamma},$$

where $I_\gamma = \int_0^{\pi/2} \cos^\gamma \theta d\theta$, $\gamma = (m+2)/m$. What happens when $m \gg 1$? Is this consistent with the equation? Can you think of an asymptotic approach to get round the problem (if you did not have the exact solution)? What if $m \ll 1$?

Continuum models

PROBLEM SHEET 3.

5.8 (new) Two steel plates are welded by passing an electric current normally through them, causing a constant Joule heating . A non-dimensional model to represent this is given by

$$T_t = T_{xx} + q \quad \text{in } 0 < x < 1,$$

where x is the coordinate normal to the plates, $x = 1$ represents the outer boundary of one plate, and $x = 0$ represents the centre. The boundary conditions are

$$T = -1 \text{ on } x = 1, \quad \frac{\partial T}{\partial x} = 0 \text{ on } x = 0,$$

(because of symmetry). If $T = 0$ is the melting temperature (remember it is desired to weld the plates), show that welding requires $q > 2$.

Write down a suitable model of Stefan type for the temperature in solid and liquid phases after melting commences, and derive an approximate evolution equation for the free boundary $s(t)$ if the Stefan number St is large. Show also, by considering the temperature gradient at $s(t)+$, that this solution is unphysical. (It is in fact not well-posed. In reality a mixed solid-liquid zone appears, termed a *mushy zone* (see Tayler 1986, page 161).) How might you model the problem correctly?

5.7 Consider a saturated liquid at (freezing) temperature T_f of infinite extent outside a sphere of radius R_w. At $t = 0$, the surface at R_w is suddenly lowered to a subfreezing temperature T_0. Write down a suitable model, and show that it can be non-dimensionalised as

$$\theta_t = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\theta),$$

$$\theta = 0 \text{ on } r = 1,$$

$$\theta = 1, \ \frac{\partial \theta}{\partial r} = St\dot{R} \text{ on } r = R(t),$$

where the Stefan number is $St = L/c_p(T_f - T_0)$.

If $St \gg 1$, define $\varepsilon = 1/St$, $t = (St)\tau$, and show that the freezing front $R(\tau)$ is described by

$$\tau = \frac{2R^3 - 3R^2 + 1}{6} + \frac{1}{6}\varepsilon(R-1)^2 - \frac{1}{45}\varepsilon^2\frac{(R-1)^2}{R} + O(\varepsilon^3).$$

Write down the equivalent problem for freezing saturated liquid in the interior of a sphere from the outside, and deduce that the corresponding expansion for $\tau(R)$ is non-uniform for all τ. (See Lunardini and Aziz (1993) for further discussion and other examples where perturbation methods are useful in freezing problems.)

6.2 *Shape of a meniscus.* Hydrostatic pressure in a liquid is given by $p = p_a - \rho g z$, where p_a is atmospheric pressure, and $z = 0$ is the liquid surface far from a wall. If the liquid surface is given by $z = \eta(x, y)$, show that η satisfies

$$\boldsymbol{\nabla}.\left[\frac{\boldsymbol{\nabla}\eta}{[1+|\boldsymbol{\nabla}\eta|^2]^{1/2}}\right] = \eta,$$

where lengths are made dimensionless with $d = (\gamma/\rho g)^{1/2}$. If $\mathbf{n}'$ is a unit normal to a contact line at a wall, parallel to the wall and pointing away from the fluid, and $\hat{\mathbf{n}}$ is a unit normal to the contact line which is normal to the liquid air interface and directed into the air, show that $\hat{\mathbf{n}}.\mathbf{n}' = \sin\theta$, where θ is the contact angle. Assuming the contact line is horizontal, show that at a vertical wall,

$$(1+|\boldsymbol{\nabla}\eta|^2)^{1/2} = \operatorname{cosec}\ \theta,$$

and thus $\partial\eta/\partial n = \cot\theta$, where $\mathbf{n}$ is a unit normal to the wall, pointing away from the liquid. Solve this problem for the case of a plane wall at $x = 0$ with $\eta \to 0$ as $x \to \infty$, and hence deduce that the capillary rise of the fluid is

$$2(\gamma/\rho g)^{1/2} \sin\left[\frac{\pi}{4} - \frac{\theta}{2}\right].$$

Continuum models

PROBLEM SHEET 4.

