b6 Viscous flow

PROBLEM SHEET 1.

1.1 Show that if $\dot{a} = u(a,t)$, $\dot{b} = u(b,t)$ ($\dot{} = d/dt$) then

$$\frac{d}{dt}\int_a^b F(x,t)\,dx = \int_a^b \left\{\frac{\partial F}{\partial t} + \frac{\partial}{\partial x}(Fu)\right\} dx.$$

Write down the generalisation of this result for the derivative

$$\frac{d}{dt}\int_{V(t)} F(\mathbf{x},t)\,dV(\mathbf{x})$$

when the boundary of $V(t)$ moves with speed $\mathbf{u}$.

Prove this result by changing to Lagrangian variables (ξ,η,ζ) where $d\mathbf{x}/dt = \mathbf{u}$, $\mathbf{x} = (\xi,\eta,\zeta)$ at $t = 0$ and showing that

$$\frac{d}{dt}\left(\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}\right) = \frac{dJ}{dt} = J\mathrm{div}\,\mathbf{u}.$$

1.2 Show that if orthonormal basis vectors for two coordinate systems are $\mathbf{e}_i$ and $\mathbf{e}'_j$, then the corresponding position vectors $\mathbf{x}$ and $\mathbf{x}'$ are related by

$$\mathbf{x}' = L\mathbf{x},$$

where $L = (l_{ij})$ and $l_{ij} = \mathbf{e}'_i.\mathbf{e}_j$.

Show that

$$l_{ij}l_{kj} = \delta_{ik},$$

where the *Kronecker delta* $\delta_{ij} = 1$ if $i = j$, $= 0$ if $i \neq j$. Deduce that L is orthogonal, i.e. $LL^T = I$.

Show that if $\mathbf{u}$ is a vector field, the components of $\mathbf{u}$ must transform as

$$u'_i = l_{ij}u_j.$$

Deduce that the strain rate tensor

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

satisfies the transformation rule

$$\dot{\varepsilon}_{pq} = l_{pi}l_{qj}\dot{\varepsilon}_{ij}.$$

1.3 Use the definitions of

$$\operatorname{curl}\mathbf{u} = \varepsilon_{ijk}\mathbf{e}_i\frac{\partial u_k}{\partial x_j},$$

$$\operatorname{grad}\phi = \mathbf{e}_r\frac{\partial \phi}{\partial x_r},$$

$$\operatorname{div}\mathbf{u} = \frac{\partial u_s}{\partial x_s},$$

$$\nabla^2 = \frac{\partial^2}{\partial x_l \partial x_l},$$

to prove the formula (in cartesian coordinates)

$$\nabla^2\mathbf{u} = \mathbf{e}_i\nabla^2 u_i = \operatorname{grad}\operatorname{div}\mathbf{u} - \operatorname{curl}\operatorname{curl}\mathbf{u}.$$

[Hint: $\varepsilon_{ijk}\varepsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$.]

1.4 Define the stress tensor σ_{ij} and the rate-of-strain tensor $\dot{\varepsilon}_{ij}$. By consideration of force and torque balance for an infinitesimal fluid element, show that

$$\sigma'_{ij} = l_{ip}l_{jq}\sigma_{pq}$$

and

$$\sigma_{ji} = \sigma_{ij},$$

where $L = (l_{ij})$ is the transformation matrix taking the orthonormal basis $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_j\}$, and σ_{pq} and σ'_{ij} are the corresponding tensor components.

State the physical assumptions that are needed for a fluid to be *Newtonian*, that is

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\dot{\varepsilon}_{ij},$$

where p is the pressure and μ the viscosity. Hence derive the Navier-Stokes equations for an incompressible, constant viscosity, Newtonian fluid, in the form

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\boldsymbol{\nabla})\,\mathbf{u}\right) = -\boldsymbol{\nabla}p + \mu\nabla^2\mathbf{u},\ \ \boldsymbol{\nabla}.\mathbf{u} = 0.$$

Given that

$$\mathbf{u} = \frac{1}{r}\left(1 - e^{-\rho r^2/4\mu t}\right)\mathbf{e}_\theta$$

(where $\mathbf{e}_\theta$ is the unit azimuthal vector in cylindrical polar coordinates) satisfies the Navier-Stokes equations, show that as $t \to 0$ this flow satisfies the Euler equations, and interpret the flow physically. What happens as $t \to \infty$?

