Dynamics of Glaciers and Ice Sheets

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Sliding, drainage and subglacial geomorphology

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Lecture 1. Sliding over hard beds.

1.1 Weertman sliding

Glaciers slide at their base, and this sliding can be observed, e. g. in cavities or subglacial tunnels. Sometimes, sliding accounts for most of the observed surface velocity, for example in the Siple Coast ice streams.

Glacier motion is modelled as a slow, viscous flow, but when sliding occurs the usual *no-slip* boundary condition appropriate to a frozen base must be replaced by a relation between *basal shear stress* τ_b and *basal velocity* u_b. This relation is called the *sliding law*.[1]

The mechanisms of sliding were enunciated by Weertman (1957). When the basal ice is *temperate* (at the melting point), a film of water is conceived to exist between the ice and the underlying bedrock.[2] This allows slip. The resistance is then provided by the roughness of the bedrock. Weertman identifies another mechanism, that of *regelation*. Regelation (literally, re-freezing) occurs because as the ice flows over a bedrock obstacle, the higher pressure upstream causes the ice to melt at the interface (because the melting temperature depends on pressure, an effect known as the *Clapeyron relation*). The resultant water squirts round the rock, and correspondingly refreezes on the downstream side. Weertman includes both mechanisms in his discussion, as follows.

Consider a bed consisting of an array of (cubical) obstacles of dimension a a distance l apart, and suppose the ice flow exerts an (average) shear stress τ at the bed. The drag on each obstacle is therefore τl^2, and thus the pressure increase upstream of an obstacle is (approximately) $\tau l^2/2a^2$, while the decrease downstream is $-\tau l^2/2a^2$. The pressure difference causes a temperature difference (due to the Clapeyron effect) of

$$\Delta T \approx C\tau l^2/a^2 \tag{1.1}$$

where C is the slope of the Clapeyron curve, $-dT_m/dp = C \approx 0.0074$ K Pa^{-1}. Let u_R be the regelative ice velocity: then $u_R a^2$ is the regelative water flux. The latent heat required to melt this is $\rho_i L u_R a^2$, where ρ_i is ice density and L is latent heat. The heat transfer is effected through the obstacle, at a rate $(k\Delta T/a)a^2 = k\Delta T a$, where k is the thermal conductivity of the bedrock. Equating these suggests that

$$u_R = \left(\frac{kC}{\rho_i L a}\right)\frac{\tau}{\nu^2}, \tag{1.2}$$

[1] It is interesting to note that the more usual no slip condition was a matter of controversy throughout the nineteenth century, see Goldstein (1938, page 676), and was adopted finally on account of agreement with experiment. On the other hand, no slip at a contact line between two fluids at a wall leads to a discontinuous velocity and thus a non-integrable stress singularity (Dussan V. and Davis 1974), and it has been proposed that some slip occurs there; Navier's slip condition $\beta u = \mu\partial u/\partial n$ is in fact a sliding law of the type discussed here.

[2] Such a film is in fact maintained to quite low temperatures by the thermodynamic mechanism of *pre-melting*: see Dash (1989).

where $\nu = a/l$ is a measure of the roughness of the bedrock. Regelation is thus effective at *small* wavelengths.

On the other hand, let u_V be the velocity due to viscous shearing past the obstacle. The upstream stress generated is $\approx \tau/2\nu^2$, and for a nonlinear (Glen's) flow law $\dot{\varepsilon} = A\tau^n$, the resulting strain rate is $\approx A(\tau/2\nu^2)^n$, with $n \approx 3$. Hence we infer

$$u_V \approx aA(\tau/2\nu^2)^n. \tag{1.3}$$

Weertman added these velocities, thus

$$u = \frac{C_1}{a}\frac{\tau}{\nu^2} + C_2 a(\tau/\nu^2)^n, \tag{1.4}$$

where C_1 and C_2 are material coefficients. It can also be argued that the stresses should be added, thus

$$\tau = \nu^2[R_r a u + R_v (u/a)^{1/n}], \tag{1.5}$$

where R_r and R_v are material roughness coefficients, given approximately by

$$R_r \approx \frac{\rho_i L}{kC}, \quad R_v \approx 2A^{-1/n}. \tag{1.6}$$

We see that motion past small obstacles occurs mainly by regelation, while motion past larger obstacles occurs largely by viscous deformation. There is a *controlling obstacle size* at which the stresses are comparable,[3] and if we take a as this value, we obtain the Weertman sliding law

$$\tau \approx \nu^2 R u^{\frac{2}{n+1}}. \tag{1.7}$$

1.2 Nye-Kamb theory

Nye (1969, 1970) and Kamb extended Weertman's scale analysis by means of a mathematically precise model. Their analysis is limited to a linear viscous flow law, and therefore inevitably leads to a linear sliding law, $\tau \propto u$. However, the analysis is not limited to a single roughness scale, and the rôle of the controlling obstacle size is made more explicit in this theory.

Nye's approach is to solve the Stokes flow equation $\nabla^4\psi = 0$ for the stream function in the ice, in $z > \nu h(x)$, and Laplace's equation for the temperature θ in the rock, $z < \nu h(x)$. By expanding for small ν, one obtains problems in upper and lower half spaces respectively, which are linear and can be solved easily via a Fourier transform. Nye's result is

$$\tau = \frac{\eta k_*^2 u}{2\pi}\int_{-\infty}^{\infty} \frac{P_h(k)k^3}{k^2 + k_*^2} dk, \tag{1.8}$$

[3]More specifically, consultation of (1.5) indicates that, *for fixed slopes* ν, the stress is minimised at a certain value of amplitude a: this is the controlling amplitude. Weertman actually chose a to minimise u in (1.4); this gives essentially the same result (check this!).

where $P_h(k)$ is the *power spectral density* (formally we define $h_L = h$ for $|x| < L$, $h_L = 0$ for $|x| > L$,and then $P_h(k) = \lim_{L\to\infty} \hat{h}_L^2/L$, where $\hat{h}_L$ is the Fourier transform of h_L). Here the controlling wavenumber k_* is defined by

$$k_*^2 = \frac{\rho_i L}{4CK\eta}, \tag{1.9}$$

where K is the thermal conductivity, and η is ice viscosity. This can be compared to the controlling wave number in (1.5), which for $n = 1$ would be given by

$$\left(\frac{2\pi}{a}\right)^2 = 4\pi^2 \frac{R_r}{R_v} = \frac{2\pi^2 \rho_i L A^{-1}}{KC}, \tag{1.10}$$

of the same form since $(2A)^{-1}$ here is the viscosity.