6.7 *The Blasius boundary layer*

Fluid flows past a flat plate $y = 0$, $x > 0$ at high Reynolds number, such that the two-dimensional velocity (u, v) satisfies $u = v = 0$ on $y = 0$, $x > 0$, $u \to 1$, $v \to 0$, $p \to 0$ as $y \to \infty$. Show that the outer (inviscid) flow away from $y = 0$ satisfying the (scaled) Navier-Stokes equations with $1/Re = 0$ is $(u, v) = (1, 0)$, $p = 0$. Show that a boundary layer exists near $y = 0$, where $v \sim \delta$, $y \sim \delta$ where $\delta = Re^{-1/2}$, and show that the corresponding equations for u and (rescaled) V are

$$u_x + V_Y = 0,$$

$$uu_x + Vu_Y = u_{YY}.$$

By introducing a stream function $u = \psi_Y$, $V = -\psi_x$, deduce that

$$\psi_Y \psi_{xY} - \psi_x \psi_{YY} = \psi_{YYY}.$$

Show that a similarity solution of the form $\psi = (2x)^{1/2} f(\eta)$, $\eta = Y/(2x)^{1/2}$, exists, where f satisfies

$$f''' + ff'' = 0,$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$

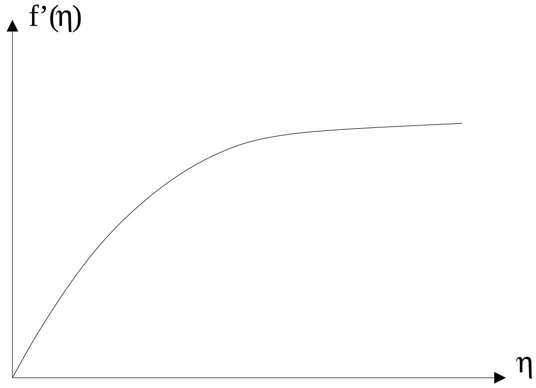


Figure 2: Blasius boundary layer profile

$f(\eta)$ must be found numerically, and in common with many similarity solutions, there is a trick to do this by rescaling. Solve

$$F'''(\xi) + F(\xi)F''(\xi) = 0,$$

$$F(0) = F'(0) = 0, \quad F''(0) = 1;$$

(this can be done easily as an initial value problem, providing (as is the case) F' cannot blow up at ∞)). Put $f(\eta) = bF(a\eta)$, thus

$$f''' + ff'' = ba^3F''' + b^2a^2FF'' = 0$$

if $b = a$, and then

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = a^2F'(\infty);$$

thus we compute F, and then define $f(\eta)$ as above with $a = \{F'(\infty)\}^{-1/2}$. As computed, f' is as shown in the figure, resembling an error function.

6.9 *Bubble collapse.*

Consider the collapse of a vacuous bubble of radius a and pressure p_g in an incompressible fluid of viscosity μ and with far-field pressure p_l. The radius of the bubble is determined by

$$\Delta p = -\rho\left[\frac{3}{2}\dot{r}^2 + r\ddot{r}\right] - 4\mu\dot{r}/r,$$

where Δp is the pressure difference $p_l - p_g$ and ρ is the density of the fluid. By scaling r and t appropriately, show that a dimensionless form of this equation can be written as

