Mathematical ecology and biology finals questions 1997

1. (i) A population $n \geq 0$ satisfies the ordinary differential equation $\dot{n} = f(n)$, $f(0) = 0$. Explain what is meant by a *steady state* of this equation, and state (and prove) a criterion which determines the stability of a steady state.

   (ii) If $f(n) = n(n_1 - n)(n_2 - n)(n_3 - n)$, $0 < n_1 < n_2 < n_3$, write down the steady states and determine their stability.

   (iii) This population is harvested at a rate h. Show that if h is constant and sufficiently small, there are two possible stable steady states, but that for sufficiently large h, extinction will occur. Give a graphical representation of the possible steady states n^* as a function of h.

   (iv) If $h = En$, show that there are two stable steady states for small E. Suppose that initially, the population is at the larger of these, and that E is increased gradually from zero. Show that extinction can occur in one of two ways; illustrate your answer by giving bifurcation diagrams of n^* versus E for the two cases (a) $n_3 - n_1 \ll n_1$; (b) $n_1 \ll n_2 \ll n_3$.

2. What is meant by an *allosteric* enzyme?

   Consider the model of the MWC dimer with two conformational states R and T, which consists of two protomers, each having a binding site for a substrate S. Assuming T is inactive and cannot bind the substrate, explain why the kinetics of the enzyme system can be written as

$$T_0 \underset{k_b}{\overset{k_f}{\rightleftharpoons}} R_0,$$

$$R_0 \underset{k_-}{\overset{2k_+ s}{\rightleftharpoons}} R_1,$$

$$R_1 \underset{2k_-}{\overset{k_+ s}{\rightleftharpoons}} R_2.$$

   where s denotes the concentration of S.

   Write down the rate equations for the concentrations of T_0, R_0, R_1 and R_2, and by assuming a quasi-steady state, deduce that the fraction of bound S-sites is given by

$$Y = \frac{Q(1+Q)}{L + (1+Q)^2},$$

   where

$$Q = k_+ s / k_-, \quad L = k_b / k_f.$$

3. Write down the SIR model for epidemics in terms of three populations S (susceptible), I (infected) and R (recovered), where the rate of infection is β

(individuals^{-1} time^{-1}) and the rate of recovery is γ (time^{-1}), and the total population $N = S + I + R$ is constant.

Hence show that an epidemic will occur for $I(0) > 0$ if $S(0) > \rho$, where ρ should be defined in terms of β and γ.

If $I(0)$ is very small and $S(0) - \rho = \delta\rho$, where $\delta \ll 1$, show that the infective population for this *weak epidemic* varies with time according to the approximate relationship

$$I \approx \frac{1}{2}\delta^2 \rho \operatorname{sech}^2 \left[\frac{1}{2}\delta\gamma t\right].$$

Now suppose that the recovered population becomes susceptible again at a rate λ (time^{-1}). Non-dimensionalise the resulting weak epidemic model by writing $S = \rho(1 + \delta u)$, $I = \delta^2 \rho v$, $t = \tau/\delta\gamma$, and show that if δ is small and $N > \rho$, the resulting model can be written in the approximate form

$$\begin{aligned} \dot{u} &= \varepsilon - v, \\ \dot{v} &= uv, \end{aligned}$$

where $\dot{u} = du/d\tau$, and

$$\varepsilon = \frac{\lambda\beta}{\delta^2\gamma^2}(N - \rho).$$

Hence show that if $\varepsilon > 0$, epidemics occur repetitively.

4. Describe the genetic basis of sexual reproduction, and give Mendel's laws of inheritance for the offspring of a population of 2-allele genotypes.

For the one locus, two allele model, let n_{ij} be the number of A_iA_j genotypes ($A_1 = A$, $A_2 = a$) in the population, and $n = n_{11} + n_{12} + n_{22}$. If the population birth rate is b, and the death rate of A_iA_j is d_{ij}, show how to derive the equations (ignoring mutations)

$$\begin{aligned} \dot{n}_{11} &= b\left(n_{11} + \frac{n_{12}}{2}\right)(D + H) - d_{11}n_{11}, \\ \dot{n}_{12} &= b\left[n_{11}(H + R) + \frac{n_{12}}{2} + n_{22}(D + H)\right] - d_{12}n_{12}, \\ \dot{n}_{22} &= b\left[\frac{n_{12}}{2} + n_{22}\right](H + R) - d_{22}n_{22}, \end{aligned}$$

where $D = n_{11}/n$, $2H = n_{12}/n$, $R = n_{22}/n$.

If $d_{ij} = d + \varepsilon\Delta_{ij}$, where $\varepsilon \ll 1$, derive equations for D, H and R, and hence show that D, H and R satisfy the Hardy-Weinberg law

$$D \approx p^2, \quad 2H \approx 2pq, \quad R \approx q^2,$$

where p and q are the proportion of A and a alleles in the population.