Nonlinear systems

Problem sheet 1.

1.1 The pair of equations

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y),$$

has an equilibrium point at $x = x_0$, $y = y_0$. If the matrix A is given by

$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

evaluated at (x_0, y_0), and its trace and determinant are denoted by $T = \operatorname{tr} A$, $D = \det A$, show that the stability of (x_0, y_0) is determined by T and D, and delineate the curves in (T, D) space which separate regions in which the critical point is a saddle, (stable or unstable) node, etc.

1.2 The Lorenz equations are given by

$$\dot{x} = -\sigma x + \sigma y,$$
$$\dot{y} = (r - z)x - y,$$
$$\dot{z} = xy - bz,$$

where $r, \sigma, b > 0$. Show that the origin is a stable equilibrium point if $r < 1$, and unstable if $r > 1$. Show that if $r > 1$, two further equilibria $C_\pm$, given by $x = y = \pm\sqrt{b(r-1)}$, $z = r - 1$, and the eigenvalues of the Jacobian of the linearised equations at either equilibrium satisfy

$$p(\lambda, r) = \lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2b\sigma(r - 1) = 0.$$

Deduce that at least one root is negative, and that if r is close to one, the other two are as well. (You may need to use the fact that as r varies, the roots $\lambda(r)$ of $p = 0$ vary continuously — why?) Show that the two roots other than the negative one can be an imaginary pair $\pm i\Omega$ only if $r = r_c$, where

$$r_c = \frac{\sigma(\sigma + b + c)}{\sigma - b - 1}.$$

Deduce that the equilibria $C_\pm$ are stable if $1 < r < r_c$, and unstable if $r > r_c$.

1.3 If $x \in \mathbf{R}$ satisfies $\dot{x} = f(x, \mu)$, with

$$f(0, 0) = 0, \quad f_x(0, 0) = 0,$$

write down an approximate equation for x near 0 by expanding f in a Taylor series, retaining terms up to degree 3. By considering the relative magnitude of these terms when x and μ are small, show that:

(i) if $f_\mu \neq 0$, then approximately

$$\dot{x} = \mu f_\mu + \tfrac{1}{2}x^2 f_{xx};$$

(ii) if $f_\mu = 0$, approximately

$$\dot{x} = \mu x f_{x\mu} + \tfrac{1}{2}x^2 f_{xx} + \tfrac{1}{2}\mu^2 f_{\mu\mu};$$

(iii) if $f_\mu = 0$ and $f_{xx} = 0$, approximately

$$\dot{x} = \mu x f_{x\mu} + \tfrac{1}{6}x^3 f_{xxx} + \tfrac{1}{2}\mu^2 f_{\mu\mu},$$

and that in the last case, the μ^2 term can also be ignored.

Deduce that the bifurcation is either of saddle-node, transcritical, or pitchfork type.

1.4 Show that the system

$$\dot{x} = \mu - x^2$$

$$\dot{y} = -y$$

has a bifurcation at $\mu = 0$ where a saddle in $x < 0$ joins a node in $x > 0$ (hence the name saddle-node bifurcation).

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PROBLEM SHEET 2.

2.1 Show that if $x \in \mathbf{R}^n$, $x^m = x_1^{m_1} \dots x_n^{m_n}$, $m = (m_1, \dots m_n)$, $\lambda = (\lambda_1, \dots \lambda_n)$, $\Lambda = \operatorname{diag}(\lambda_i)$, and $\{e_k\}$ is a normal basis, then

$$L_\Lambda x^m e_s = [(m, \lambda) - \lambda_s] x^m e_s,$$

where $L_\Lambda h = h' \Lambda x - \Lambda h$, and h' is the Jacobian of h.

Deduce that if

$$g_r = \sum_{\Sigma m_i = r} c_{ms} x^m e_s,$$

then the solution of the homological equation $L_\Lambda p = g_r$ is

$$p = \sum_{\Sigma m_i = r} \frac{c_{ms}}{[(m, \lambda) - \lambda_s]} x^m e_s.$$

(The solution of the homological equation determines the form of the near identity transformation which removes the non-resonant term g_r of degree r.)