$$-1 = \frac{3}{2}\dot{r}^2 + r\ddot{r} + \varepsilon\dot{r}/r,$$

where $\varepsilon = 4\mu/a\{\rho\Delta p\}^{1/2}$. Deduce that $\varepsilon \ll 1$, if $Re \gg 4$, where $Re = Ua/\nu$ and $U = (\Delta p/\rho)^{1/2}$. Estimate the bubble size for which this inequality is valid if $\Delta p = 1$ bar ($= 10^5$ Pa), $\rho = 10^3$ kg m^{-3}. Show that for such bubbles, collapse occurs at a finite time given dimensionlessly by

$$t_c = \sqrt{\frac{3}{2}}\int_0^1 \left(\frac{r^3}{1-r^3}\right)^{1/2} dr.$$

Is this affected by $\varepsilon \neq 0$? What if the bubble is not vacuous?

6.10 An ice sheet of thickness h satisfies the equation

$$h_t = \frac{\partial}{\partial x}\left[\frac{1}{3}h^3h_x\right] + a(x),$$

where at the margins, the accumulation rate is negative. Show that, if the margin $x_m(t)$ is non-stationary, then (if $h > 0$ for $x < x_m$)

$$\begin{aligned} h &\sim c(x_m - x), \\ \dot{x}_m &\sim -|a(x_m)|/c, \end{aligned}$$

if $\dot{x}_m < 0$ (retreat), while

$$\begin{aligned} h &\sim c(x_m - x)^{1/3}, \\ \dot{x}_m &\sim c^3/9, \end{aligned}$$

if $\dot{x}_m > 0$. Thus the slope is singular in advance but finite in retreat. This distinction causes the degenerate diffusion equation above to have waiting time behaviour, since following a retreat, the margin slope must rebuild itself before another advance is possible. (See Lacey et al. 1982, Lacey 1983 for further discussion.)

PROBLEM SHEET 5.

10.3 The B-Z reaction is set up in a thin layer of *unstirred* fluid in a petri dish. Justify modelling this situation by the reaction-diffusion equation

$$\mathbf{c}_t = \mathbf{f}(\mathbf{c}) + D\nabla^2\mathbf{c},$$

where $\mathbf{c}$ is the vector of reactant concentrations, and $\mathbf{f}$ is given by the Field-Noyes model. If the dimensionless concentrations of $HBrO_2$, Br^- and Ce^{4+} are denoted by u, v, w, respectively, derive the model

$$\varepsilon u_t = u + v - uv - qu^2 + \varepsilon\nabla^2 u,$$

$$v_t = 2fw - v - uv + \nabla^2 v,$$

$$pw_t = u - w + p\nabla^2 w,$$

for motions with a length scale $l \sim (D/K_1 A)^{1/2}$. If $D \sim 10^{-5}$ cm^2 s^{-1}, show that $l \sim 10^{-1}$ mm.

Discuss the following description of a propagating wavefront for this system: we anticipate rapid regions of change which join regions where process I and process II apply, that is, in process I we expect $u, v, w \sim 1$, whereas in process II, $u, w \sim 1/q$. We ignore variation of w at the wave front, and put $u = \tilde{u}/q$, $w = \tilde{w}/q$; then, approximately,

$$\varepsilon\tilde{u}_t = \tilde{u} - \tilde{u}v - \tilde{u}^2 + \varepsilon\nabla^2\tilde{u},$$

$$qv_t = 2f\tilde{w} - \tilde{u}v + q\nabla^2 v,$$

$$p\tilde{w}_t = \tilde{u} - \tilde{w} + p\nabla^2\tilde{w};$$

rescale t via $t = \varepsilon\tilde{t}$ and space via $\mathbf{x} = \varepsilon^{1/2}\tilde{\mathbf{x}}$, drop tildes, to obtain (with $\tilde{w}$ constant)

$$u_t = u(1 - u - v) + \nabla^2 u,$$

$$(q/\varepsilon)v_t = 2f\tilde{w} - uv + (q/\varepsilon)\nabla^2 v.$$

If $q \ll \varepsilon$, we may put $v \approx 2f\tilde{w}/u$, so that u satisfies (with $\lambda = 2f\tilde{w}$)