1.5 Viscous incompressible fluid of constant density ρ and coefficient of viscosity μ lies at rest in the region $0 < y < \infty$. At $t = 0$ the rigid boundary $y = 0$ is jerked into motion in the x-direction with constant speed U. Explain briefly why the velocity $\mathbf{u}$ of the fluid takes the form $\mathbf{u} = (u(y,t), 0, 0)$ and show that

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

where $\nu = \mu/\rho$.

Write down the initial and boundary conditions satisfied by u. By writing $u = f(\eta)$ where $\eta = y/(\nu t)^{1/2}$, show that the solution to (1) with the above conditions is

$$u = U\left[1 - \frac{1}{\sqrt{\pi}} \int_0^\eta e^{-s^2/4} ds\right].$$

Sketch the solution, illustrating carefully how the profile changes with time.

1.6 Incompressible fluid occupies the half space $y > 0$ above a plane rigid boundary $y = 0$ which oscillates backwards and forwards in the x-direction with velocity $U \cos \omega t$. Assuming that $\mathbf{u} = (u(y,t), 0, 0)$ satisfies

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

show, by writing $u = \mathrm{Re}\,(f(y)e^{i\omega t})$, that

$$u = Ue^{-\sqrt{\omega/2\nu}\,y} \cos\left(\sqrt{\frac{\omega}{2\nu}} y - \omega t\right).$$

What is the width of the 'boundary layer', i.e. the order of magnitude of y for which the velocity is significant?

b6 Viscous flow

PROBLEM SHEET 2

2.1 A fluid flows steadily along a straight cylindrical pipe of uniform cross-section D. The x-axis is along the pipe and the velocity in the x-direction is u. Show that u satisfies the equation

$$\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=-G \text{ in } D$$

with $u=0$ on the boundary ∂D, where $-G$ is the (constant) pressure gradient in the x-direction. Show that the drag per unit length on the pipe wall is G times the area of the cross-section. Show that if D is a circle of radius a (Poiseuille flow), the total flux down the pipe is

$$\frac{\pi G a^2}{8\mu}$$

If the cross-section of the pipe is an equilateral triangle bounded by $y=0$, $y\pm\sqrt{3}z=\sqrt{3}a$, find an explicit polynomial solution for u.

2.2 Write down (look them up) the Navier Stokes equations in cylindrical polar coordinates. Hence find an exact solution for the shearing azimuthal flow (Taylor-Couette flow) induced between two concentric cylinders of radius r_1 and r_2 when they rotate with angular velocities Ω_1 and Ω_2 respectively.

Find also an exact solution for the shearing flow (Couette flow) induced by the motion of two parallel planes a distance d apart, say at $z=0$ and $z=d$, when these move at speeds v_1 and v_2 in the x–direction. (Assume there is no pressure gradient.)

Find an approximation to the Taylor-Couette solution when the gap width between the cylinders is small (in a sense you should define, i.e. small compared to what?), and show that this approximate solution is the same as the exact solution for Couette flow with suitably defined parameters.

2.3 A viscous fluid is in two-dimensional flow between two planes at $\theta=\pm\alpha$ (in cylindrical coordinates). Show that in a purely radial flow, the radial velocity is $u=F(\theta)/r$, and deduce carefully that F satisfies the equation

$$F''+4F+\frac{F^2}{\nu}+c=0,$$

where c is constant. What are suitable boundary conditions for the flow?

