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On the flow of polythermal glaciers
I. Model and preliminary analysis

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Many interesting phenomena have been observed in the flow of glacial ice masses. In order to establish a rational theory for the study of these phenomena, we develop here a detailed continuum model for the flow of polythermal (i.e. partially cold and partially temperate) glaciers in wide mountain valleys. As a first step in the analysis of the nonlinear double free boundary problem posed by this model, the structure and stability of solutions are studied in a particular very simple limiting situation. Further analysis is deferred to part II of the paper.

1. Introduction

On a time scale of years, glacial ice flows like an incompressible non-Newtonian fluid, a typical velocity being 100 metres per year. The flow is driven by gravity and is supplied by the accumulation of new ice formed on the glacial surface from packed fallen snow. In general, climatic conditions will be such that over certain regions of the surface (typically those at lower altitudes) the ice is ablating rather than accumulating and for these regions the ice accumulation rate is simply considered to be negative. Glaciers will also in general be polythermal, i.e. consist of two zones, 'cold' and 'temperate', in which the ice is respectively below and at the melting point. Particular thermal limits are the so-called 'polar' and 'temperate' glaciers which consist almost exclusively of ice in one or the other of these two states. In the cold zone, the ice is basically frozen to the bedrock. When the ice becomes temperate, however, it can slide over the bedrock by maintaining a thin water film there which lubricates the flow. In this case, the ice experiences a frictional drag as it flows over the underlying bedrock undulations.

Many interesting phenomena have been observed in the flow of glacial ice masses. Of those that occur on a global scale, one should mention slow 'kinematic' waves, fast 'seasonal' waves and 'surges'. The kinematic waves appear as undulations in the glacial surface which travel along at about three to four times the surface speed of the ice (see, for example, Nye (1960) and the references listed there). The seasonal

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waves are waves of velocity which travel at about 20 to 150 times the surface speed of the ice (Deeley & Parr 1914; Hodge 1974). Surging valley glaciers exhibit periodic motions of a relaxation oscillation type. This oscillation consists of two phases: an active (or surging) one, when the ice velocity increases by an order of magnitude and a substantial part of the glacier travels down the valley to lower altitudes, and a quiescent one, when the elongated glacier withdraws slowly up the valley towards its original pre-surge state. These phases typically last about one to two years and 30–40 years respectively (Meier & Post 1969). Of the three flow phenomena mentioned here, the first has been studied by Nye (1960, 1963) using Whitham & Lighthill's (1955) theory of kinematic waves. His analysis predicts wavespeeds which are similar to those actually observed. For seasonal waves, there is no quantitative theory which can predict the high propagation rates of velocity disturbances. A similar difficulty besets the study of surges, although for this case the physics is much better understood (Weertman 1969) and various numerical models (e.g. Budd 1975) have exhibited surge-like oscillations, the periods and amplitudes of which are similar to those observed. It remains to be shown, however, that one of the various postulated physical theories (e.g. Robin 1955, 1969; Weertman 1969) can mathematically predict the occurrence of surges.

In this paper, we develop a self-consistent, non-dimensionalized continuum model for the flow of polythermal glaciers in wide mountain valleys in order to establish a rational theory for the study of the above-mentioned (and other) flow phenomena. In so doing, we have minimized the use of semi-physical arguments and empirical laws of the type that often appear in presently existing glacier flow models. In §2 we introduce the full set of differential equations and boundary conditions which serves as a basis for our flow model. In §3 these equations and boundary conditions are made non-dimensional in a consistent manner and the important dimensionless parameters of polythermal ice flow are identified. This provides us with a rational basis for neglecting numerous terms in the full model set of equations and boundary conditions. In §4 this procedure is used to develop a vastly simplified 'reduced' flow model which we expect to be a reasonable approximation to the full model. It is this reduced model that will be analysed in detail in Part II of this paper. In §5 we begin this analysis by considering the structure and stability of solutions in a particular limiting case where the viscosity is independent of the temperature and the glacier is completely cold. In §6 we summarize the results of the present paper. For ease of reference, a nomenclature is given in §7.