$$u_t = u(1 - u) - \lambda + \nabla^2 u;$$

discuss what should be plausible boundary conditions for this equation. Examine this equation in one space dimension for possible travelling wave solutions of the form $u = u(x - ct)$. (See, for example, J.D. Murray, Lectures on nonlinear differential equation models in biology, OUP, Oxford, 1977.) Show that the dimensional wave speed is then $c(K_3 AD)^{1/2}$ (experiments confirm this, with $c = 1.7$, see J.J. Tyson, A quantitative account of oscillations, bistability, and traveling waves in the Belousov-Zhabotinskii reaction; in R.J. Field and M. Burger, eds., Oscillations and travelling waves in chemical systems, John Wiley, New York, pp. 93–144).

10.6 The B-Z experiment is carried out in a CSTR (continuously stirred tank reactor), so that the reaction mixture is drained at a constant rate, and replaced by bromate at the same rate. Derive the corresponding Oregonator model, and analyse the dynamics as a function of the residence time.

10.8 The Lotka system of chemical reactions is governed by the scheme

$$A + X \xrightarrow{k_1} 2X,$$

$$X + Y \xrightarrow{k_2} 2Y,$$

$$Y \xrightarrow{k_3} P,$$

where k_1, k_2, k_3 denote rate constants, and $2X$ means $X + X$, etc.

Write down a set of equations for the concentrations $[A]$, $[X]$, $[Y]$, $[P]$ of the chemical species.

If, during the reaction, $[A]$ is approximately constant, show that $[X]$ and $[Y]$ are determined by two coupled differential equations, and show that a non-trivial steady state $[X] = X_0$, $[Y] = Y_0$ exists for these.

Use the steady states to non-dimensionalise the equations, and show that dynamic behaviour of $[X]$ and $[Y]$ is determined by a single parameter $\delta = k_3/k_1[A]$.

Show that small perturbations to the steady state are oscillatory, and show that the *dimensional* period of oscillation is $2\pi/(k_1k_3[A])^{1/2}$.

If k_3 is known, how could you use this result to find k_1 experimentally? What is the restriction on the value of k_1 which allows $[A]$ to be considered as constant? (You may assume $\delta = O(1)$.)

Continuum models

PROBLEM SHEET 6.

12.4 Solve explicitly the Gel'fand equation

$$\nabla^2\theta + \lambda e^\theta = 0, \quad \theta = 0 \ \text{ on } \ r = 1,$$

in a cylindrical geometry, and show that there are no solutions if $\lambda > \lambda_c$, and calculate λ_c.

By rewriting the Gel'fand equation (in three dimensions, with spherical symmetry) as a pair of equations for $p = \lambda r^2 e^\theta$, $q = 2 + r\theta'$, in terms of $t = \ln(1/r)$, use a phase plane analysis to show that, if $\theta = 0$ on $r = 1$ (and $\theta_r = 0$ on $r = 0$), then multiple solutions exist if λ is near a critical value λ_c, and that when $\lambda = \lambda_c$, infinitely many solutions exist. Show that $\lambda_c = 2$. Compute or sketch the bifurcation diagram of $\theta(0)$ versus λ.

12.5 A first order exothermic chemical reaction is described by the equations

$$T_t = \nabla^2 T + \beta c \exp\left[\frac{\beta T}{1+\delta T}\right],$$

$$c_t = \frac{1}{Le}\nabla^2 c - \mu c \exp\left[\frac{\beta T}{1+\delta T}\right],$$

where T is temperature and c is reactant concentration. If $Le = 1$, show that the Shvab-Zeldovich variable $c \pm \mu T$ is constant, and derive the resultant single equation for T. Deduce (for an exothermic reaction) that if $\mu \gg 1$, then $T = O(1/\mu)$. Deduce further, that, if $\beta \gg \mu$ (and $\beta \gg \delta$), then the reaction will proceed slowly, but finish with a flourish.