Suppose now the flow is symmetric about $\theta=0$. By scaling F with $F(0)$ and θ with α, write down the equation and boundary conditions for the scaled function $\phi=F(\theta)/F(0)$, and show that if $u>0$ in the channel, ϕ is given by

$$\int_\phi^1\frac{d\phi}{[R(1-\phi)\{K+(1+\frac{6}{R})\phi+\phi^2\}]^{1/2}}=\alpha\sqrt{\tfrac{2}{3}}\eta,$$

where K is given by

$$\int_0^1\frac{d\phi}{[R(1-\phi)\{K+(1+\frac{6}{R})\phi+\phi^2\}]^{1/2}}=\alpha\sqrt{\tfrac{2}{3}},$$

and you should define R. By consideration of this integral equation for K, show that no such solution exists if

$$\alpha > \pi/2,$$

or if

$$\alpha\sqrt{\frac{2R}{3}} > \int_0^1 \frac{d\phi}{[\phi(1-\phi^2)]^{1/2}}.$$

2.4 Write down the second order differential equation satisfied by the functions $F(\theta)$ and its scaled version $\phi(\eta)$ which describe the radial velocity in a Jeffery–Hamel flow (cf. question 2.3), together with suitable boundary conditions, in terms of parameters α and R which you should define.

Assume the flow is symmetric. Consider separately the cases where the centreline velocity is inwards and outwards. Show that when R is large, the inflow case becomes characterised by a cross-sectionally uniform velocity, together with thin 'boundary layers' of thickness $\sim R^{-1/2}$ at the walls, but that the corresponding solutions for the outflow case oscillate rapidly in θ, and are not unique. (*Your arguments should not be precise, but based on consideration of the form of approximate solutions of the equation.*)

Is the assumption of symmetry necessary for these conclusions? What about the assumption of large R?

[*It may help to think of the equation for ϕ as that of a nonlinear oscillator,* $\phi'' + V'(\phi) = 0$.]

b6 Viscous flow

PROBLEM SHEET 3

3.1 Explain what the first and second laws of thermodynamics imply about the internal energy and the entropy of a material, and deduce that

$$TdS = de + p\,dv,$$

where p is the pressure and v is the specific volume.

The Gibbs free energy is defined as $G = e + pv - TS$; by calculating $\partial^2 G/\partial T \partial p$ in two ways, show that

$$-\left.\frac{\partial S}{\partial p}\right|_T = \left.\frac{\partial v}{\partial T}\right|_p;$$

hence deduce that

$$T\,dS = c_p\,dT - \alpha v T dp,$$

where c_p is the specific heat at constant pressure, and α is the thermal expansion coefficient.

Derive the equation of conservation of energy for a fluid in the form

$$\rho c_p \frac{dT}{dt} - \alpha T \frac{dp}{dt} = \boldsymbol{\nabla}.(k\boldsymbol{\nabla}T) + \tau_{ij}\dot{\varepsilon}_{ij}.$$

3.2 Write down the equations governing the motion and temperature of a linearly viscous, incompressible fluid (do not assume the viscosity is constant). Such a fluid flows between two parallel plates $z = 0$ and $z = h$, the latter of which is moved relative to the other under the action of a prescribed constant shear stress τ_w. Suppose that the plates are held at constant temperature $T = 0$, the thermal conductivity is constant, and that $\mu = \mu_0 e^{-\beta T}$, where μ_0 is constant. Show that a possible solution is of the form $\mathbf{u} = (u(z), 0, 0)$, $T = T(z)$, and write down the ordinary differential equations and boundary conditions satisfied by u and T. (you should assume that there is no pressure gradient.)

Show that a solution of the above form is only possible for a certain range of values of τ_w. (You may use the fact that $\displaystyle\int_0^{T_m} \frac{dT}{\sqrt{e^{\beta T_m} - e^{\beta T}}}$ increases from zero at $T_m = 0$ to a maximum value I_m and then decreases to zero as $T_m \to \infty$.)

3.3 Write down the energy equation for a viscous, compressible fluid flow. Use suitable estimates to estimate the importance of the adiabatic heating term, the heat conduction term, and the viscous dissipation term for (i) flow of water in a tap; (ii) flow of rock in the Earth's mantle. Show also that an adiabatic temperature gradient could be expected to be a good approximation in the Earth's atmosphere (depth 10 km, horizontal wind speed 20 m s^{-1}, horizontal length scale 1000 km).