Nye's theory emphasises the rôle of the power spectrum of h in determining the roughness. Unfortunately, there is no generalisation available for nonlinear flow laws.

1.3 Nonlinear sliding laws

Lliboutry (1968, 1979) is the principal exponent of more complicated theories associated with a nonlinear flow law and a fairly general bed. The key to a more precise model is a variational principle for slow non-Newtonian flows. Fowler (1981) and Meyssonnier (1983) used this to derive bounds for the roughness in the sliding law, where regelation was neglected (i.e., the bedrock varies on a longer length scale than $2\pi/k^*$). One obtains the sliding law $\tau = Ru^n$, with bounds for R, which are however, rather wide for general h. For a pure sine wave, however, a very good approximation can be derived. Research on these clean problems continues using numerical as well as analytic methods. For a useful summary and an account of some new results, see Gudmundsson (1997a,b).

1.4 Cavitation

Lliboutry was the person who emphasised the importance of cavitation in the sliding law. Cavities exist in the lee of obstacles and have the effect of reducing the effective roughness of the bed. Lliboutry framed his theory in terms of a pseudo-empirical shadowing function, which determined the fraction of cavity free bed in terms of the cavity roof slopes. He derived various forms for the sliding law, and raised the possibility that the functional relation for $u(\tau)$ could be multi-valued. Importantly, he also demonstrated that the sliding law would depend on the *effective pressure* $N = p_i - p_c$, where p_i is overburden pressure and p_c is cavity pressure (which will be equal to the hydraulic drainage pressure at the bed). Iken (1981) suggested that as N decreases (given τ) there is a critical $N_c > 0$ below which unstable sliding will occur. The argument is based only on a force balance;[4] the result can be interpreted in terms of Fowler's (1986) extension of Nye's earlier theory. By reformulating Nye's

[4]This is not quite enough, since determination of the forces in a viscous flow problem also requires consideration of the rheological flow law.

model (neglecting regelation) as a Hilbert problem, Fowler showed that for typical 'unimodal' bedrocks (i.e. with one hump per period), the sliding law was

$$\tau = Nf(u/N), \tag{1.11}$$

where f first increases from zero to a maximum f^* and then decreases to zero as $u \to \infty$. Hence $\tau < Nf^*$ for all u, i.e. $N > \tau/f^*$ which corresponds to Iken's separation pressure. Examination of the results indicates that the maximum drag Nf^* is reached when the cavity from one bump begins to reach the next. In reality, where bumps of varying sizes exist, Fowler (1987) suggested that in this case the drag is simply shifted to larger bumps as the smaller ones become drowned. Since small bumps are all cavitated in this theory, this justifies neglecting the effect of regelation. An approximate method led to the generalised Weertman law

$$\tau = cu^r N^s \tag{1.12}$$

with typical values $0 < r,\ s < 1$.

1.5 Comparison with experiment

Sliding laws are not easily compared with data. There is a hint of field and experimental support for (1.12). Budd *et al.* 1979 found that experimental data on ice sliding over solid slabs of various materials could be described by (1.12), with $r = s = \frac{1}{3}$. Bindschadler 1983) tested various sliding laws against measured data from Variegated Glacier, and also found a best fit when $r = s = \frac{1}{3}$. Bentley (1987) tested a variety of sliding laws against data from Whillans ice stream in Antarctica, and found them all wanting.

The main problem with (1.12) (assuming it is correct) is in the assessment of a sensible value for c; this must rely on small scale details of the bed configuration, which by its nature is not readily available for inspection.

Lecture 2. Subglacial drainage theory

2.1 Weertman films

Weertman (1972) conceived of water flowing at the base of an ice sheet or glacier as a thin water film. Walder (1982) showed that such a flow would be unstable. The mechanism is pervasive to channel forming flows: a local increase in film thickness leads to increased flow, hence increased frictional heating, increased meltback, and thus further channel widening. This positive feedback provides an initial instability, which is limited at short wavelengths by a dissipative mechanism, for example heat conduction. Even though a uniform film is not feasible, one can argue that an uneven 'patchy' film may exist, and this is one possibility under the Antarctic Siple Coast ice streams (Alley 1989).

The water flux Q per unit width through a film of mean thickness h is given by

$$Q = \frac{h^3}{12\mu}\left(\rho_w g \sin\theta - \frac{\partial p}{\partial s}\right), \tag{2.1}$$

where μ is the viscosity of the water, θ is the inclination of the bed to the horizontal, p is water pressure, and s is distance downstream. For a patchy film, this relation would be modified by a pre-multiplicative tortuosity coefficient. If α is the ice surface inclination, then the ice pressure is $p_i = \rho_i g H$, where H is ice thickness, and $\partial H/\partial s \approx \tan\theta - \tan\alpha$, so $\partial p_i/\partial s \approx -\rho_i g[\tan\alpha - \tan\theta]$, and (2.1) is

$$Q \approx \frac{h^3}{12\mu}\left[\Phi + \frac{\partial N}{\partial s}\right], \tag{2.2}$$

where Φ is the gravitational head,

$$\Phi = \rho_w g \sin\theta + \rho_i g(\tan\alpha - \tan\theta) \tag{2.3}$$

(cf. equation (2.2)), and $N = p_i - p$ is the effective pressure. Normally, the hydraulic (gravitational) head is much bigger than the effective pressure gradient term.

2.2 Röthlisberger channels

Outlet streams from glaciers frequently flow from a single channel, often carved as a large tunnel in the ice. The basic theory of drainage through such channels was elaborated by Röthlisberger (1972), followed by Nye (1976). The mechanism of flow is that, firstly, the channel water pressure is below the ice overburden pressure, thus the *effective pressure* $N = p_i - p_w$ is positive. (This is an observation, but is also necessary for integrity of the ice.) As a consequence of this excess pressure, viscous ice creep tends to close the channels. This is counteracted by the frictional heat release by the turbulent flow in the channel. This balance enables channels to be maintained in a steady state.