12.6 For an exothermic reaction governed by the model in question 12.5, with $\beta \gg 1$ and $\delta \ll \beta$, give reasons why $c < 1/\beta$ for times of $O(1)$ if thermal runaway is to be prevented. What does this imply about μ? Suppose now $\mu \gg 1$ and $Le \gg 1$. Show that $c \approx \exp[-\mu t e^\theta]$, where $\theta = \beta T$, and hence, by writing $\tilde{t} = \mu t$ and considering $\tilde{t} = O(1)$, show that $\theta \sim \beta/\mu$, and that if $\theta = \beta\Theta/\mu$, Θ satisfies (approximately) $\Theta_{\tilde{t}} \approx e^{-\tilde{t}}$. Hence find the maximum temperature rise.

PROBLEM SHEET 7.

14.2 *Lapwood convection.* Consider the onset of convection in a porous medium, where the base is impermeable and the top is permeable (for example, convection in sea-floor crustal rock at mid-ocean ridges). Specify the appropriate boundary conditions for this case, and show that the appropriate form of the stream function $\psi = f(z)e^{ikx+\sigma t}$ in a linear stability analysis when $\sigma = 0$ is determined by

$$f(z) = \sinh\alpha_+ \sin\alpha_- z + \sin\alpha_- \sinh\alpha_+ z,$$

where (with $R > k^2$)

$$\alpha_+ = [k^2 + k\sqrt{R}]^{1/2}, \quad \alpha_- = [k\sqrt{R} - k^2]^{1/2}.$$

Hence show that the dispersion relation is given by

$$\frac{\tanh\alpha_+}{\alpha_+} + \frac{\tan\alpha_-}{\alpha_-} = 0.$$

Denoting $F(\alpha) = [\tanh\alpha]/\alpha$, indicate graphically the form of $F(\alpha)$ and thus also $F^{-1}(\theta)$ (for $0 < \theta < 1$); by plotting $[\tan\alpha]/\alpha$, show that (for $\alpha > 0$) $F^{-1}[(-\tan\alpha)/\alpha] = G(\alpha)$ is defined as a positive monotone increasing function on strips $[\alpha_n, n\pi)$, where $(2n-1)\pi/2 < \alpha_n < n\pi$, $n = 1, 2, ...$, and $G(\alpha_n) = 0$, $G(n\pi) = \infty$. Deduce that for given k, there are an infinite number of critical values of R where $\sigma = 0$ given by the intersections in the first quadrant of the hyperbola $\alpha_+^2 - \alpha_-^2 = 2k^2$ with $\alpha_+ = G(\alpha_-)$. If the intersection values are denoted by $(\alpha_-^{(n)}, \alpha_+^{(n)})$, with $\alpha_-^{(1)} < \alpha_-^{(2)} < \ldots$, show that $\alpha_+^{(1)} < \alpha_+^{(2)} < \ldots$, $\alpha_+^{(n)} \approx \alpha_-^{(n)} \approx n\pi$ as $n \to \infty$, and that the corresponding values of R, denoted R_n, satisfy $R_1 < R_2 < \ldots$. If $\alpha_1^* < \alpha_2^* < \ldots$ denote the roots of $\alpha = G(\alpha)$, show that $\alpha_n < \alpha_n^* < n\pi$, $\alpha_n^* \to n\pi$ as $n \to \infty$, and that $\alpha_-^{(n)}$ increases monotonically with k, with $\alpha_-^{(n)}(0) = \alpha_n^*$, $\alpha_-^{(n)}(\infty) = n\pi$. Deduce that each critical curve $R_n(k)$ is convex, $R_n \to \infty$ as $k \to 0$ and $k \to \infty$, so that the critical value for the onset of convection is $R_c = \min_k R_1(k)$. Show that

$$\left[\frac{\alpha_1^{*2}}{k} + k\right]^2 < R_1(k) < \left[\frac{\pi^2}{k} + k\right]^2,$$

and deduce that $4\alpha_1^{*2} < R_c < 4\pi^2$. Devise a numerical method to compute R_c (the actual value is 27.1). This problem was studied by Lapwood (1948).

