Continuum models finals questions 1998

1. The Lorenz equations are given by

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= (r - z)x - y, \\ \dot{z} &= xy - bz, \end{aligned}$$

where b, r, σ are dimensionless parameters.

Suppose that $r \gg 1$, while $\sigma, b = O(1)$. By anticipating the dominant balances $\dot{y} \sim rx$, $\dot{x} \sim \sigma y$, $\dot{z} \sim xy$, show that a distinguished limit exists when $z \sim r$, and by suitably rescaling the equations, show that they can be written in the form

$$\begin{aligned} \dot{\xi} &= \eta - \varepsilon\sigma\xi, \\ \dot{\eta} &= (1 - \zeta)\xi - \varepsilon\eta, \\ \dot{\zeta} &= \xi\eta - \varepsilon b\zeta, \end{aligned}$$

and give the definition of ε.

Hence find two approximate first integrals when $\varepsilon \ll 1$, and describe what they imply about the solutions.

2. A lake of depth h is cooled at a rate q (i.e. the heat flux from the surface is q). Initially it is at the freezing temperature T_f, but is not frozen. Write down a model for the lake temperature, and by suitably non-dimensionalising it, show that it depends on the single dimensionless parameter

$$St = \frac{kL}{c_p qh},$$

where k is the thermal conductivity, L is the latent heat, and c_p is the specific heat.

If $St \gg 1$, find approximate expressions for the freezing front and the surface temperature, and hence show that the lake freezes completely at a time $t_f = \rho Lh/q$, at which time the surface temperature is $T_f - (qh/k)$.

3. A simple chlorite-iodide reaction in acid solution is given by

$$\begin{aligned} ClO_2^- + HOI + H^+ &\to HIO_2 + HOCl, \\ HOCl + I^- &\to HOI + Cl^-, \\ HIO_2 + I^- + H^+ &\to 2HOI. \end{aligned}$$

Show that the reaction can be modelled by the scheme

$$A + Z \xrightarrow{k_3} X + P,$$

$$P + Y \xrightarrow{k_4} Z,$$

$$X + Y \xrightarrow{k_5} 2Z;$$

identify the species involved, and explain why the others can be ignored.

Use the law of mass action to write down suitable equations for the concentrations a, x, y, z, p of a homogeneous reaction mixture. If the initial values of the concentrations of a and y are $a = y = a_0$, show that the system can be written in the non-dimensional form

$$\begin{aligned} \dot{a} &= -az, \\ \mu_1 \dot{x} &= az - xy, \\ \dot{y} &= -py - xy, \\ \mu_3 \dot{z} &= py + 2xy - az, \\ \dot{p} &= az - py, \end{aligned}$$

by appropriate choice of scales for the variables. Define the parameters μ_1 and μ_3.

Find two first integrals (conservation laws) of these equations. Supposing that $x, p, z \approx 0$ at $t = 0$, deduce that a, y, p approximately satisfy the equations

$$\begin{aligned} \mu_3 \dot{a} &= -a[2a + p - 2y], \\ \mu_1 \dot{y} &= -y[(\mu_1 - 1)p + 1 + y - 2a], \\ \mu_3 \dot{p} &= -\mu_3 py + a[2a + p - 2y]. \end{aligned}$$

4. A non-porous pellet reacts exothermically with an external gas stream. If the pellet is spherical, of radius $r = S(t)$, where $S(0) = a$, then its temperature T satisfies

$$T_t = \kappa \nabla^2 T,$$

with

$$-k\frac{\partial T}{\partial r} = -\rho_s v|\Delta H| + h_T(T - T_\infty) \quad \text{on} \quad r = S,$$

and

$$\dot{S} = -v = -v_0 \exp[-E/RT]|_{r=S}, \quad S(0) = a.$$

Explain briefly the meaning of the terms in the model, and derive the non-dimensional form

$$\theta_t = \nabla^2\theta \quad \text{in} \quad 0 < r < S,$$

$$\dot{S} = -\nu v, \quad v = \exp\left(\frac{\theta}{1+\varepsilon\theta}\right)\bigg|_{r=S}, \quad S(0) = 1,$$

$$-\frac{1}{Nu}\frac{\partial\theta}{\partial r} = -\gamma v + \theta \quad \text{on} \quad r = S,$$

and give expressions for ν, ε, Γ and Nu.

Now suppose $\varepsilon \ll 1$ and $\nu \ll 1$. By putting $t = \tau/\nu$, show that θ is given approximately by

$$\theta \approx \theta_0(\tau) + \nu[\theta_1(\tau) + \frac{1}{6}\theta_0'(\tau)r^2] + \ldots,$$

and deduce that if $\zeta = \ln(1/S)$, then

$$\frac{\nu}{3Nu}\frac{d\theta_0}{d\zeta} \approx \gamma - \theta_0 e^{-\theta_0}.$$

Show that thermal runaway occurs if $\gamma > e^{-1}$. What happens if $\gamma < e^{-1}$?