[*For flow in a tap, take* $\Delta p \sim \rho U^2$, *where* U *is the velocity scale; in the mantle, take* $\Delta p \sim \rho g l$, *where* l *is the length scale. Properties of water can be found in the book by Batchelor cited below; for properties of the Earth's mantle, see, for example, D. L. Turcotte and G. Schubert (1982), Geodynamics, Wiley.*]

3.4 Estimate the size of the Reynolds number in the following fluid flows:

a pebble thrown in a pond;
a bubble rising in a glass of Champagne;
pouring golden syrup;
a mountain stream;
a 747 at take off;
an Olympic swimmer;
the atmospheric circulation;
rain running down a window pane;
waves on a beach;
flow of a glacier;
basaltic magma (that you see in rapid lava flows);
rock convecting in the earth's mantle.

[*For viscosities and other properties of air and water, see, for example G.K. Batchelor (1967), An introduction to fluid mechanics, C.U.P. You may take viscosities of magma, ice, syrup and rock as being* 10 Pa s, 10^{13} Pa s, 10^3 Pa s, *and* 10^{21} Pa s *respectively, and* 1 Pa = 1 N m^{-2}.

b6 Viscous flow

Problem sheet 4

4.1 Show that the momentum equation for an incompressible viscous fluid can be written as

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \wedge \boldsymbol{\omega} + \boldsymbol{\nabla} H = -\nu \boldsymbol{\nabla} \wedge \boldsymbol{\omega}$$

where $H = p/\rho + \frac{1}{2}|\mathbf{u}|^2$, $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u}$, and hence show that

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}.\boldsymbol{\nabla})\,\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

If $\Gamma(t) = \int_{C(t)} \mathbf{u}.\,d\mathbf{s}$ is the circulation round a closed contour $C(t)$ which is moving with the fluid , derive Kelvin's result

$$\frac{d\Gamma}{dt} = -\nu \int_C (\boldsymbol{\nabla} \wedge \boldsymbol{\omega}).\,d\mathbf{s}.$$

Show that, for two-dimensional flow in the (x, y)-plane with the stream function ψ, the vorticity is in the z-direction and has z-component ω where

$$\omega = -\nabla^2 \psi.$$

Deduce that the Navier-Stokes equations lead in this case to the equation

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial(\psi, \nabla^2 \psi)}{\partial(y, x)} = \nu \nabla^4 \psi.$$

4.2 State and prove the Prandtl-Batchelor theorem concerning high Reynolds number flow bounded by closed streamlines.

4.3 Assuming that $T = \frac{2}{\sqrt{\pi}} \int_\eta^\infty e^{-s^2} ds$ satisfies

$$\frac{\partial T}{\partial x} = \epsilon \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right),$$

with $T = 1$ on $y = 0, x > 0$, and $T \to 0$ at infinity, where $\epsilon > 0$ and $(\xi + i\eta)^2 = (x + iy)/\epsilon$, indicate on a diagram the regions of the (x, y)-plane where $T \not\to 0$ as $\epsilon \downarrow 0$. Deduce that as $\epsilon \downarrow 0$, there is a boundary layer of thickness $O(\sqrt{\epsilon})$ in which

$$\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial Y^2},$$

with $T(x, 0) = 1$, $T(x, \infty) = 0$, to lowest order, in suitably scaled coordinates.

4.4 An incompressible fluid of small kinematic viscosity ν flows along a plane wall $y = 0$ with mainstream velocity $U(x)$. Explain how to obtain the two-dimensional boundary layer equations

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial Y} &= \frac{\partial^2 u}{\partial Y^2} + UU', \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial Y} &= 0. \end{aligned}$$

Let ψ denote the stream function. Show that a 'similarity' solution of the form

$$\psi = F(x)f(\eta), \text{ with } \eta = \frac{Y}{g(x)},$$

is possible if

$$\text{either } U(x) \propto (x - x_0)^m \text{ or } U(x) \propto e^{cx},$$

where x_0, m and c are constants. In the first of these two cases, for what value of m is the boundary layer thickness independent of x?