The Nye/Röthlisberger model is

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{m}{\rho_w} + M, \tag{2.4}$$

where S is channel cross section, Q is the water flux, m is the melt rate, and M is the water supply (e.g. from surface meltwater through moulins, etc.). This expresses conservation of mass. The closure condition for the channel is actually a kinematic condition[5] for the ice flow, and is

$$\frac{\partial S}{\partial t} = \frac{m}{\rho_i} - KSN^n, \tag{2.5}$$

where N is effective pressure and K is proportional to the constant in Glen's flow law.

Momentum balance is expressed through a turbulent friction law,

$$\Phi + \frac{\partial N}{\partial s} = \frac{FQ^2}{S^{8/3}}, \tag{2.6}$$

where F is a roughness coefficient[6]; equation (2.6) is due to Manning, and is an empirical correlation.

Lastly the frictional heat generated is due to potential energy release, and the consequent melting is given by an energy equation in the form[7]

$$mL = Q\left(\Phi + \frac{\partial N}{\partial s}\right), \tag{2.7}$$

where L is latent heat. Elimination of Q and m gives the hyperbolic equation for S:

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial s}\left[\frac{S^{4/3}\left(\Phi + \frac{\partial N}{\partial s}\right)^{1/2}}{F^{1/2}}\right] = \frac{\left(\Phi + \frac{\partial N}{\partial s}\right)^{3/2}}{\rho_s F^{1/2}} S^{4/3} + M, \tag{2.8}$$

together with the closure condition

$$\frac{\partial S}{\partial t} = \frac{\left(\Phi + \frac{\partial N}{\partial s}\right)^{3/2}}{\rho_i F^{1/2}} S^{4/3} - KSN^n. \tag{2.9}$$

Röthlisberger solved the steady state model numerically. We see two space derivatives (for S (or Q) and N), and we therefore require two boundary conditions: these can be taken as

[5]The *kinematic* condition at the interface between two fluids is a consequence of their continuity, and states that fluid elements in the interface remain there, i. e., the speed of the interface is the same as that of the fluid which is at the interface. There are two such equations, one each for ice and for water, which is why we get two evolution equations for S.

[6]In terms of Manning's roughness coefficient n', $F = \rho_w g(2\pi)^{2/3}n'^2$ for a semi-circular channel.

[7]This assumes that the water temperature is isothermal, and at the melting temperature. This is a reasonable assumption under normal circumstances, but may not always be accurate during jökulhlaups.

$$
\begin{aligned}
&Q = 0 \text{ at } s = 0, \qquad \text{(stream head)},\\
&N = 0 \text{ at } s = l, \qquad \text{(outlet)}.
\end{aligned} \tag{2.10}
$$

If we neglect $\partial N/\partial s$ (a *singular perturbation*) and approximate (2.4) as $\partial Q/\partial s \approx M$, then $Q \approx Ms$ is known, and in a steady state

$$
\begin{aligned}
S &\approx \frac{F^{3/8}}{\Phi} Q^{3/4},\\
\frac{Q\Phi}{\rho_i L} &\approx KSN^n,
\end{aligned} \tag{2.11}
$$

whence

$$
N \approx \left(\frac{\Phi}{\rho_i L K}\right)^{1/n} \left(\frac{\Phi}{F}\right)^{3/8n} Q^{1/4n}. \tag{2.12}
$$

We see that N increases with Q, and this explains the arterial nature of Röthlisberger channels (they like to be on their own).

2.3 Jökulhlaups

A spectacular success of Nye and Röthlisberger's theory was in its application to the study of jökulhlaups, glacial outburst floods, and in particular those from the subglacial lake Grímsvötn in Iceland (Nye 1976). A compendium of information on Icelandic jökulhlaups is given in the review by Björnsson (1992). The drainage model is now supplemented by an upstream boundary condition on the discharge Q which is related to changes in the upstream lake volume. Changes in lake volume (and thus elevation) cause alteration in the hydrostatic inlet pressure, and thus to the effective pressure N. It turns out that the new inlet condition can be written as

$$
-\frac{A_L}{\rho_w g}\frac{\partial N}{\partial t} = m_L - Q, \tag{2.13}
$$

where A_L is lake surface area and m_L is the lake refilling rate (at Grímsvötn, this is by subglacial geothermal melting). When this boundary condition is applied, and in addition a hydraulic seal exists near the inlet (in the case of Grímsvötn, this is due to the caldera rim surrounding the lake) then regular oscillations occur as observed, and they share many of the observed properties of the floods, in particular that flood initiation occurs when the lake is below the level necessary to float the overlying ice (Fowler 1999).

2.4 Linked cavities

When ice flows over rough bedrock, cavities form in the lee of obstacles. These cavities may be full of subglacial water, and the possibility exists that subglacial drainage may occur through such cavities. This possibility was advanced by Kamb *et al.* (1985) based on observations of the surging Variegated Glacier, and was studied theoretically by Kamb (1987) and Walder (1986). The basic idea is that the cavities will be linked by orifices in the bed which act like little Röthlisberger channels. The flow rate is controlled by these orifices, which are described by a similar theory to that for channels. A difference is that the local overburden stress in the orifices P is given by

$$sP + (1 - s)p_w = p_i, \tag{2.14}$$

where s is the shadowing function defined by Lliboutry (1979): it is the proportion of uncavitated bedrock, and importantly depends on the sliding velocity. Therefore the drainage relation between $P - p_w$ and local water flux Q becomes a relation between $N = p_i - p_w$ and Q which also involves the sliding velocity u (which is itself determined in terms of shear stress τ and N). The drainage effective pressure N is thus determined implicitly in terms of u.

Which drainage system?