In the past, a variety of models have been proposed to explain certain features of the flow (e.g. Nye 1960; Budd 1975). The most detailed analytical model is that of Grigoryan, Krass & Shumskiy (1976), who consider a simpler model set of differential equations and boundary conditions (in curvilinear coordinates), non-dimensionalize this set, and then qualitatively discuss the behaviour of solutions to the resulting set in the case where the heat transport by convection dominates that by conduction. It should be emphasized that there are numerous fundamental differences between their work and that presented here. For example, as will be made clear in § 2 here, we provide a more complete set of boundary conditions and, following Lliboutry (1976), we allow the stress–strain rate relationship for temperate ice to be a function of its moisture content. Secondly, our non-dimensionalization of the model is determined explicitly by the dimensional parameters actually present in the problem and is therefore more consistent. Thirdly, we show how to reduce our model to a much more manageable (though still complex) form, which enables us in § 5 and part II to carry out detailed mathematical analysis and obtain quantitative as well as qualitative results.

2. The basic model

Before describing the general model, we would like to emphasize that a table of nomenclature is given in § 7. This table may be of considerable use to the reader in view of the fact that quite a large number of symbols are introduced in this paper.

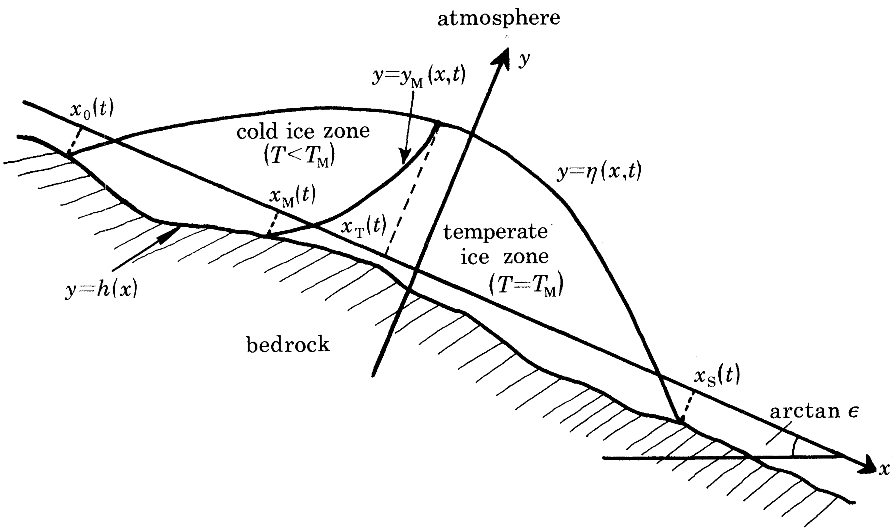

Figure 1. Typical polythermal glacier profile.

The important physics of polythermal glacial ice flow has been outlined in § 1; further details may be found in Paterson (1969) and will be presented as required in the remainder of this paper. In wide mountain valley glaciers, the flow is essentially two-dimensional, and so we restrict attention here to glaciers whose profiles are the same for all longitudinal sections; a typical geometry of such a profile is shown in figure 1. We take a coordinate system a sshown where the x axis makes an angle arctan ϵ with the horizontal and ϵ is the mean bedrock slope. The origin may be chosen arbitrarily. We denote by $y = \eta(x, t)$ the top surface, by $y = h(x)$ the (given) bedrock profile, and by $y = y_M(x, t)$ the surface dividing the cold and temperate zones, which will be called the melting surface. On this surface $T = T_M$, where T and T_M are the local ice temperature and pressure melting temperature respectively. The intersections of the melting surface with the atmosphere and the bedrock are denoted by $x_T(t)$ and $x_M(t)$ and will be called the top and bottom melting points respectivley. We denote the ends of the glacier by $x_0(t)$ and $x_S(t)$, x_0 being the 'head' and x_S being the 'snout'.