2.2 A real, 2-D system $\dot{x} = f(x, \mu)$, $x \in \mathbf{R}^2$, has a Hopf bifurcation at $\mu = 0$. By embedding the system with the equation $\dot{\mu} = 0$, show that the normal form can be written as

$$\dot{z} = \lambda(\mu) z + \sum_m c_m(\mu) |z|^{2m} z,$$

where $\lambda(0) = i\Omega$, and z is an appropriately defined complex variable.

2.3 (Wiggins p. 239) Calculate the general form of the normal form expansions, and give an explicit transformation which removes the quadratic terms from the following systems:

(i)
$$\dot{\theta} = -\theta + v^2,$$
$$\dot{v} = -\sin\theta.$$

(ii)
$$\dot{x} = \tfrac{1}{2} x + y + x^2 y,$$
$$\dot{y} = x + 2y + y^2.$$

2.4 (Wiggins, p. 240) By suspending the following systems with the equation $\dot{\mu} = 0$, find the general form of the normal form expansion of the following systems:

(a)
$$\dot{x} = \tfrac{1}{2} x + y + x^2 y,$$
$$\dot{y} = x + 2y + \mu y + y^2.$$

(b)
$$\dot{x} = x - 2y + \mu x,$$
$$\dot{y} = 3x - y - x^2.$$

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PROBLEM SHEET 3.

3.1 Give a formal justification that a fixed point x^* is stable if $|f'(x^*)| < 1$. (That is, prove that if f is continuously differentiable and x_0 is close enough to x^*, then $x_n \to x^*$ as $n \to \infty$.) Show also that the stability of the periodic orbit $\{x_1, ...x_p\}$ of f is determined by $|\prod_1^p f'(x_j)|$.

3.2 Use the implicit function theorem (see, e.g., Devaney) to show that if $x \to f(x, \epsilon)$ is a smooth (C^1) map, and $x^* = f(x^*, 0)$ with $f'(x^*, 0) \neq 1$, then for sufficiently small ϵ, there is a fixed point $x(\epsilon)$ of f with $x(0) = x^*$. Deduce (with some care!) that the map

$$\theta \to f_\epsilon^s(\theta) = \theta + s\epsilon \sin s\theta + O(\epsilon^2) \pmod{2\pi}$$

defined on the unit circle, i.e. $\theta \in [0, 2\pi]$, has $2s$ fixed points for small ϵ, where $O(\epsilon^2)$ is a C^2 perturbation.

[*The notation* $g \in C^n$ *means the function* g *and its first* n *derivatives are continuous. Hint: consider, in the statement of the implicit function theorem, the functions* $g(x, \varepsilon) = f(x, \varepsilon) - x$, $g(\theta, \varepsilon) = (f_\epsilon^s(\theta) - \theta)/\varepsilon$.]

3.3 By expanding in Taylor series in the vicinity of the bifurcation point, show in detail that a period-doubling bifurcation generates a single periodic orbit, and give a criterion for its stability in terms of the Schwarzian derivative at the fixed point,

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

3.4 Classify all bifurcations of fixed points (stating the bifurcation parameter values) of the following maps:

$$\begin{array}{ll} \text{(i)} & x \to \mu - x^2; \\ \text{(ii)} & x \to \mu x(1 - x); \\ \text{(iii)} & x \to \mu x - x^3. \end{array}$$

Sketch the local bifurcation diagrams.

3.5 (Drazin p. 111) If $x \to f(x, \mu) = x[\mu/\{x^2 + (\mu - x^2)e^{-4\pi\mu}\}]^{1/2}$, show that if $\mu < 0$, f has a stable fixed point $x = 0$, and if $\mu > 0$ then f has an unstable fixed point $x = 0$, and two stable fixed points $x = \pm\sqrt{\mu}$. Find $f^n(x, \mu)$ (*the notation* f^n *denotes the* n*-th iterate of* f). What are the domains of attraction of these stable fixed points?

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PROBLEM SHEET 4.

