Mathematical ecology and biology finals questions 1998

1. Explain briefly the one-locus two-allele model of genetic evolution.

   Let $n_{11}(x,t), n_{12}(x,t), n_{22}(x,t)$ be the spatially distributed populations of genotypes AA, Aa, aa, in such a model, where $-\infty < x < \infty$ is the spatial coordinate. Explain briefly the derivation of the equations

$$\begin{aligned}
\frac{\partial n_{11}}{\partial t} &= b\left(n_{11}+\frac{n_{12}}{2}\right)p - d_{11}n_{11} + \nu n_{12} - 2\mu n_{11} + \kappa\frac{\partial^2 n_{11}}{\partial x^2},\\
\frac{\partial n_{12}}{\partial t} &= b\left[n_{11}q+\frac{n_{12}}{2}+pn_{22}\right] - d_{12}n_{12} + 2\nu n_{22} + 2\mu n_{11} - (\mu+\nu)n_{12} + \kappa\frac{\partial^2 n_{12}}{\partial x^2},\\
\frac{\partial n_{22}}{\partial t} &= b\left(\frac{n_{12}}{2}+n_{22}\right)q - d_{22}n_{22} + \mu n_{12} - 2\nu n_{22} + \kappa\frac{\partial^2 n_{22}}{\partial x^2},
\end{aligned}$$

   where $p = D+H$, $q = H+R$, $D = n_{11}/n$, $2H = n_{12}/n$, $R = n_{22}/n$, $n = n_{11}+n_{12}+n_{22}$, and ν, μ, κ are non-negative constants.

   Assuming $d_{12} = d_{22} = d$, $d_{11} = d - \Delta$, deduce that

$$\begin{aligned}
\frac{\partial n}{\partial t} &= n[b-d+\Delta D] + \kappa\frac{\partial^2 n}{\partial x^2},\\
\frac{\partial p}{\partial t} &= \Delta D(1-p) + \nu(1-p) - \mu p + \kappa\left(\frac{\partial^2 p}{\partial x^2} + \frac{2}{n}\frac{\partial n}{\partial x}\frac{\partial p}{\partial x}\right).
\end{aligned}$$

   Assuming the Hardy-Weinberg distribution in a spatially uniform population n, and ignoring mutations, deduce the equation

$$\frac{\partial p}{\partial t} = \Delta p^2(1-p) + \kappa\frac{\partial^2 p}{\partial x^2}.$$

   If p satisfies $p = 1$, $x < 0$ and $p = 0$, $x > 0$ when $t = 0$, explain briefly how you expect the solution to behave for $t > 0$.

2. The production of a substance P from a substrate S is facilitated by a cooperative enzyme D during glycolysis, and is described by the rate equations

$$\begin{aligned}
\frac{ds}{dt} &= v - 2ke_0Y,\\
\frac{dp}{dt} &= -k_p p + 2ke_0Y,
\end{aligned}$$

   where Y is the fraction of bound sites on the enzyme. Explain the meaning of the terms in these equations.

   Assuming

$$Y = \frac{(S+S^2)(1+P)^2}{L+(1+S)^2(1+P)^2},$$

where $S = s/K_m$, $P = p/K_p$, and K_m, K_p, and L are constants, show that the equations can be written in the non-dimensional form

$$\begin{aligned}\dot{S} &= \mu - \phi(S,P),\\ \dot{P} &= -P + K\phi(S,P),\end{aligned}$$

with

$$\phi = \frac{\beta(S+S^2)(1+P)^2}{L+(1+S)^2(1+P)^2},$$

and μ, β and K should be defined.

Find an approximation for ϕ if $1 \ll S, P \ll \sqrt{L}$, and hence find an approximate steady state if $1 \ll \mu K \ll \sqrt{L}$ and $1 \ll (\mu\beta)^{1/2}K \ll \sqrt{L}$. By analysing the nullclines in the vicinity of this steady state, show that it must be a spiral point or a node, and that it is unstable if, approximately,

$$L > 4\beta\mu^3K^2.$$

What do you expect happens in this case?

3. A model for leaf vein formation is partially represented by the activator-inhibitor system

$$\begin{aligned}\frac{\partial A}{\partial t} &= \frac{bA^2}{H} - \mu A + D_a\nabla^2 A,\\ \frac{\partial H}{\partial t}t &= cA^2 - \nu H + D_h\nabla^2 H,\end{aligned}$$

where b, μ, D_a, c, ν, D_h are positive constants. Show that a suitably non-dimensionalised form of the model is

$$\begin{aligned}\frac{\partial a}{\partial t} &= \frac{a^2}{h} - a + \nabla^2 a,\\ \frac{\partial h}{\partial t} &= \lambda(a^2 - h) + \delta\nabla^2 h,\end{aligned}$$

and identify the parameters λ and δ.

Show that the non-zero steady state is diffusively unstable if $\lambda > 1$ and $\delta > (3+2\sqrt{2})\lambda$.

4. The epidemiological model for AIDS represents the infection through the interacting populations of T cells (T), virus (V) and infected T cells (I). T cells are produced at a rate σ, die naturally with specific mortality rate d_T, and become infected through the reaction $T + V \to I$ with rate constant γ. The virus dies naturally at a rate d_V, and is produced by infected T cells at a rate p_V per infected cell. The infected cells die naturally at a rate d_I.

Use this description to explain the model

$$\begin{aligned}\frac{dT}{dt} &= \sigma - d_T T - \gamma V T,\\ \frac{dV}{dt} &= d_V[cI - V],\\ \frac{dI}{dt} &= \gamma V T - d_I I,\end{aligned}$$

where $c = p_V/d_V$, and explain what term has been neglected in the model.

If d_V is sufficiently large, explain why it is reasonable to take $V \approx cI$, and deduce that a viral outbreak will occur if $T > d_I/\gamma_c$. Deduce also that viral infection will always occur in a healthy person if $\sigma > d_V d_I d_T/\gamma p_V$.