Let $\boldsymbol{q} = (q_1, q_2) = (u, v)$ be the velocity field and p the pressure in the ice. Then the continuum equations which represent the conservation of mass and vector momentum in the ice flow are the following ones:

$$\left.\begin{aligned} u_x + v_y &= 0, \\ \rho[u_t + uu_x + vu_y] &= -p_x + \tau_{1x} + \tau_{2y} + \rho g' \epsilon, \\ \rho[v_t + uv_x + vv_y] &= -p_y + \tau_{2x} - \tau_{1y} - \rho g'. \end{aligned}\right\} \tag{2.1}$$

Here, letter subscripts denote partial differentiation, ρ is the (constant) density of the ice, and $g' = g(1+\epsilon^2)^{-\frac{1}{2}}$, where g is the acceleration due to gravity. Also, τ_1 and τ_2 are the longitudinal and tangential stress deviators and are defined for an isotropic, incompressible fluid in terms of the stress tensor, $[\sigma_{ij}]$, in the following way:

$$\left.\begin{aligned} \sigma_{ij} &= -p\delta_{ij} + \tau_{ij}, \\ \tau_1 = \tau_{11} &= -\tau_{22}, \\ \tau_2 = \tau_{12} &= \tau_{21}, \end{aligned}\right\} \tag{2.2}$$

δ_{ij} being the Kronecker delta.

As a constitutive equation for the cold ice, we assume that the stress and strain rate second invariants are related by a power law of the following commonly accepted type (Glen 1953, 1955; Paterson 1969):

$$e_{ij} = A\tau^{n-1}\tau_{ij}, \quad e = A\tau^n, \quad n \approx 3 \quad \text{or} \quad 4, \tag{2.3}$$

where

$$\begin{aligned} 2\tau^2 &= \tau_{ij}\tau_{ij}, \quad 2e^2 = e_{ij}e_{ij}, \\ e_{11} &= -e_{22} = e_1 = u_x = -v_y, \\ e_{12} &= e_{21} = e_2 = \tfrac{1}{2}(u_y + v_x). \end{aligned} \tag{2.4}$$

In (2.4) and throughout this paper, the standard tensor summation convention is to be employed. In (2.3), A is found experimentally to be a function of the temperature; in common with other authors, e.g. Barnes, Tabor & Walker (1971), we assume it is of the form

$$A = B\exp(-Q/RT), \tag{2.5}$$

where Q is an activation energy, R is the universal gas constant, and B is a constant which is determined experimentally. Finally, conservation of energy in the cold ice zone leads to the equation

$$\rho C_v[T_t + uT_x + vT_y] = k\nabla^2 T + \tau_{ij}e_{ij}, \tag{2.6}$$

where C_v and k are the assumed constant heat capacity and conductivity for the ice.

In (2.6) we have neglected the change in internal energy stored in the ice as plastic work.

In the temperate zone, equations (2.5) and (2.6) are not suitable models for energy conservation and (together with (2.3)) material constitution. In this zone, the heat produced by viscous dissipation is used to partially melt the ice rather than increase the temperature, and the presence of trapped meltwater there can significantly affect the resulting relationship between stress and strain rate (Lliboutry 1976). Lliboutry indicates that (2.3) is still valid with $n \simeq 3$, and so we adopt the following equations as appropriate replacements for (2.5) and (2.6) in this zone:

$$\rho L[w_t + uw_x + vw_y] = \tau_{ij} e_{ij} \tag{2.7}$$

and

$$A = r(w), \tag{2.8}$$

where w, the moisture content, is defined to be the local mass fraction of water in the ice, L is the latent heat of melting of ice and $r(w)$ is a rheology-dependent function which must be determined experimentally. In (2.7), we have omitted certain terms in Lliboutry's actual energy equation which arise from the variation of melting temperature with pressure and salt concentrations, and which appear from his own work to be negligible. Terms containing T and its derivatives are similarly absent from (2.7). This approximation is justified *a posteriori* in § 3, where it is shown that T is very nearly 0 °C throughout the temperate ice zone. A more serious and possibly unjustified omission is the neglect of a term describing the heat transport due to water flow *through* the ice (as opposed to convected *with* the ice). It is felt that the present state of understanding of glacial hydrology is insufficient to predict on rational grounds the form that such a term should take for the long time scales of interest in this paper, and so, rather than attempt an *ad hoc* description of water transport, we simply choose to omit this term. The mathematical consequences of this omission are discussed briefly on page 226 below.

To go with the equations (2.1)–(2.8), we must prescribe a set of initial conditions and boundary conditions. The initial conditions will be left unspecified, but we require the following boundary conditions to be satisfied.