4.5 A flat plate of length L is aligned with a free stream flow $U\mathbf{i}$. Starting from the Navier-Stokes equations in the form

$$\boldsymbol{\nabla}.\mathbf{u} = 0, \quad (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\mathbf{u},$$

show that if $UL/\nu \gg 1$, the flow near the plate can be described by the boundary layer equations

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned} \tag{2}$$

when written in suitable non-dimensional variables. What boundary conditions have to be imposed? Verify that if the plate is semi-infinite, with the leading edge at the origin, then $u = f'(\eta)$, $v = -x^{-1/2}(f(\eta) - \eta f'(\eta))/2$, where $\eta = yx^{-1/2}$, satisfy (2) as long as

$$f''' + \tfrac{1}{2}ff'' = 0,$$

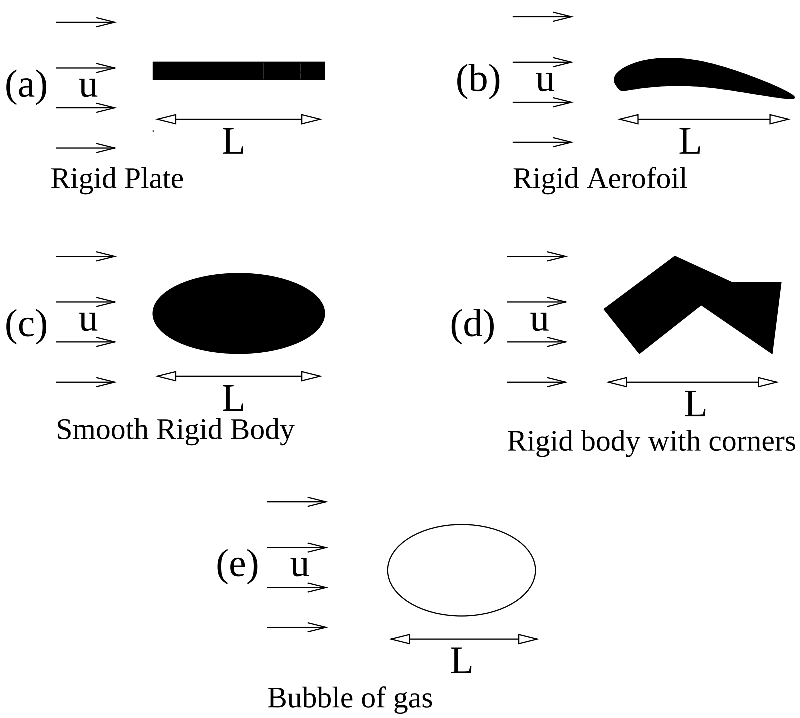
with $f(0) = f'(0) = 0$ and $f' \to 1$ as $\eta \to \infty$.

Give a rough sketch of the streamlines and compare them with those predicted by an inviscid irrotational model.

b6 Viscous flow

Problem sheet 5

5.1 Viscous fluid flows past the objects shown below with $UL/\nu \gg 1$. Indicate on a sketch where you would expect boundary layer theory to apply/break down and also where the equations of irrotational inviscid flow will be a good approximation



5.2 Starting from the Navier-Stokes equations in the form

$$(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\mathbf{u}, \quad \boldsymbol{\nabla}.\mathbf{u} = 0,$$

which describe the steady flow of a stream with speed U_∞ past a two-dimensional body of a typical dimension L, derive Prandtl's dimensionless boundary-layer equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2}, \ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

in suitable dimensionless boundary layer variables which you should define. Show that $u = v = 0$ for $y = 0$ and $u \to U(x)$ as $y \to \infty$, where $p + \frac{1}{2}U^2 = \text{const}$.

Assuming that the body is smooth and blunt, explain how to derive p. Show that, for a circular cylinder of radius L, $U(x) = 2\sin x$, where x is the (dimensionless) arc-length around the cylinder measured from the forward stagnation point.

Give a criterion for the validity of Prandtl's equations in terms of $(\partial u/\partial y)_{y=0}$, and indicate by a diagram what happens in the vicinity of a point where this derivative vanishes. Does the existence of such a point affect your estimate of p?

5.3 Show that the scaling $f = \alpha F(\xi)$, $\eta = \xi/\alpha$ leaves the Blasius equation

$$\frac{d^3 f}{d\eta^3} + \tfrac{1}{2} f \frac{d^2 f}{d\eta^2} = 0$$

unchanged but changes the boundary condition at ∞. Suppose that the solution of the initial-value problem

$$F''' + \tfrac{1}{2} F F'' = 0$$

with $F(0) = F'(0) = 0$, $F''(0) = 1$ is calculated and it is found that $F'(\infty) = C$ ($\simeq (0.332)^{-2/3}$). How can f be found in terms of F and what is the value and physical significance of $f''(0)$?