A channel system is possible, but if cavities exist, drainage can occur through a linked cavity system. Indeed, observations on Variegated Glacier indicate that the surge in 1982-3 was initiated by a switch in drainage systems. This can be understood as follows. Suppose a linked cavity system with flux Q_K and effective pressure N_K coexists with a channel system with flux Q_R and effective pressure N_R. In equilibrium, $N_R = N_K$ (otherwise water flows from one system to the other *up* effective pressure gradients); if the water flux is perturbed, say increased by ΔQ in the cavity system (and therefore decreased by ΔQ in the channel system), then the effective pressures are perturbed by $-(dN_R/dQ_R)\Delta Q$ in the channel system and $(dN_K/dQ_K)\Delta Q$ in the cavity system. The system will be unstable, leading to further decrease in channel flux, providing

$$-\frac{\partial N_R}{\partial Q} < \frac{\partial N_K}{\partial Q}. \tag{2.15}$$

We have seen that $\partial N_R/\partial Q > 0$, and theory indicates that $\partial N_K/\partial Q < 0$. (The reason for this is that a decrease in N_K corresponds to an increase in cavity pressure, and thus a corresponding decrease in pressure over the bed; the resulting smaller closure rate of the inter-cavity conduits allows an increased transmissive flow between cavities.) Since N_K also depends on sliding velocity u, this instability really depends on u: if u is large enough, there will be a transition from channel drainage to cavity drainage. (Warning: this is a dangerous argument, and requires care.) Kamb (1987) studied the detailed mechanism of the opposite transition, from cavity drainage to channel flow, which is effected through frictional heating induced enlargement of the interconnecting orifices. The switch from channel to cavity systems is associated with the onset of a surge.

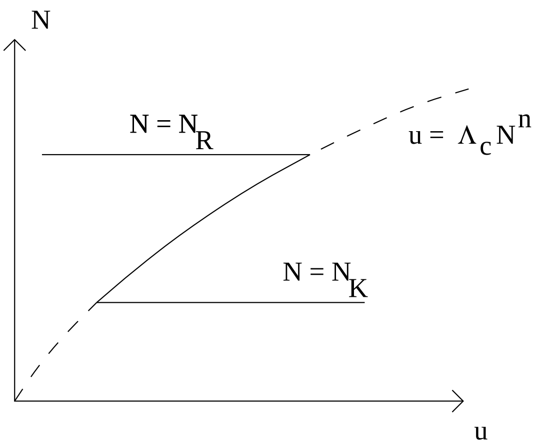


Figure 2.1: Variation of drainage effective pressure with sliding velocity

2.5 Drainage transitions and surges

In the simplified drainage theory of Fowler (1987), N takes values N_R or $N_K < N_R$, depending on the mode of drainage, and there is an instability which is flow dependent, and causes the transition from channels to linked cavities if a parameter $\Lambda = u/N^n$ (n is the exponent in Glen's law) is larger than some critical value. Kamb (1987) describes the opposite instability, when a cavity system is unstable to a channel system. The Kamb instability can be interpreted as occurring when $\partial N/\partial Q$ shifts from negative (stable cavity drainage) to positive (channelised flow), and this occurs as Λ decreases through a second critical value (lower than the first).

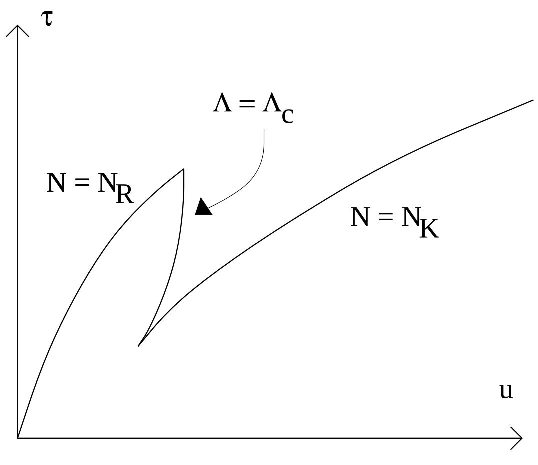


Figure 2.2: A drainage-induced multi-valued sliding law

The result of these two instabilities is the diagram in figure 2.1 which shows that N is a multivalued function of u. When this is coupled with a sliding law of the type

$$\tau = cu^r N^s, \tag{2.16}$$

we find the multi-valued sliding law shown in figure 2.2. The existence of such a multi-valued sliding law can explain glacier surges, and is consistent with observations on Variegated Glacier.

Lecture 3. Basal processes and geomorphology

3.1 Soft glacier beds

Glaciers and ice sheets erode their beds. Stress fracturing plucks boulders from the bed, and these are ground up to cobbles, gravel, and finally to a glacial flour which gives (as suspended sediment) proglacial streams their milky colour. This mass of eroded sediment is carried along at the base of a glacier as a kind of moving conveyor belt, if the ice is temperate, and is known as *till*. It can form a layer several metres thick, and its deformation can be responsible for the bulk of a glacier's motion. One of the best known examples is Trapridge Glacier, a surge-type glacier which is underlain by about six metres of till (Clarke *et al.* 1984). Part of the interest in the basal till in this context is that the surge *must* be associated with sliding, and the mechanism cannot be identical to that responsible for Variegated Glacier surges.

A deforming basal layer also occurs where an ice sheet overrides sediments, such as in the lowlands of Europe or the plains of North America (in the last ice age). A modern example of till-based ice flow is in the Siple coast of Antarctica, where five ice streams (A to E)[8] exist, which are (except for C) zones of fast flowing ice underlain by several metres of wet deforming till (Alley et al. 1987): almost all of the deformation is due to the conveyor-belt sliding of the till. It is therefore important to understand the corresponding sliding law, particularly as this may be instrumental in explaining why ice streams exist, and also how large scale oscillations occurred in the Pleistocene events, associated with Heinrich events (MacAyeal 1993).