4.1 Draw a graph of $F(x) = 4x^3 - 3x$ on the interval $[-1, 1]$. Show that the map $x \to F(x)$ has three fixed points, and examine their stability. By using a suitable trigonometric transformation, show that the map is chaotic, and construct a suitable symbolic representation for orbits of F. [*Hint: write x as a ternary fraction*]. Use the symbolic representation of x to find how many fixed points and period two cycles there are, and verify your answer using the map.

4.2 (Jordan and Smith p. 146) Use the Poincaré-Lindstedt method to find approximate periodic solutions of

$$\begin{aligned} &\text{(i)} \qquad \ddot{x} - \epsilon x \dot{x} + x = 0, \\ &\text{(ii)} \qquad (1 + \epsilon \dot{x})\ddot{x} + x = 0, \quad \epsilon \ll 1. \end{aligned}$$

4.3 Use the Poincaré-Lindstedt method to find approximate periodic solutions to

$$\ddot{x} + x + \epsilon x^2 = 0, \quad \epsilon \ll 1.$$

4.4 Show that the system

$$\dot{x} = -y + \mu x + xy^2,$$
$$\dot{y} = x + \mu y - x^2$$

has a Hopf bifurcation at $\mu = 0$. Use the Poincaré-Lindstedt method to find periodic solutions when $\mu = -\varepsilon^2$, and show that their amplitude is approximately $2(-2\mu)^{1/2}$.

4.5 Use the method of averaging in its simple form (e.g. Drazin p. 192) to approximate the solutions of the equation

$$\ddot{x} + \varepsilon[|\dot{x}| - 1]\dot{x} + x = 0$$

as $\varepsilon \to 0$. Could you use Poincaré-Lindstedt?

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PROBLEM SHEET 5.

5.1 Show that the equation

$$\ddot{q} + G(q)\dot{q}^2 - F(q) = 0$$

may be put in Hamiltonian form by writing $p = f(q)\dot{q}$ for some appropriate function $f(q)$.

5.2 Suppose that a Hamiltonian system has Hamiltonian $H(q,p)$, and that a function $S(q,I)$ can be found such that $p = \partial S/\partial q$, *and* S satisfies the *Hamilton-Jacobi* equation

$$H\left(\frac{\partial S}{\partial q}, q\right) = \tilde{H}(I),$$

for some function $\tilde{H}$.

Derive an expression for $\dot{p}$ in terms of $\dot{q}$, $\dot{I}$ and derivatives of S.

Derive another expression for $\dot{p}$ by differentiating $\tilde{H}$ with respect to q.

Hence show that

$$\dot{I} = 0.$$

By differentiating $\tilde{H}$ with respect to I, show also that

$$\dot{\theta} = \frac{\partial \tilde{H}}{\partial I}.$$

(I, θ are *action angle* coordinates for the integrable Hamiltonian $\tilde{H}$.)

5.3 Use perturbation theory to solve for the motion described by the perturbed Hamiltonian

$$H = \tfrac{1}{2}(p^2 + \omega_0^2 q^2) + \varepsilon q^3, \quad \varepsilon \ll 1.$$

(You may assume that suitable action angle coordinates when $\varepsilon = 0$ are $q = (2I/\omega_0)^{1/2}\sin\theta$, $p = (2\omega_0 I)^{1/2}\cos\theta$.)

Hence show that the perturbed frequencies are given by

$$\omega(J) = \omega_0 - \varepsilon^2\left(\frac{15J}{2\omega_0^4}\right)\ldots$$

5.4 Write a short (2–3 page) essay describing how stochastic behaviour arises in perturbed Hamiltonian systems.

Nonlinear systems

PROBLEM SHEET 6.

6.1 For the symmetric Lorenz map

$$\xi \to \operatorname{sgn}(\xi)[\alpha\,|\xi|^{\delta} - \mu],$$

$\alpha > 0$, $\delta < 1$, find the values of μ for which (a) the bifurcating fixed point (from $\xi = 0$ at $\mu = 0$) has a saddle-node bifurcation; (b) the strange invariant set becomes attracting.

Describe what happens as μ varies if $\alpha < 0$.

6.2 (i) By consideration of the Lyapounov function $V = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$, show in detail that trajectories of the Lorenz equation enter the ellipsoid D_ε: $rx^2 + y^2 + b(z - r)^2 = b^2r^2 + \varepsilon^2$ after a finite time. Deduce that thereafter trajectories remain inside the ellipsoid $V \leq \max_{D_\varepsilon} V$. Can we put $\varepsilon = 0$ in the above? Why? Or why not?