5.4 (*Jeffery-Hamel again*)
Starting from the Navier-Stokes equations, show that for steady radial flow in a two-dimensional wedge ($0 < \theta < \alpha$) due to a source of strength (i.e. flux) Q, a suitably defined dimensionless velocity is $(g(\theta)/r)\,\mathbf{e_r}$ where g satisfies

$$\frac{1}{Re}\left(g'' + 4g\right) + g^2 = \text{constant}$$

with $g(0) = g(\alpha) = 0$, $\int_0^\alpha g(\theta)\,d\theta = 1$, and the Reynolds number is $Re = Q/\nu$. Show that an approximate "outer" solution when $|Re| \gg 1$ might be $g \approx 1/\alpha$. Show further that in any boundary layer that might exist on $\theta = 0$ the appropriate equation for g is

$$\pm \frac{d^2 g}{d\phi^2} + g^2 = \frac{1}{\alpha^2}$$

where $\phi = |Re|^{1/2}\theta$, and plus and minus signs refer to outflow and inflow, respectively. Write down the boundary and matching conditions for g and deduce that such a solution is only possible for inflow (i.e., $Q < 0$).

Now suppose $|Re| \ll 1$. Show that g satisfies the approximate equation

$$g'' + 4g \approx \text{constant},$$

Hence find an expression for g in terms of three constants A, B, and C, and write down three conditions satisfied by these constants. Show that a unique solution exists unless $\alpha = \pi$ or $\tan\alpha = \alpha$. Write down the general solution in the former case.

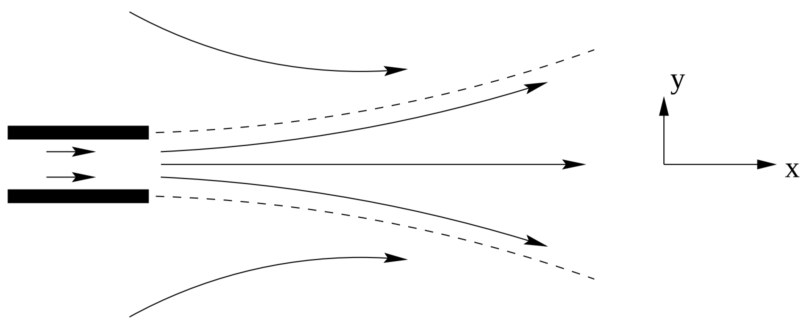
b6 Viscous flow

PROBLEM SHEET 6

6.1 Consider a two-dimensional jet far enough downstream of a nozzle that it can be described by the boundary layer equations

$$\psi_y \psi_{xyy} - \psi_x \psi_{yyy} = \psi_{yyyy}$$

with $\psi_y \to 0$ at ∞ :



What are the boundary conditions on $y = 0$?

You need some extra information to ensure that the solution is not just $\psi \equiv 0$. One possibility is to look at the mass flux $\int_{-\infty}^{\infty} u\, dy$ but this is an unknown function of x because of entrainment (see figure above). What *is* constant for a jet is the momentum flux $\int_{-\infty}^{\infty} \psi_y^2\, dy = M$.

Show that $\psi = x^{1/3} f(\eta)$, $\eta = y/3x^{2/3}$ is a solution as long as

$$\begin{aligned} f''' + ff'' + f'^2 &= 0, \\ f(0) = f''(0) &= 0, \\ f'(\infty) &= 0, \end{aligned}$$

and that

$$f = 2\alpha \tanh \alpha\eta,$$

where α is a constant to be found in terms of M.

6.2 Explain what the difference is between *laminar* and *turbulent* flow.

Hydrodynamic stability theory aims to explain transition to turbulence by linearising the Navier-Stokes equations. Explain how this is done, and derive the Orr-Sommerfeld equation for two-dimensional stream function disturbances to a plane parallel flow $U(y)$ of the form $\psi = \phi(y)\exp[i\alpha(x - ct)]$, and give suitable boundary conditions on ϕ for plane Poiseuille flow.