Till rheology

When basal ice resting on subglacial sediments is temperate, basal meltwater floods the till, so that it is water-saturated. The motion of wet, granular materials is complicated but is often simply modelled as a Herschel-Bulkley fluid. That is, subjected to a shear stress τ, there is a yield stress τ_0: the particulate medium cannot deform unless it overcomes the frictional resistance between particles. A common assumption is the Coulomb yield criterion:

$$\tau_0 = c_0 + N \tan \psi, \tag{3.1}$$

where N is the effective pressure, ψ is the angle of friction, and c_0 is a small cohesion due essentially to the clay fraction of the till. If this stress is exceeded, then a non-zero strain rate $\dot{\varepsilon}$ results, which depends nonlinearly on the stress; for a pseudo-plastic (Herschel-Bulkley) material, $\dot{\varepsilon} \propto (\tau - \tau_0)^a$, where $a > 1$. Experiments on sub-Antarctic sediments (Kamb 1991) suggest that the creep is better described by a sharp exponential dependence on stress.

The rôle of the effective pressure is also important in the flow behaviour. The effective pressure is $N = P - p_w$, where P is overburden pressure and p_w is pore water pressure, and is essentially a measure of the pressure transmitted between the

[8] Ice stream B has been renamed Whillans ice stream, in memory of the glaciologist Ian Whillans.

solid particles. As we expect the frictional resistance to the flow to increase with N, this suggests that for given τ, $\dot{\varepsilon}$ will decrease as N increases. The simplest realistic *type* of viscous rheology is then the Boulton-Hindmarsh (1987) law

$$\dot{\varepsilon} = A\tau^a N^{-b}, \tag{3.2}$$

if we neglect τ_0, or

$$\dot{\varepsilon} = A(\tau - \tau_0)^a N^{-b}, \tag{3.3}$$

if a yield stress is included.

Dilatancy

A further complication to the rheology is that when a till deforms, it dilates (the particles have to move round each other) so that the porosity ϕ (which under normal consolidation conditions will be a function of N) depends on both τ and N; this may be of less significance cryodynamically, however. The law (3.2) is a very useful form for practical use, although the issue of viscous versus plastic rheology is controversial, and a matter of much current debate.

Boulton and Hindmarsh's (1987) values for (3.2) were $a = 1.33$, $b = 1.8$, $A = 3 \times 10^{-5}$ Pa^{b-a} s^{-1} ($= 4$ bar^{b-a} y^{-1}). Measurements on other glaciers typically give a value of viscosity of about 10^{10} Pa s (Fischer and Clarke 2001). One physically inappropriate inference from (3.2) is that the strain rate becomes infinite (or the viscosity tends to zero) as $N \to 0$. This would be appropriate for an unconfined mixture, where the limit $N \to 0$ is associated with fluidisation and slurry like behaviour. However, under a glacier, it seems unlikely that even in free flotation (with $N = 0$) the resistance would be negligible, since the deformation of the till still requires cobbles and clasts to move past each other. Furthermore, lubricated flow over bed topography produces resistance, just as in classical sliding theory.

Geotechnical theory

Boulton and Hindmarsh's law is purportedly based on seven data points, although the primitive data is not available, and (3.2) cannot be considered to be experimentally substantiated. The creep behaviour of saturated granular materials such as soils has been extensively studied. As stated above, there is a yield stress (which depends on effective pressure and porosity, i.e. consolidation history), and when this is exceeded, plastic deformation occurs. According to Kamb (1991), longer term creep at large strains is typically described by a constitutive law of the form

$$\dot{\varepsilon} = A \exp[\alpha\tau/\tau_f] \tag{3.4}$$

for $\tau > \tau_f$, the stress at failure (for example given by (3.1) above), with very large values of α. In effect, this would imply almost *perfectly plastic* behaviour, that is, $\tau \approx \tau_f$ if $\dot{\varepsilon} > 0$. Kamb used till from Whillans ice stream (ice stream B) in Antarctica to measure $\tau_f \approx 0.02$ bars, and this low value has prompted the idea (Whillans and van der Veen 1993, 1997, Alley 1993) that the Whillans ice stream till is so weak that the resistance to flow must be due to 'sticky spots' or lateral drag. However, Kamb's experiment was effectively at zero confining pressure. If the yield stress is given by $\tau_f = \mu N$, with $N = 0.4$ bar and $\mu = 0.4$, we find $\tau_f = 0.16$ bars, close to the observed basal value. In addition, it is not clear how the deformation of clast-rich till relates to Kamb's small pot experiment. Hooke *et al.*'s (1997) recent results are inconclusive, but may be consistent with an almost perfectly plastic rheology. Iverson et al. (1998) report results of ring shear tests consistent with a Coulomb plastic law, $\tau_f \propto N$; they also remove larger clasts from the apparatus.

Since one expects on average that effective pressure would increase with depth in till (because of hydrostatic effects), so also would the strength, or yield stress, of a perfectly plastic till; in which case one might expect failure at the ice-till interface only. This is inconsistent with observations of deformation with depth (e. g., Porter and Murray 2001), and has led to somewhat *ad hoc* theories to explain this within the context of a plastic rheology (Iverson and Iverson 2001, Tulaczyk 1999). The controversy continues to rage. A useful account of differing points of view is in the issue of Quaternary International on 'Glacier Deforming-bed Processes, volume 86 (2001).

Ice sliding over till

If we neglect the vertical variation of effective pressure due to gravity (an unwarranted assumption: the effective gravitational head over 5 metres of till with $\Delta\rho = \rho_s - \rho_w = 2$ kg m^{-3} is 1 bar, compared to inferred values of $N \sim 0.4$ bar at the base of ice stream B) then a sliding velocity u of ice over a till layer of thickness h_T will give a strain rate

$$u/h_T = A\tau^a N^{-b}, \tag{3.5}$$

following (3.2). It follows that the sliding law in this case is

$$\tau = cu^r N^s, \tag{3.6}$$

with $c = (h_T A)^{-1/a}$, $r = 1/a$, $s = b/a$. If we include the gravitational head, this relation is modified slightly. Most importantly, the yield stress criterion (3.1) suggests that the till does not deform if

$$N > (\tau - c_0)/\tan\psi, \tag{3.7}$$

and this leads to the concept of a deforming 'A' horizon overlying a non-deforming 'B' horizon (Boulton and Hindmarsh 1987). In a glacier, with $\tau \sim 1$ bar and if $\tan\psi \sim 1$, then the A horizon corresponds to a depth of ~ 5 metres. In an ice stream with $\tau \sim 0.1$ bar, it may be only 0.5 metres. Notice that the effect of the finite A horizon is to increase the roughness c in (3.6) (or decrease u, or decrease A).