1. *On the top surface*

We specify that (i) the stress tensor is continuous; (ii) the kinematic condition for a free boundary is satisfied; (iii) over the cold ice zone, the temperature is equal to the (assumed known) atmospheric temperature. That is,

$$\sigma_{ij} n_j = -p_{\mathrm{A}}(x,t)\, n_i, \tag{2.9}$$

$$\mathrm{d}\eta/\mathrm{d}t - v = a(x,t), \tag{2.10}$$

and

$$T = T_{\mathrm{A}}(x,t) \quad \text{for} \quad x < x_{\mathrm{T}}(t). \tag{2.11}$$

Here, $\boldsymbol{n} = (n_1, n_2)$ denotes the unit outward normal to the ice surface, $\mathrm{d}/\mathrm{d}t$ is the material derivative, and p_{A} and T_{A} are atmospheric pressure and temperature functions, which are considered known. For reasonable climates, T_A is typically monotonic increasing in x for each fixed t; we shall generally assume this to be the case. Also, we shall assume for convenience that p_A is constant; this assumption in no way limits the results presented here. More importantly, $a(x,t)$ denotes the top surface ice accumulation (or ablation) rate per unit length, which depends in a complicated way upon the local weather conditions but which is also considered known. A graph of the annual average of this function against x is shown in figure 2 (after Nye 1963). We shall assume that this graph is a typical one in the sense that $a(x,t)$ is monotonically decreasing in x for $x \geqslant x_0(t)$ and identically zero for $x < x_0(t)$. The point where $a(x,t) = 0$ and the surface ice begins to ablate is denoted by $x_E(t)$ and is called the 'equilibrium point'.

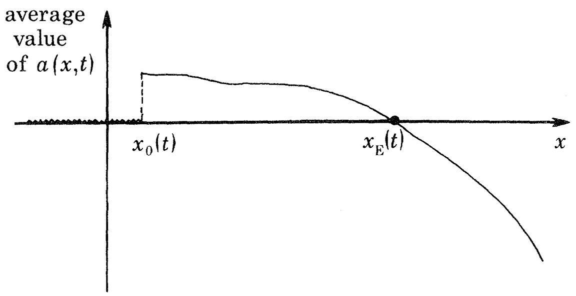

FIGURE 2. Typical ice accumulation function.

2. *Along the bedrock surface*

The development of a suitable set of boundary conditions for this surface requires more consideration than did that for the top surface. As mentioned earlier, temperate ice slides over its bedrock, but the corresponding flow is not 'inviscid' because the ice experiences a resistive drag near the bedrock due to the small-scale roughness of the bedrock. In attempting to develop flow field boundary conditions which adequately represent this flow behaviour, it is appropriate to consider the flow as consisting of two parts, an outer one and an inner one, as in normal boundary layer theory (Batchelor 1967) and as illustrated in figure 3.

The detailed small-scale flow of the ice over the roughness of the bedrock profile is essentially contained in the inner flow layer, while the flow in the outer region appears to be that of an ice mass sliding over a smoothed-out form of the bedrock profile. The inner flow 'feels' the outer flow as a uniform shearing flow at infinity, and the outer flow 'feels' the inner flow as a tangential stress at the smoothed bedrock boundary. This formulation is explained further by Fowler (1977) and may be made more precise by using the ideas of matched asymptotic expansions (Van Dyke 1975; Cole 1968). The basic mathematical requirement is that the limiting velocity and stress fields of the outer flow on the smoothed bedrock be equal to the limiting velocity and stress fields of the inner flow far from the rough bedrock. In principle, one can therefore determine the basal shear stress for the outer flow in terms of the basal sliding velocity from an examination of the inner flow problem; thus appropriate bedrock boundary conditions for the outer flow are the following:

$$\tau_{\mathrm{NT}} = f(q_{\mathrm{T}}); \quad q_{\mathrm{N}} = -W_{\mathrm{M}}(x,t). \tag{2.12}$$