Describe briefly why this theory is largely irrelevant to the phenomenon of transition.

6.3 A turbulent flow has velocity components $u_i = \bar{u}_i + u_i'$, where the overbar denotes a (local) time average. Write down the Navier-Stokes equations, and hence derive an averaged equation for $\bar{u}_i$.

Deduce that the viscous stresses are supplemented in the mean flow equation by the Reynolds stress $-\rho\overline{u_i' u_j'}$.

What is meant by an eddy viscosity?

A one-dimensional turbulent shear flow flows between two plane walls a distance $2h$ apart, under a constant pressure gradient $-p'$. If the maximum averaged velocity is u_m at $z = h$, and the eddy viscosity is assumed to be of the form

$$\varepsilon_T = \rho u_m^2 \phi(z/h),$$

show that

$$u_m = \left[\frac{-p' h^2 c}{\rho}\right]^{1/3},$$

and give an expression for the constant c.

Evaluate c if $\phi(\zeta) = \zeta^\alpha(1-\zeta)$, $\alpha < 1$. Why might this be a reasonable choice? What happens if $\alpha = 1$? How might the problem be resolved?

6.4 Starting from the steady state Navier-Stokes equations

$$(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\mathbf{u}, \quad \boldsymbol{\nabla}.\mathbf{u} = 0,$$

explain how the slow flow approximation

$$\mathrm{curl}^3\mathbf{u} = 0, \quad \boldsymbol{\nabla}.\mathbf{u} = 0 \tag{3}$$

can be determined when the Reynolds number is small.

In spherical polar coordinates (r, θ, ϕ) an axisymmetric solution of (3) can be written as

$$\mathbf{u} = \mathrm{curl}\left(0, 0, \frac{\psi}{r\sin\theta}\right) = \left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, 0\right).$$

Given that $\mathrm{curl}^2(0, 0, \dfrac{\psi}{r\sin\theta}) = (0, 0, -\dfrac{D^2\psi}{r\sin\theta})$ where $D^2 = \dfrac{\partial^2}{\partial r^2} - \dfrac{\cot\theta}{r^2}\dfrac{\partial}{\partial\theta} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$, show that ψ satisfies the equation

$$D^4\psi = 0.$$

A fluid of viscosity ν flows with constant velocity U past a sphere of radius a where $Ua \ll \nu$. State the boundary conditions satisfied by ψ in this case and show that a suitable solution is

$$\psi = U\left(\tfrac{1}{2}r^2 - \tfrac{3}{4}ar + \tfrac{a^3}{4r}\right)\sin^2\theta.$$

6.5 Starting from the Navier-Stokes equations, formulate the problem of a 2-dimensional steady flow of stream velocity $\mathbf{u}$ and viscosity ν past a cylinder of radius a when $|\mathbf{u}|a/\nu \ll 1$. Show that, if the stream-function ψ is written in the form $f(r)\sin\theta$, then it will satisfy the slow flow equations provided

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)^2 f = 0.$$

Write down the boundary conditions which f must satisfy at $r = a$ and show that if f satisfies these conditions then ψ cannot approach the free stream flow at infinity.

Show that a regular expansion in powers of ϵ ($\epsilon \ll 1$) does not give a complete solution to the model problem

$$u'' + \epsilon u' = 0 \text{ with } u(0) = 0 \text{ and } u(\infty) = 1,$$

and use this analogy to explain the difficulty that has arisen above.

b6 Viscous flow

VACATION PROBLEM SHEET

VAC 1 Write down the solution of the equation

$$y'' + \epsilon y' = 0$$

with $y = 0$ at $x = 0$ and $y = 1$ at $x = L$. Expand this solution asymptotically as $\epsilon \to 0$ to show that

$$y \sim \frac{x}{L} + \frac{\epsilon}{2L}x(L-x) + O(\epsilon^2).$$

This formula gives $y'(0) \sim \epsilon/2$ as $L \to \infty$; explain why this approximation is invalid for sufficiently large L.