14.3 The Boussinesq equations for convection in a fluid layer heated from below can be written in the dimensionless form

$$\begin{aligned} \boldsymbol{\nabla}.\mathbf{u} &= 0, \\ \frac{1}{\sigma}\frac{d\mathbf{u}}{dt} &= -\boldsymbol{\nabla} p + \nabla^2 \mathbf{u} + Ra\theta\mathbf{k}, \\ \frac{d\theta}{dt} &= \nabla^2\theta, \end{aligned}$$

where $\theta = 1$ on $z = 0$, $\theta = 0$ on $z = 1$, and where the *Prandtl* and *Rayleigh* numbers are

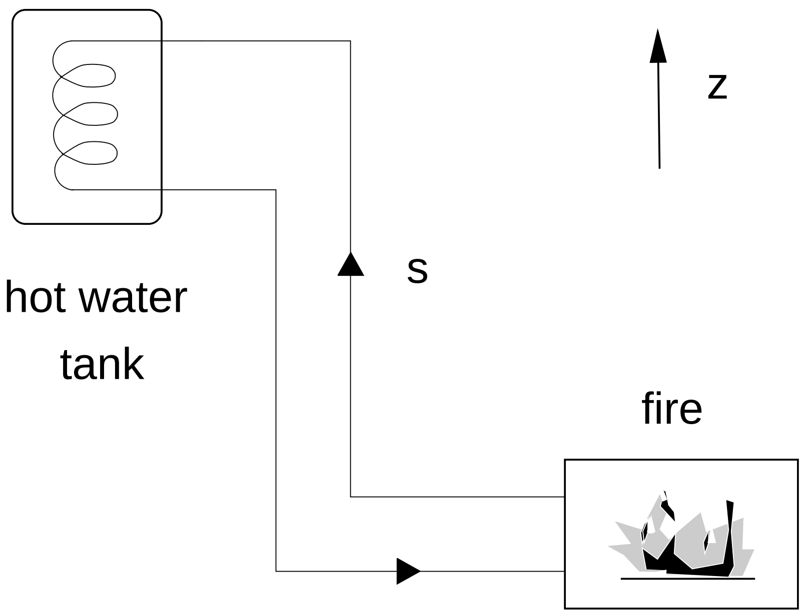
$$\sigma = \frac{\mu}{\rho_0\kappa}, \quad Ra = \frac{\alpha g d^3 \rho_0 \Delta T}{\mu\kappa}.$$

Write down the steady state with $\mathbf{u} = \mathbf{0}$, and show that for free slip conditions ($\mathbf{u}.\mathbf{n} = \partial(\mathbf{u}.\mathbf{t})/\partial n = 0$ on the boundary), this is unstable to convection if

$$Ra > Ra_c = 27\pi^4/4 \approx 657.5,$$

and the critical wavenumber is $k_c = \pi/\sqrt{2}$.

14.6 *Convection in a fluid loop.* A domestic back boiler is configured as shown in the figure below.



Show that in a Boussinesq model of the flow, the mean fluid velocity (across a cross section of the pipe) v is a function only of t, $v = v(t)$, and by assuming that the flow in the pipe is of Poiseuille type, show that v satisfies

$$\rho_0 \dot{v} = -\frac{\partial p}{\partial s} - \rho g \frac{\partial z}{\partial s} - \frac{8\mu}{a^2} v,$$

where we assume the pipes are circular, of radius a, s is the distance along the pipes, and z is vertical height. By assuming $\rho = \rho_0[1 - \alpha(T - T_0)]$, derive the equation in the form

$$\rho_0 \dot{v} = -\frac{\partial \Pi}{\partial s} + \alpha \rho_0 g (T - T_0)\frac{\partial z}{\partial s} - \frac{8\mu}{a^2} v,$$

where $p + \rho_0 g z = \Pi$, and by integrating round the closed loop C of length L, show that v satisfies

$$\rho_0 L \dot{v} = \alpha \rho_0 g \oint_C (T - T_0)\, dz - \frac{8\mu L}{a^2} v.$$