Here, N and T subscripts refer to components of the stated variables which are normal and tangential to the smoothed bedrock surface. In the second condition, $W_{\mathrm{M}}(x,t)$ represents a suction per unit length due to removal of melt water from the bedrock surface. Actually (Lliboutry 1968), typical values for $W_{\mathrm{M}}(x,t)$ are orders of magnitude less than those for $|a(x,t)|$, and so henceforth we simply set $W_{\mathrm{M}}(x,t) \equiv 0$. Several authors (Weertman 1957, 1964; Lliboutry 1968, 1975; Nye 1969, 1970; Kamb 1970; Morland 1976; Fowler 1977) have studied the problem of determining

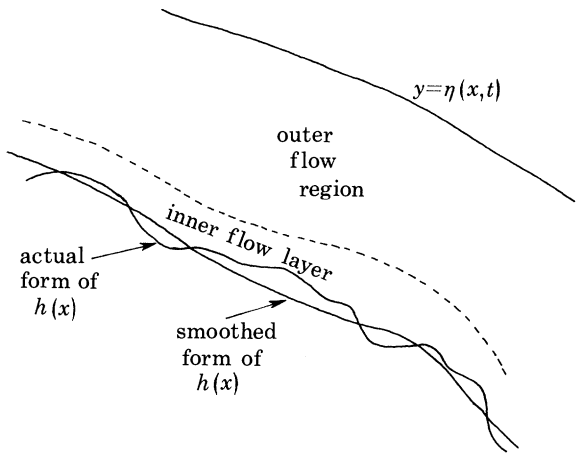

FIGURE 3. Typical small-scale flow geometry.

the functional form of $f(\cdot)$. In this paper, we leave it arbitrary, restricting it only minimally in the discussion below. We note that the sliding law in (2.12) cannot be expected to retain its simplistic form (i.e. where f depends only upon q_{T}) in regions where the ice thickness and the inner flow layer thickness are of the same order of magnitude. Since the only temperate ice region of this type that will be encountered in our study consists of a few metres of ice at the very snout of the glacier, this inadequacy is ignored here. A detailed discussion of the flow problem near the snout is presented by Fowler (1977).

As mentioned earlier, cold ice freezes to its underlying bedrock; thus the natural flow field boundary conditions in this case are

$$u = v = 0. \tag{2.13}$$

Since realistic sliding laws have $f(0) = 0$, (2.12) and (2.13) predict a discontinuity in either the stress or the velocity at the bottom melting point. In practice, such a discontinuity will not occur since the no-slip conditions in (2.13) become invalid when the temperature is near the pressure melting point. This is because the ice will start to slide when the pressure variation on the bedrock is sufficient to cause a small part of the basal ice to reach the pressure melting point, so that the ice becomes lubricated in patches (cf. Robin 1976). It is therefore appropriate to replace the conditions in (2.13) by the second condition in (2.12) and a temperature-dependent sliding law of the following type:

$$q_{\mathrm{T}} = F(\tau_{\mathrm{NT}}, T), \tag{2.14}$$

where, for any given τ_{NT} and T_{M}, F depends on $T \leqslant T_{\mathrm{M}}$ as indicated in figure 4.

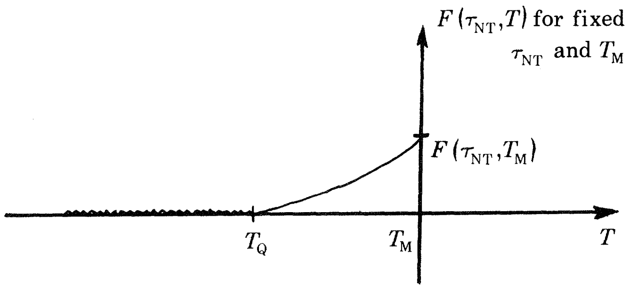

FIGURE 4. Assumed form of the glacier sliding law.

Here T_{Q} is a constant which is independent of τ_{NT} and near T_{M}. As $T_{\mathrm{Q}} \rightarrow T_{\mathrm{M}}$, the sliding law expressed in (2.14) tends to the limits given in (2.13) and (2.12) for $T < T_{\mathrm{M}}$ and $T = T_{\mathrm{M}}$, provided one can invert $f(\cdot)$ to define τ_{NT} implicitly in terms of q_{T}. However, we shall see in Part II that the behaviour of solutions to our model problem in this limit is *not* the same as that when $T_{\mathrm{Q}} \equiv T_{\mathrm{M}}$; clearly, it is the former case that is of physical interest and should be considered here.

