Continuum models M. Sc. exam questions 1998

1. A glacier of viscosity η and density ρ flows slowly down a plane slope inclined at an angle α to the horizontal. Supposing the motion to be two-dimensional, use the lubrication approximation to derive the evolution equation for the depth h in the form
$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left[\frac{\rho g\sin\alpha}{3\eta}\left\{1-\cot\alpha\frac{\partial h}{\partial x}\right\}h^3\right]=a,$$
where x is the downslope coordinate, and specify the meaning of the term a.

   If $a(x)=a_0(1-2x/l)$, show that a non-dimensional form of the equation can be written as
$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left[(1-\mu h_x)h^3\right]=1-2x,$$
where μ should be given. If $\mu\ll 1$, find an approximate steady state (you may assume $h=0$ at $x=0$), and explain how perturbations to this might behave.

2. *This version is more complicated than the one actually set.*
   A simple chlorite-iodide reaction is given by
$$\begin{aligned}
H^+ + ClO_2^- + I^- &\to HOCl + HOI, && (1)\\
H^+ + HOI + I^- &\to I_2 + H_2O, && (2)\\
ClO_2^- + HOI + H^+ &\to HIO_2 + HOCl, && (3)\\
HOCl + I^- &\to HOI + Cl^-, && (4)\\
HIO_2 + I^- + H^+ &\to 2HOI, && (5)\\
2HIO_2 &\to HOI + IO_3^- + H^+, && (6)\\
HIO_2 + HOI &\to I^- + IO_3^- + 2H^+, && (7)\\
HIO_2 + HOCl &\to IO_3^- + Cl^- + 2H^+. && (8)
\end{aligned}$$
   Extract six model reactions from these for the reactants $A=ClO_2^-$, $X=HIO_2$, $Y=I^-$, $Z=HOI$, $P=HOCl$, explaining why it may be reasonable to ignore the other reactants, and why reactions (2) and (8) are not included.

   Write down the rate equations for the homogeneous concentrations a of A, x of X, etc., assuming the rate of the model equation (i) is k_i.

   By choosing a balance in which
$$k_3az\sim k_5xy\sim k_6x^2\sim k_7xz\sim k_4py,$$
   and $a\sim a_0$, show that the equations can be written in the non-dimensional form
$$\begin{aligned}
\mu_a\dot a &= -az-\varepsilon ay\\
\dot x &= az-(2x+y+z)x,\\
\mu_y\dot y &= xz-(x+p)y-\varepsilon ay,\\
\mu_z\dot z &= 2xy+x^2-(a+x)z+py+\varepsilon ay,\\
\mu_p\dot p &= az-yp+\varepsilon ay,
\end{aligned}$$

where ε should be defined, and the reaction time scale is

$$t_r = \frac{k_7}{k_6 k_3 a_0}.$$

Show also that

$$\mu_a = \frac{k_7}{k_3}, \quad \mu_y = \frac{k_6}{k_5}, \quad \mu_z = \frac{k_6}{k_7}, \quad \mu_p = \frac{k_5}{k_4},$$

and show that for values $k_4 = 1.4 \times 10^8$ M^{-1}s^{-1}, $k_3 = 6.6 \times 10^6$ M^{-1}s^{-1}, $k_5 = 10^4$ M^{-1} s^{-1}, $k_7 = 2.3 \times 10^2$ M^{-1} s^{-1}, $k_6 = 10^2$ M^{-1}s^{-1}, $k_1 = 2.5$ M^{-1}s^{-1}, $\varepsilon \sim 10^{-8}$, $\mu_a \sim 10^{-4}$, $\mu_p \sim 10^{-4}$, $\mu_y \sim 10^{-2}$, $\mu_z \sim 1$.