If heat losses are neglected in the pipe, show that T satisfies

$$T_t + v T_s = 0.$$

Show further, that if the boiler at $s = 0$ delivers heat at a rate q_H (e.g. in kilowatts), then a suitable jump condition for the temperature across $s = 0$ is

$$v \Delta T = \frac{q_H}{\rho_0 c_p A},$$

where $A = \pi a^2$ is the tube cross sectional area and c_p is the specific heat. Similarly, show that the jump in T across the hot water tank at $s = L/2$ is

$$v \Delta T = -\frac{q_c}{\rho_0 c_p A},$$

where q_c is the cooling rate. What is the final boundary condition?

By choosing suitable scales, non-dimensionalise the model and show that it depends on the single dimensionless parameter

$$\delta = \frac{\rho_0 a^2}{8\mu L}\left(\frac{q_H \alpha g}{8\mu\pi c_p}\right)^{1/2}$$

(you should assume $q_H = q_c$). By considering the particular case where the connecting pipes are vertical, show that there is a unique steady state (up to an additive constant for the temperature) and examine its stability.

(This is the celebrated Welander loop, studied by P. Welander (J. Fluid Mech. **29**, 17–30, 1967), J.B. Keller (J. Fluid Mech. **26**, 599–606, 1966) and more recently by M.A. Herrero and J.J.L. Velazquez (Euro. J. Appl. Math. **1**, 1–24, 1990).

Continuum models

PROBLEM SHEET 8 (VAC WORK).

16.2 A one-dimensional homogeneous model for flow in a heated tube is given by

$$\rho_t + u\rho_z + \rho u_z = 0,$$

$$p_z = -\kappa\rho u^2,$$

$$\rho(h_t + uh_z) = q,$$

where we neglect gravity and inertia. Assuming ρ $(= \rho_w)$ is constant in the sub-cooled region $0 < z < r(t)$, and that $h = h_0$, $u = u_0$ on $z = 0$, $h = h_{sat}$ on $z = r(t)$, derive the relations

$$z = \int_{t'}^{t} u_0(s)ds,$$

$$h = h_0 + \frac{q}{\rho_w}(t - t'),$$

where q is assumed constant, and deduce that

$$r(t) = \int_{t-\tau}^{t} u_0(s)ds,$$

where

$$\tau = \rho_w(h_{sat} - h_0)/q.$$

Show also that the sub-cooled pressure jump is

$$\Delta p_{sc} = \kappa\rho_w u_0^2 r.$$

Let α be the volume fraction of steam in $z > r$. Show that

$$\rho = \rho_w(1 - \alpha) + \rho_s\alpha,$$

$$\rho h = \rho_w h_w(1 - \alpha) + \rho_s h_s\alpha,$$

where h_s and h_w $(= h_{sat})$ are steam and water enthalpies, and deduce that

$$h = (\beta/\rho) + \gamma,$$

where

$$\beta = \frac{\rho_w\rho_s L}{\rho_w - \rho_s}, \quad \gamma = \frac{\rho_w h_w - \rho_s h_s}{\rho_w - \rho_s},$$

L being the latent heat, $h_s - h_w$. Hence show that, in the two-phase region

$$\frac{\partial u}{\partial z} = \frac{q}{\beta},$$

and thus that

$$u = u_0 + \frac{q}{\beta}(z - r)$$

there. Hence show that the two-phase pressure drop between $z = r$ and the end of the tube $z = l$ (assume $\alpha < 1$ there) is

$$\Delta p_{tp} = \kappa \int_r^l \rho u^2 dz,$$

where $u = u_0 + q(z - r)/\beta$, and $\rho_t + u\rho_z + \rho u_z = 0$ with $\rho = \rho_w$ on $z = r$.