Along the bedrock surface of the cold ice, we also assume that a geothermal heat flux supplies the ice with heat at a rate Φ which depends on temperature in a known way. Following the discussion of the previous paragraph, we assume that this dependence is roughly as illustrated in figure 5. That is, in regions where $T \leqslant T_{\mathrm{Q}}$, the heat flux is relatively constant ($= G$) while in regions where $T_{\mathrm{Q}} < T < T_{\mathrm{M}}$ and the bedrock is beginning to be lubricated, part of the natural bedrock heat flux is used up in this lubrication process and the cold ice 'feels' a heat flux which is monotonically decreasing in T. The appropriate thermal boundary condition in this case is then

$$\sigma_{\mathrm{NT}} q_{\mathrm{T}} - k \partial T / \partial N = -\Phi(T), \tag{2.15}$$

where $\partial T/\partial N = \boldsymbol{n} \cdot \boldsymbol{\nabla} T$. It should be noticed that (2.15) contains a source term due to viscous heating at the bedrock surface. This term vanishes for $T < T_{\mathrm{Q}}$, but provides a non-zero contribution to the heat flux when $T_{\mathrm{Q}} < T < T_{\mathrm{M}}$. Let us denote by $x_{\mathrm{Q}}(t)$ and $x_{\mathrm{Z}}(t)$ the points where $T = T_{\mathrm{Q}}$ and where T first equals T_{M}. For $x > x_{\mathrm{Z}}(t)$, we specify that $T = T_{\mathrm{M}}$ until $\partial T/\partial N$ reaches the value which is prescribed on the melting surface (see below). At this point $x = x_{\mathrm{M}}(t)$, the melting surface 'breaks away' from the bedrock and the basal ice becomes temperate. The final set of bedrock surface boundary conditions that we prescribe are then the following:

$$q_{\mathrm{N}} = 0, \quad q_{\mathrm{T}} = F(\tau_{\mathrm{NT}}, T),$$
$$\sigma_{\mathrm{NT}} q_{\mathrm{T}} - k\partial T/\partial N = -\Phi(T) \quad \text{for} \quad x < x_{\mathrm{Z}}(t),$$
$$T = T_{\mathrm{M}} \quad \text{for} \quad x_{\mathrm{Z}}(t) < x < x_{\mathrm{M}}(t). \tag{2.16}$$

A formal derivation of the thermal boundary conditions and their relation to the inner and outer flows discussed previously is given by Fowler (1977).

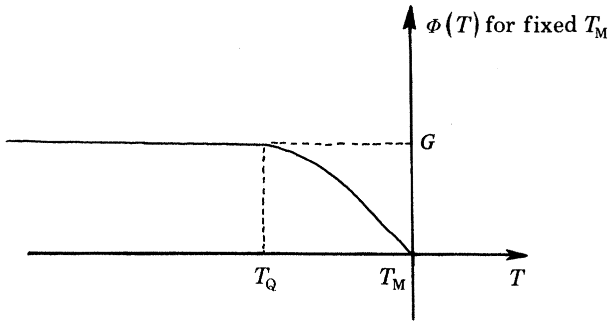

FIGURE 5. Assumed form of the geothermal heat flux function.

3. *Along the melting surface*

We specify that all dependent variables in the model problem together with the local heat flux vector $(-k\nabla T)$ must be continuous. In particular, we specify that the moisture content of the ice is zero and that the temperature and pressure satisfy the Clausius–Clapeyron condition. The boundary conditions along this line are therefore the following:

$$\text{all variables and } \partial T/\partial N \text{ are continuous across the line,} \tag{2.17}$$

$$w = 0, \tag{2.18}$$

and
$$T = T_{\mathrm{M}} = T_{\mathrm{f}} - \theta(p - p_{\mathrm{A}}), \tag{2.19}$$

where θ is the Clausius–Clapeyron constant for water, and T_{f} is the freezing point of water at atmospheric pressure.