It is clear that different choices of rheological behaviour will lead to different forms of sliding law. In particular, failure may occur at the ice-till interface and the till-bedrock interface, if one exists, and prescription of corresponding slip rates is problematical.

3.2 Drainage over till

If ice slides over till, the sliding law depends on N; hence we need a drainage law to determine N. Unfortunately, there is little to constrain how water drains over a till bed. One possibility is by Darcy flow downwards through the till to an underlying aquifer. Indeed, if we assume such a downward flow to be q (volume flux per unit area) then Darcy's law implies

$$q = -\frac{k}{\eta_l}\left[\frac{\partial p_w}{\partial z} - \rho_w g\right], \tag{3.8}$$

where z is depth, k is permeability, and η_l is water viscosity. With hydrostatic overburden pressure $\partial P/\partial z = \rho_s g$, where ρ_s is the till density, then

$$\frac{\partial N}{\partial z} = \frac{\eta_l q}{k} + \Delta\rho g. \tag{3.9}$$

To estimate these terms, we take $q \sim G/\rho_w L$, where G is geothermal heat flux, L is latent heat. With $G \sim 0.05$ W m^{-2}, $\rho_w \sim 10^3$ kg m^{-3}, $L \sim 3.3 \times 10^5$ J kg^{-1}, we have $q \sim 1.5 \times 10^{-10}$ m s^{-1} ~ 5 mm y^{-1}. Then for $\eta_l \sim 2 \times 10^{-3}$ Pa s, and $k \sim 3 \times 10^{-14}$ m^2, we have $\eta_l q/k \sim 10^{-4}$ bar m^{-1}, which is insignificant compared to $\Delta\rho g \sim 10^{-1}$ bar m^{-1}.

On the other hand, if subglacial till is underlain by bedrock, so that the meltwater must be evacuated along a flow line, then along this flow line, the integrated water flux is

$$qx = -\frac{kh_T}{\eta_l}\left[\frac{\partial p_w}{\partial x} - \rho_w g \sin\theta\right], \tag{3.10}$$

where θ is the bedslope in the x direction. If we take ice pressure to vary as $\partial p_i/\partial x = -\rho_i g[\tan\alpha - \tan\theta]$, where α is the surface slope, then

$$\frac{\partial N}{\partial x} = \left(\frac{x}{h_T}\right)\left(\frac{\eta_l q}{k}\right) - [\rho_w g \sin\theta + \rho_i g\{\tan\alpha - \tan\theta\}]. \tag{3.11}$$

Consider, for example, the Siple Coast ice streams, for which $x \sim 1000$ km, $h_T \sim 10$ m, $\alpha \sim 10^{-3}$. Then (with $\theta = 0$), $(x/h_T)(\eta_l q/k) \sim 10$ bar m^{-1}, while $\rho_i g\alpha \sim 10^{-4}$ bar m^{-1}. In this case we see that the required flux will lead to negative effective pressures, and in this case we infer the existence of some kind of channelised flow.

Two possibilities have been suggested for this flow. A patchy Weertman type film has been advocated by Alley (1989). Essentially, the water collects in puddles and would have a Darcy type law governing its behaviour. As described earlier, this flow may be subject to instability. If that is the case, then a channelised flow could occur. Now, two distinct end-members are possible: Röthlisberger channels as before, and also 'canals': channels cut down into the till. The mechanism governing their behaviour is similar to that of Röthlisberger channels. However, till creep replaces ice creep and sediment erosion replaces melting.

Walder and Fowler (1994) analysed these types of channel flow, and suggested that the end-member states were distinguished by a critical value N^* of the effective pressure. Essentially, for $N > N^*$, Röthlisberger channels would exist if α were large enough, while for $N < N^*$, canals would exist (for any α). Thus for ice sheets (or ice streams) with small α, canals would provide the drainage mechanism. These have low N (as inferred for Whillans ice stream) and importantly they have N decreasing with increasing water flux Q, which provides for a stable anastomosing pattern (as opposed to the isolated, arterial R channels) and also is a possible source of instability. It ought to be said that the Walder/Fowler theory was rather primitive, but has in certain features received support from the much more detailed analysis of Ng (1998). Engelhardt and Kamb (1997) report consistency of the canal description with observations on Whillans ice stream.

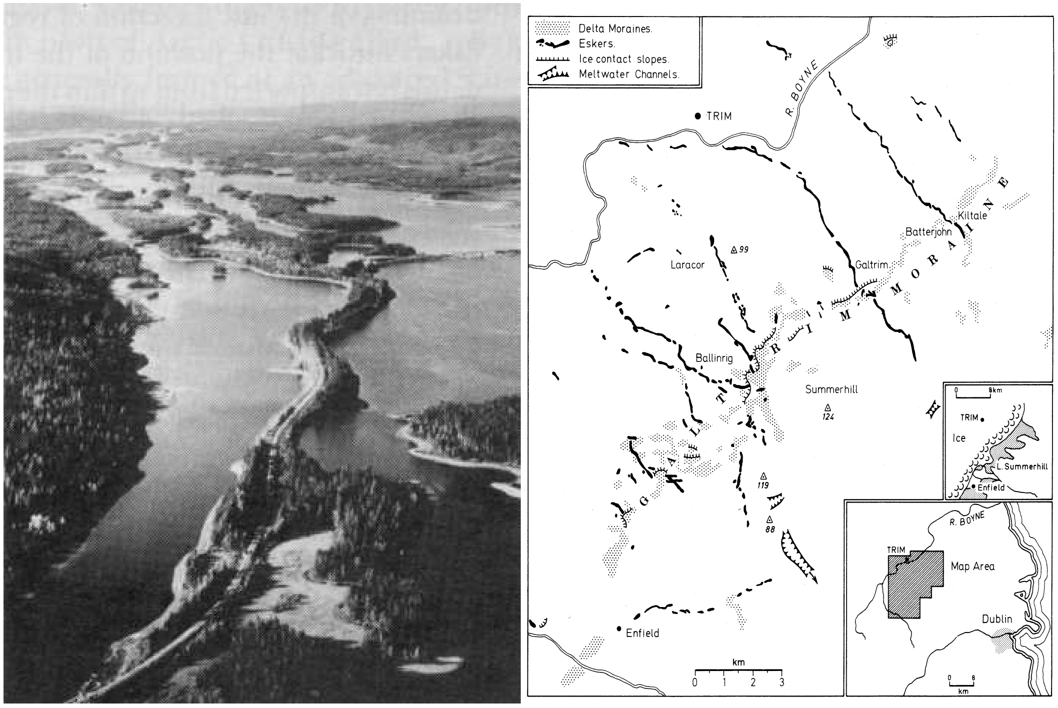
The property that $\partial N/\partial Q < 0$ is fundamentally due to the assumption that subglacial canals, like sub-aerial rivers, choose their own depth, essentially by a balance between erosional shear stress and the critical Shields stress. The film versus canals argument cannot then solely rely on the Walder (1983) stability argument, since that invoked ice melting. In forming canals, it is essential to thicken the film sufficiently to enable the sediments to be eroded. Otherwise, we would speculate that the Walder instability would deform the ice upwards but that in the absence of erosion, the till would simply creep into the uplifted ice regions. A possibility is thus a water film over a wavy till interface. The 5 mm y^{-1} geothermal basal melting would give a film thickness over a 700 km long flow line of 0.6 mm. A puddly base is then feasible under ice streams, but perhaps less likely for the larger melt-water fluxes below glaciers.