16.3 By choosing scales

$$u, u_0 \sim ql/\beta;\ z, r \sim l;\ t \sim l/u;\ \rho \sim \rho_w;\ \Delta p \sim \kappa \rho_w u^2 l;$$

show that the dimensionless homogeneous model of exercise 16.2 can be written

$$\begin{aligned}
r &= \int_{t-\tau^*}^{t} u_0^*(s)ds, \\
\Delta p^* &= u_0^{*2} r + \int_r^1 \rho u^2 dz, \\
u &= u_0^* + (z - r), \\
\rho_t &+ u\rho_z + \rho u_z = 0,\ \rho = 1 \text{ on } z = r;
\end{aligned}$$

and the dimensionless quantities are defined by

$$\begin{aligned}
\Delta p^* &= \Delta p \left[\frac{\beta^2}{\kappa \rho_w q^2 l^3}\right]; \\
u_0^* &= u_0 \left[\frac{\beta}{ql}\right]; \\
\tau^* &= \rho_w \frac{\Delta h_{sc}}{\beta},
\end{aligned}$$

where $\Delta h_{sc} = h_{sat} - h_0$ is the inlet sub-cooling.

Dropping the asterisks, show that in the steady state Δp is given as a function of u_0, by

$$\Delta p = \tau u_0^2 + (\frac{1}{2}\tau^2 - \tau)u_0^2 + (1 - \tau)u_0 + \frac{1}{2},$$

and find conditions under which Ledinegg instability (*i.e.*, due to multiple steady states) can occur. Show also that dryout does not occur (i.e. the volume fraction α of steam remains positive) providing

$$1 + (1 - \tau)u_0 < 1/\delta,$$

where $\delta = \rho_s/\rho_w$.

16.6 When a pint of Guinness is first poured, many tiny bubbles are nucleated. As they rise towards the surface, waves can be seen propagating downwards. To explain this, write down a one-dimensional model for bubbly two-phase flow with variables α (void fraction), u, v (liquid and gas velocities), p_l, p_g (liquid and gas pressures), which includes gravity and an interactive drag term M. If the bubbles have radius a, show that the assumption of Stokes drag (e.g. G.K. Batchelor, An introduction to fluid dynamics, CUP, Cambridge, 1967, equation (4.9.30)) on each bubble leads to

$$M = \frac{\mu D(\alpha)(v-u)}{a^2}, \qquad (*)$$

where $D(\alpha) = 3\alpha$, and explain why this should be appropriate for small α. Conversely, if the bubble density is so large that the liquid drains as in a foam, show that the assumption of Darcy flow (for a porous medium) leads to

$$M \approx \frac{\mu}{k}(1-\alpha)^2(v-u),$$

and for the Carman-Kozeny relation, we again derive $(*)$ with $D(\alpha) \approx 180\alpha^2/(1-\alpha)$. Show that inertia terms can be neglected if $v^2/gh \ll 1$, where h is a typical length scale of the system, and show that, if $(*)$ is generally applicable, with D an increasing function of α, then the model can be reduced to the first order equation

$$\frac{\partial \alpha}{\partial t} + \frac{\partial Q}{\partial z} = 0,$$

where $Q = \alpha v$ is given by

$$Q = \frac{\Delta\rho g a^2}{\mu}\frac{\alpha^2(1-\alpha)^2}{D(\alpha)}.$$

Deduce that voidage waves will travel downwards if α is large enough, and show that the speed is of order $\Delta\rho g a^2/\mu D$. Show also that this is in accord with observation of wave speeds ~ 1 cm s^{-1}, if $D \sim 10$, $\mu/\Delta\rho \sim 10^{-6}$ m^2 s^{-1}, $g \sim 10$ m s^{-2}, $a \sim 10^{-4}$ m (100 microns). Why do you think these waves form in the first place?