Perturbation Methods

Problem Sheet 5

1. Show that the problem

$$\varepsilon x^m y' + y = 1 \text{ in } 0 < x < 1,$$

$$y(0) = 0, \quad 0 < m < 1,$$

has a boundary layer near $x = 0$, and find the appropriate rescaling for x near $x = 0$. What do you think happens if the equation is $\varepsilon x^m y' + y = x$, $y(0) = 0$?

2. The function $y(x; \varepsilon)$ satisfies

$$\varepsilon y'' + x^{1/2} y' + y = 0 \text{ in } 0 \le x \le 1,$$

and is subject to boundary conditions $y = 0$ at $x = 0$ and $y = 1$ at $x = 1$. Show that there may be a boundary layer at $x = 0$, but not at $x = 1$. Next, find the leading order outer expansion. By appropriately rescaling the equation near $x = 0$, find the leading order inner approximation. Finally, match the inner and outer expansions at leading order.

3. Find two term inner and outer expansions (and match them) of the solution of

$$\varepsilon^2 y'''' - y'' = -1, \quad -1 < x < 1,$$

with $y = y' = 0$ at $x = \pm 1$.

4. The function $y(x)$ satisfies

$$\varepsilon y'' + y y' - y = 0 \quad \text{in } 0 < x < 1.$$

(i) Show that if $y(0) = 0$, $y(1) = 3$, a solution can be found with a boundary layer at $x = 0$, and give leading order approximations for the inner and outer expansions.

(ii) Next suppose $y(0) = -\frac{3}{4}$ and $y(1) = \frac{5}{4}$. Show that a solution is possible having an interior layer where y jumps from $-M$ to $+M$, for some M, and find the leading order matched asymptotic expansions.

Perturbation Methods

Problem Sheet 6

1. Find a boundary layer approximation for u satisfying

$$\varepsilon(u_{xx} + y_{yy}) = u_y, \quad \text{in} \quad 0 < x < 1, \; 0 < y < 1,$$

with $\varepsilon \ll 1$, and

$$u_y = 0 \quad \text{on} \quad y = 0,$$

$$u = 0 \quad \text{on} \quad y = 1,$$

$$u = 1 \quad \text{on} \quad x = 0,$$

$$u = -1 \quad \text{on} \quad x = 1.$$

2. Heat transfer to a fluid flowing in a pipe $0 < r < 1$, $x > 0$, is described by the equation

$$u\frac{\partial T}{\partial x} = \varepsilon \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2} \right],$$

$\varepsilon \ll 1$, with boundary conditions $T = 0$ at $x = 0$, $T = 1$ on $r = 1$ (and $T \to 1$ as $x \to \infty$). Use boundary layer methods to compute the heat flux

$$q = -\left.\frac{\partial T}{\partial r}\right|_{r=1}$$

as a function of x, in the two cases (i) $u = 1$; (ii) $u = 1 - r^2$. (Use similarity solutions to solve the boundary layer equations.)

3. Burgers' equation is

$$u_t + uu_x = \varepsilon u_{xx}, \quad \varepsilon \ll 1, \quad u(\pm\infty, t) = 0,$$

in $0 < x < \infty$. Show that the 'outer' solution for this equation with initial condition $u = u_0(x)$ develops a discontinuity in general (and perhaps several), and show that the propagation of the resulting 'shock' can be described in terms of a moving transition layer at $x = x_s(t)$. Find the solution in the shock, and determine the shock speed $\dot{x}_s$ in terms of limits $\lim_{x \to x_s \pm} u$ of the outer solution.

4. Let u satisfy the equation

$$\varepsilon\left(\frac{\partial}{\partial t}+c_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c_2\frac{\partial}{\partial x}\right)u+\left(\frac{\partial}{\partial t}+b\frac{\partial}{\partial x}\right)u=0,$$

in $x>0$, $t>0$, with $\varepsilon\ll 1$, $c_1>0$, $c_2<0$.

(i) What is the condition on b for stable solutions?

(ii) In the stable case, find an approximate solution to the signalling problem where

$$u=1 \quad \text{on} \quad x=0,$$

$$u=u_t=0 \quad \text{on} \quad t=0,$$

assuming $b>0$.

What happens if $b<0$?

Perturbation Methods

Problem Sheet 7

1. Suppose that $xQ(x) > 0$, and that $Q'(0) > 0$. Show that a uniform asymptotic approximation to the function w satisfying

$$\varepsilon^2 w'' = Q(x)w, \quad -\infty < x < \infty, \quad \varepsilon \ll 1,$$

with $w \to 0$ as $x \to \infty$, is

$$w \sim C S_0^{1/6} Q^{-1/4} Ai\left[\left(\frac{3S_0}{2\varepsilon}\right)^{2/3}\right],$$

where C is an arbitrary constant, $S_0 = \int_0^x Q^{1/2}dx$, and Ai is the (bounded) Airy function. Under what conditions on Q does $w \to 0$ as $x \to -\infty$?

2. Find the large eigenvalue solutions of the equation

$$y'' + \lambda(1 - x^2)^2 y = 0$$

subject to $y = 0$ at $x = \pm 1$.

At the ends $x = \pm 1$ you will need to use turning point solutions like $(1 - x^2)^{1/2} J_{1/4}(\lambda^{1/2}(1 - x^2)^2/4)$, and then use $J_{1/4}(z) \sim (2/\pi z)^{1/2} \cos(z - 3\pi/8)$ as $z \to \infty$. For definitions and asymptotic properties of the Bessel functions J_ν, see Abramowitz and Stegun, Handbook of Mathematical Functions.