3.3 Geomorphological processes

Actually, we essentially know how drainage works in large ice sheets. Eskers (see figure 3.1(a)) are long ridges of gravel and sand deposits which are formed in subglacial channel flows. They could form from either canal-type or channel-type flows; the separated channels evident in figure 3.1(b) are suggestive of a channel type drainage system. Sediment deposition raises the channel floor (and thus also the ice roof) until the channel is pinched off and drainage shifts elsewhere.

A different evolution occurs in the formation of tunnel valleys, for example in the lowlands of Northern Germany (Ehlers 1981). These massive structures, hundreds of metres deep and kilometres wide, are evidential of an anastomosing drainage pattern (hence of canal type, sediment-controlled). However, the flow is sufficiently rapid (and thus N is low and the sediments are very mobile) that the sediments which are squeezed in to the channel can be efficiently removed, and the channel thus eats its way into the substrate. Figures 3.2 and 3.3 show plan view and section of tunnel valleys in Northern Germany.

Eskers are, in this scheme, indicative of weak flow, while tunnel valleys suggest larger flows. Shaw and co-workers (e.g. Shaw *et al.* 1989) argue that in fact many of these features are formed in huge floods by fluvial action, but there seems little necessity for supposing this.



(a) (b)

Figure 3.1: (a): aerial view of a Swedish esker (Sugden and John 1976, page 329); (b): eskers near Trim in Ireland (Embleton and King 1975, page 478).

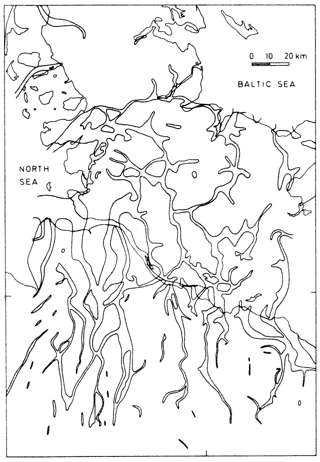


Figure 3.2: Plan view of tunnel valleys in Northern Germany, indicative of anastomosing pattern (Ehlers 1981).

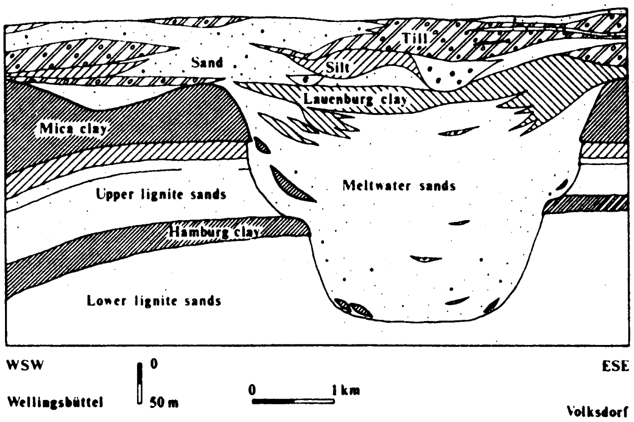


Figure 3.3: Typical section of a tunnel valley (Ehlers 1981).

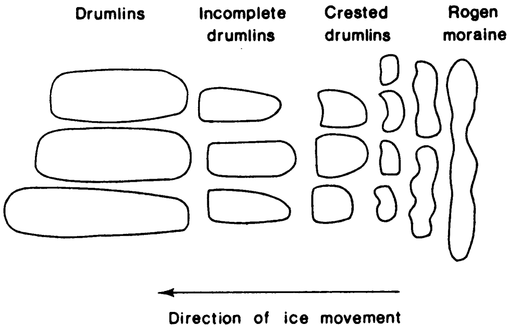


Figure 3.4: Formation of drumlins from the basic Rogen instability (Sugden and John 1976, page 246).

Sedimentary features

A variety of land forms, notably drumlins and Rogen moraine, may be formed subglacially due to the erosive properties of ice moving over deformable sediments, by analogy with dune formation in deserts and rivers. Rogen moraine is a ribbed wave-like formation transverse to ice motion, and drumlins represent a three-dimensional development of it (Lundqvist 1989), much as ripples and dunes under water evolve to bars; presumably the different subglacial land forms are associated with different parametric conditions: see figure 3.4. Figure 3.5 shows a typical swarm of drumlins.