Use the above example to explain the sense in which the dimensionless slow flow equations

$$0 = -\boldsymbol{\nabla} p + \nabla^2 \mathbf{u}, \quad \boldsymbol{\nabla}.\mathbf{u} = 0$$

are a good approximation to the Navier-Stokes equations for a flow streaming past an obstacle at small Reynolds number Re. If $\mathbf{u} \sim \mathbf{i} + O(1/r)$ as $r \to \infty$ in such a flow, where r is the spherical polar coordinate, show that the slow flow model will break down when $r \sim Re^{-1}$.

What boundary conditions apply to the slow flow equations when the obstacle is

(i) a solid sphere?
(ii) a stationary gas bubble?

VAC 2 Viscous fluid with a constant kinematic viscosity ν occupies the gap, of thickness $O(h_0)$, between a flat plate $z = 0$ moving with constant velocity U in the x-direction and a stationary rigid surface described by $z = h_0 h(x/L)$, $0 < x < L$. From the Navier-Stokes equations, taking the fluid flow to be two-dimensional and assuming that both h_0/L and the reduced Reynolds number $Uh_0^2/\nu L$ are extremely small, show that the pressure p in a suitably dimensionless form satisfies the equation

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) = 6\frac{dh}{dx}.$$

Explain why, in general, pressure variations are much larger in cases where h is non-constant than in cases where h is constant.

In a three-dimensional flow under the same conditions, the stationary surface is described by $z = h(x, y)$. Show that the equation satisfied by p is now

$$\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3 \frac{\partial p}{\partial y}\right) = 6\frac{\partial h}{\partial x}.$$

A slider bearing consists of a flat plate $z = 0$ moving with constant velocity U in the x-direction and a fixed slider at $z = h_0(1 - \lambda x/L)$ for $0 < x < L$ and $0 < \lambda < 1$.

The ambient pressure outside the bearing is p_0; show that the pressure within the bearing is

$$p_0 + \left(\frac{\mu U L}{h_0^2}\right)\left[\frac{6\lambda x(1-x)}{(2-\lambda)(1-\lambda x)^2}\right].$$

Show that the total load supported by the bearing is

$$\frac{6}{\lambda^2}\left(\frac{\mu U L^2}{h_0^2}\right)\left(\log\frac{1}{1-\lambda} - \frac{2\lambda}{2-\lambda}\right).$$

VAC 3 A thin layer of viscous fluid is in two-dimensional flow down a slope which is at an angle α to the horizontal. The fluid occupies the region $0 < x < x_N(t)$, where x denotes distance down the slope. Use thin film theory (assume the aspect ratio is much less than $\tan\alpha$) to show that the thickness of the film $h(x,t)$ satisfies the equation

$$\frac{\partial h}{\partial t} + \frac{g\sin\alpha}{\nu}h^2\frac{\partial h}{\partial x} = 0.$$

Show that, assuming the volume of fluid $\int_0^{x_N} h\,dx$ is conserved, there is a similarity solution of the form $h = t^{-\alpha}\phi(\xi)$, $\xi = x/t^\alpha$, and find α. Deduce that $h \propto (x/t)^{1/2}$ and $x_n \propto t^{1/3}$.

[You may assume that effects near the 'nose' $x = x_N(t)$, where h must be brought rapidly to zero, may be neglected.]

When $\alpha = \pi/2$ verify that $h = f\left(x - \dfrac{gh^2}{\nu}t\right)$ solves the equation, so that particular values of h propagate down with speed proportional to h^2. Hence, or otherwise, draw rough sketches of the evolution with time of a realistic initial profile $h(x,0)$.

Can you comment on the case $\alpha = 0$ (e.g. an ice sheet)?

VAC 4 A squeeze film exists between two parallel plates of length L at $z = 0$ and $z = \delta L$. Starting at $t = 0$, a constant force F is exerted downwards on the top plate. Show that the correct scaling for the horizontal velocity is $F\delta^2/\mu$. Deduce that at time t later the distance between the plates is $\delta L/(1+\alpha t)^{1/2}$ where α is to be determined. This predicts that the plates will not touch in finite time. Consider the validity of this prediction by computing roughly the time taken for the film to close when the surface roughness (the "asperity" size) has amplitude $10^{-3}\delta L$.