3. Use multiple scales methods to derive approximate equations for the (complex) amplitude of x satisfying

$$\begin{aligned} &\text{(i)} \qquad \ddot{x} + \varepsilon \dot{x}^3 + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0; \\ &\text{(ii)} \qquad \ddot{x} + \varepsilon \dot{x}(x^2 - 1) + (1 + \varepsilon k)x = \varepsilon \cos t. \end{aligned}$$

How do the solutions behave? Can you derive the amplitude equations using the method of averaging?

4. Find the leading order approximation which is valid for times $t = O(\varepsilon^{-1})$ to the solution of

$$\dot{x} + x^2 y \cos t = \varepsilon(x - 2x^2),$$

$$\dot{y} = \varepsilon\left(1 - \frac{\sin t}{x}\right),$$

with $x(0) = 1$, $y(0) = 0$.

Perturbation Methods

Vacation problems

1. *Resonance.*

   The equation
   $$\ddot{x} + \beta\dot{x} + x + \delta x^3 = \Gamma \cos \omega t$$
   represents a damped, forced, nonlinear oscillator.

   (i) Find a particular solution for x when $\beta = \delta = 0$, and plot the response amplitude a versus the forcing frequency ω. *Resonance* occurs at $\omega = 1$.

   (ii) For ω near 1, $\beta = 0$, $\delta \ll 1$, show that there is a *distinguished* limit in which $\omega = 1 + \varepsilon\mu$, $\mu = O(1)$, $\varepsilon \ll 1$, $x \sim \Gamma/\varepsilon$, and $\varepsilon = (\delta\Gamma^2)^{1/3}$. By using multiple scales, find an evolution equation for the slowly varying amplitude $A(\tau)$, $\tau = \varepsilon t$, of $x = Ae^{it} + (cc)$. Deduce the modification to the response diagram of $|A|$ versus μ.

   (iii) Repeat the above calculation, but with $\beta = b\varepsilon$, $b = O(1)$. What is the effect of $b \neq 0$?

2. *Relaxation oscillations.*

   The Van der Pol oscillator with strong damping can be written in the form
   $$\delta\ddot{x} + (x^2 - 1)\dot{x} + x = 0, \quad \delta \ll 1.$$
   By seeking appropriate 'outer' and 'transition' layers, show that an approximate periodic solution can be constructed to leading order via matched asymptotic expansions, and show that the period is approximately $3 - 2\ln 2$.

3. *An aging spring.*

   Use any method to construct approximate solutions to
   $$\ddot{x} + e^{-\varepsilon t}x = 0, \quad \varepsilon \ll 1.$$

4. Attempt to solve approximately, or say why you cannot, the equations
   $$\ddot{x} + \sin x = \varepsilon \sin t,\ \varepsilon \ll 1,\ x(0) = 1,\ \dot{x}(0) = 0;$$
   $$\ddot{x} + \varepsilon\beta\dot{x} + \sin x = \varepsilon \sin t,\ \varepsilon \ll 1,\ \beta = O(1),\ \text{as } t \to \infty.$$
   $$\ddot{x} + \sin x = \sin \varepsilon t,\ \varepsilon \ll 1,\ x(0) = 1,\ \dot{x}(0) = 0.$$

5. *Homogenisation and Darcy's law.*

The Stokes equations for viscous fluid flow are

$$\boldsymbol{\nabla}.\mathbf{u} = 0,$$

$$0 = -\boldsymbol{\nabla} p + \varepsilon^2 \nabla^2 \mathbf{u},$$

in dimensionless variables, where $\mathbf{u}$ satisfies the no-slip boundary condition $\mathbf{u} = \mathbf{0}$ on $f(\mathbf{x}/\varepsilon) = 0$ (this being a periodic surface which bounds fluid flow through a porous network of tubules or veins in a solid). Use multiple scales $\mathbf{x}$ and $\mathbf{X} = \mathbf{x}/\varepsilon$ to formulate approximate equations for $\mathbf{u} \sim \mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)} + \ldots$, $p \sim p^{(0)} + \varepsilon p^{(1)}$, and show that there are solutions $\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x})$, $p^{(0)} = p^{(0)}(\mathbf{x})$, and then

$$\mathbf{u}^{(0)} = -\frac{\partial p^{(0)}}{\partial x_j} \mathbf{w}^{(j)},$$

where $\mathbf{w}^{(j)}$ satisfies

$$\boldsymbol{\nabla}_X \mathbf{w}^{(j)} = 0,$$

$$0 = -\boldsymbol{\nabla}_X P + \nabla_X^2 \mathbf{w}^{(j)} + \mathbf{e}_j,$$

where $\mathbf{e}_j$ is the unit vector in the X_j direction, and

$$\mathbf{w}^{(j)} = \mathbf{0} \quad \text{on} \quad f(\mathbf{X}) = 0.$$

Deduce that the average velocity (over the periodically repeated volume V), denoted by $\langle \mathbf{u} \rangle$, satisfies Darcy's law in the form

$$\langle \mathbf{u} \rangle = -\mathbf{K} : \boldsymbol{\nabla} p^{(0)},$$

where the dimensionless permeability tensor is given by

$$K_{ij} = \langle w_i^j \rangle.$$