(ii) Show that, if $0 < r < 1$ in the Lorenz equations, the origin is globally stable. (Hint: consider the Lyapounov function $V = x^2 + \sigma y^2 + \sigma z^2$.)

(iii) Show that the Lorenz equations have a pitchfork bifurcation of the origin at $r = 1$, and a Hopf bifurcation of the two non-trivial steady states at

$$r = \sigma(\sigma + b + 3)/(\sigma - b - 1).$$

6.3 Shil'nikov bifurcations occur for systems of the form

$$\begin{aligned} \dot{x} &= -\lambda_2 x - \omega y + P(x, y, z, \mu), \\ \dot{y} &= \omega x - \lambda_2 y + Q(x, y, z, \mu), \\ \dot{z} &= \lambda_1 z + R(x, y, z, \mu), \end{aligned}$$

where $P, Q, R = O(r^2)$. If there is a homoclinic bifurcation at $\mu = 0$, then an appropriate Poincaré map is approximately 1-D, and is

$$\zeta \to a\zeta^{\delta} \cos\left[\frac{\omega}{\lambda_1} \ln(1/\zeta)\right] + \mu, \quad (\text{if } \zeta > 0)$$

where $\zeta \propto z$ on the Poincaré surface $r = (x^2 + y^2)^{1/2} = c$. Here $\delta = \lambda_2/\lambda_1$. Deduce that for $\delta < 1$, an infinite number of periodic orbits bifurcate at $\mu = 0$, but for $\delta > 1$, only one does. Draw a bifurcation diagram of $\ln(1/\zeta)$ versus μ in each case.

6.4 Write a short essay describing the way in which chaotic trajectories arise in the Lorenz equations.

Nonlinear systems

VACATION SHEET.

vac.1 Define what is meant by a Cantor set. The Cantor middle-thirds set is constructed by deleting successive middle thirds of intervals, as follows:

$$\begin{aligned} S_0 &= [0, 1] \\ S_1 &= [0, \tfrac{1}{3} \cup [\tfrac{2}{3}, 1] \\ S_2 &= [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{5}{9}] \cup [\tfrac{7}{9}, 1] \\ &\dots \end{aligned}$$

Show that $\lim_{n\to\infty} S_n$ exists, and is a Cantor set (closed, no interior points, no isolated points) (hint: use a ternary representation for $x \in [0, 1]$). Show that this Cantor set has (Lebesgue) measure zero.

Can you define a mapping such that $\phi(S_\infty) = S_\infty$, and which is chaotic?

vac.2 Consider the conjugacy between the Smale horseshoe ϕ and the shift map σ on Σ_2.

(i) Define a metric on Σ_2.

(ii) Show that σ is continuous and (continuously) invertible.

(iii) Construct a dense orbit for σ.

(iv) Let $\mathbf{0} = (...00.000...) \in \Sigma_2$. A sequence $\mathbf{s}$ is called *homoclinic* if $\sigma^n(\mathbf{s}) \to \mathbf{0}$ as $n \to \pm\infty$. Prove that homoclinic sequences are dense in Σ_2.

vac.3 Consider the piecewise linear map $f : [0, 4] \to [0, 4]$ defined by $f(0) = 2$, $f(1) = 4$, $f(2) = 3$, $f(3) = 1$, $f(4) = 0$, with f linear between these values.

(i) Show that f has a unique fixed point in [0,4] and that it is unstable.

(ii) Show that all periodic orbits of f are unstable.

(iii) Show that f has a period 5 orbit, but no period 3 orbit (hint: consider the actions of f on the intervals $I_i = [i - 1, i]$).

(iv) With I_i defined as above, let the matrix $A = (a_{ij})$ be such that $a_{ij} = 1$ if I_i is mapped to I_j, and 0 if not. Show that $(A^n)_{ii}$ represents the number of distinct paths which map I_i to I_i under f^n. Can this be used to infer information on periodic orbits? Calculate A^3 and A^5.