Although the literature on drumlins is extensive and goes back well over a hundred years, dynamic theories are conspicuously absent. Recent work by Hindmarsh (1996, 1998a,b) has demonstrated that the shearing flow of ice over a deformable layer of till can be unstable to the formation of landforms; the instability mechanism is that ice flow over a till protuberance causes increased normal stress on the till, and this decelerates the till flow, causing an enhancement of the bump. Hindmarsh demonstrated instability numerically in a wide variety of conditions, and Fowler (2000) derived analytic criteria for the instability in terms of two parameters, $\xi = rs/N$ and $Y = \alpha\tau/N$, where s is till thickness, N is till effective pressure, τ is basal shear stress, and r and α are two material parameters having values $r \approx 0.1$ bar m^{-1}, and $\alpha \approx 10$ is assumed (this is the same rheological exponent as in (3.4)). The instability region is shown in figure 3.6.

Figure 3.5: A field of drumlins in Saskatchewan. Ice flow was from the bottom left (Boulton 1987, page 66).

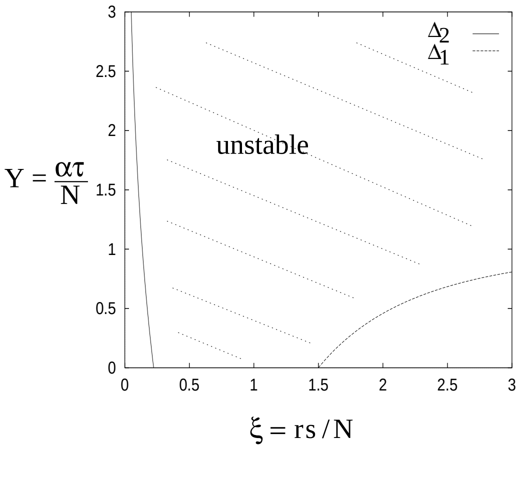


Figure 3.6: Instability region of a layer of till of thickness s. Undulations form under a wide variety of conditions.

The fact that deformable beds create their own microstructure represents a mechanism for enhancing basal friction (and thus sticky spots). Elevated drumlins have relatively high N and are thus stiff and immobile, while the sediments at lower elevations are less viscous. Indeed, this feature may well be an essential constituent of the mechanism of their formation.

Exercises

1. Ice of depth h slides over topography of wavelength l and amplitude a. Use the sliding law (1.5) to assess typical sliding velocities for various values of a, h and l relevant to valley glaciers or ice sheets. Given that shearing within the ice is given by $\frac{\partial u}{\partial z} = A\tau(z)^n$, where $\tau(z) = \rho_i g(h-z)\sin\alpha$, α is the surface slope, and z is the height above the bed $z=0$, use dimensional reasoning to estimate u_b/u_s, the ratio of sliding to surface velocity, for different scenarios, in terms of the parameters $\sigma = l/h$ and $\nu = a/l$. (Use values $\rho_i = 0.9\times 10^3$ kg m^{-3}, $L = 3\times 10^5$ J kg^{-1}, $k = 2$ W m^{-1} K $^{-1}$, $C = 0.8\times 10^{-2}$ K Pa^{-1}, $A = 6\times 10^{-24}$ Pa^{-3} s^{-1}, $n=3$. If you are inventive, use table 5.2 of Paterson (1994, page 97) and a suitable temperature profile.)

2. Use dimensional estimates to estimate the sizes of the terms in Röthlisberger's model of channel drainage, and justify (if you can) the neglect of the $\partial N/\partial s$ term. What effect does this have on the boundary conditions?

   Explain why, if N is an increasing function of Q, one might expect an arterial drainage network, whereas if it is a decreasing function of Q, one might expect a distributed drainage system. [*Hint: imagine what would happen to two neighbouring channels if there is a pressure or flow perturbation in one of them.*]

3. If N decreases as the water flux Q of a canal system draining through sediments increases, then the following positive feedback is potentially operative. An increase of u increases frictional heating, and hence increases water flux Q: this decreases N, which leads via the sliding law to a further increase of u. Suppose that

$$N = \hat{c}/Q^{1/3},$$

   and a suitable sliding law is

$$\tau = cu^r N^s.$$

   Now the meltwater at the base of an ice sheet or ice stream (or cold glacier with temperate base) is due to geothermal heat G, frictional heat τu, but is balanced by cooling into the ice q. The melt flux per unit area is thus $(G + \tau u - q)/\rho_w L$, and in a simple model we equate this with the water flux Q per unit width, thus

$$Q = \frac{(G + \tau u - q)w_d}{\rho_w L},$$

where w_d is the stream spacing. The cooling rate depends on ice thickness and flow. For fast flow, a boundary layer approximation gives $q \approx (\rho_i c_p k/\pi l)^{1/2} \Delta T\, u^{1/2}$, where l is the downstream flowline length. Show that the above equations then collapse, *in suitably scaled units*, to

$$[1 + \tau u - u^{1/2}]^{s/3} = \mu u^r/\tau,$$

with the single parameter μ (which you should define) being a measurement of roughness: small μ means slippery till. If $r = s = 1/2$, show that if μ is small enough, u will be a multi-valued function of τ. Use values $\rho_w = 10^3$ kg m^{-3}, $\rho_i = 0.9 \times 10^3$ kg m^{-3}, $L = 335$ kJ kg^{-1}, $c_p = 2$ kJ kg^{-1} K^{-1}, $w_d = 3$ km (cf. figure 3.1), $k = 2.1$ W m^{-1} K^{-1}, $l = 2000$ km, $\Delta T = 50$ K, $G = 0.05$ W m^{-2}, $\tau = 0.15$ bars, $u = 500$ m y^{-1}, to infer values of Q, and hence also appropriate estimates for c and $\hat{c}$, if $N = 0.4$ bars. Hence calculate a typical value of μ, and investigate whether the sliding law is multivalued in this case. This behaviour may play a rôle in the dynamics of ice streams and surging Pleistocene ice sheets.

4. One objection that might be raised to the idea that drumlins always form through an instability in a deformational layer of till is the observation that some drumlins appear to have a stratigraphy consistent with water deposited sediments. Is this a real objection? Can you think of a way in which till deformation could occur and yet allow such stratigraphy to occur? Or if not, can you think of another physical mechanism whereby such drumlins could be formed? And if so, how could you go about building a mathematical model (i. e., describe what physical processes would be relevant).

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