Let us note that, in writing the temperate energy equation (2.7) and the boundary condition (2.18), we have tacitly assumed that the melting surface $y_{\mathrm{M}}(x)$ as illustrated in figure 1 is a monotonically increasing function of x or, more specifically, that the direction of the streamline of the flow at all points along the melting surface is from the cold to the temperate ice. This assumption may be written in the form

$$\mathrm{d}[y - y_{\mathrm{M}}(x,t)]/\mathrm{d}t \lessgtr 0 \quad \text{when} \quad y = y_{\mathrm{M}} \quad \text{if} \quad T < T_{\mathrm{M}} \quad \text{in} \quad y_{\mathrm{M}} \pm. \tag{2.20}$$

Here, $y_{\mathrm{M}} \pm$ indicates ice regions just above and just below y_{M}, and the $+(-)$ sign is associated with the $<$ $(>)$ inequality. (2.20) can be written in the form

$$v - y_{\mathrm{M}t} - u y_{\mathrm{M}x} \lessgtr 0 \quad \text{on} \quad y = y_{\mathrm{M}} \quad \text{if} \quad T < T_{\mathrm{M}} \quad \text{in} \quad y_{\mathrm{M}} \pm. \tag{2.21}$$

The importance of this assumption is indicated by the following fact. The temperate energy equation (2.7) is hyperbolic, and has as its characteristics precisely the streamlines of the flow. Furthermore, the form of (2.7) implies that w increases monotonically along these characteristics. If (2.20) does not hold at any point on y_{M}, then the streamline through this point emanates from the temperate zone, within which w is increasing (from a minimum of zero) and so must be strictly positive. This contradicts the boundary condition prescribed in (2.18), and implies that the model presented here with this boundary condition has no solution. A typical example of such a situation is depicted in figure 6.

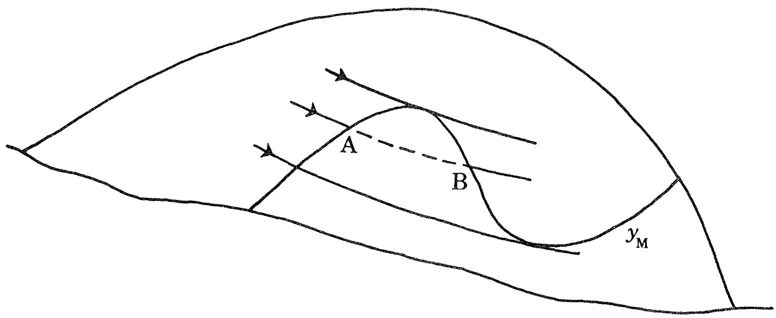

FIGURE 6. Possible melting surface geometry when no solution of (2.7) can exist. Along the dotted streamline, w must increase monotonically from zero at A and hence cannot be zero at B, as required by (2.18).

In cases where (2.20) is satisfied, this ill-posedness of the model problem disappears. In general, however, (2.20) will not be satisfied, but the resulting contradiction can be remedied by the inclusion in (2.7) of some suitable small diffusion-like term, e.g. one of the form $\nu\nabla^2 w$, $\nu \ll 1$. In this case it is also necessary to prescribe suitable moisture boundary conditions on the other parts of the temperate ice zone boundary. However, it is possible to show, using standard boundary layer arguments (e.g. Cole 1968), that the presence of a small non-zero ν will not affect to leading order the solutions with $\nu = 0$, provided we correctly pose our model problem by replacing the boundary condition (2.18) by

$$w|_{y=y_{\mathrm{M}}} = 0 \quad \text{when} \quad [v - y_{\mathrm{M}t} - u y_{\mathrm{M}x}]\,|_{y=y_{\mathrm{M}}} \lessgtr 0 \quad \text{if} \quad T < T_{\mathrm{M}} \quad \text{in} \quad y_{\mathrm{M}} \pm . \tag{2.22}$$

The details of the arguments are beyond the scope of this paper, and are not included here. (We would for example require a more detailed knowledge of the form of $r(w)$, and of the basal sliding process.)

We feel that it is reasonable and meaningful to consider the $\nu = 0$ model with the boundary condition (2.22), and so we proceed on this basis.

In summary, then, our basic model for the wide valley flow of polythermal glaciers consists of the equations (2.1) to (2.8) and the boundary conditions in (2.9) to (2.11) and (2.16), (2.17), (2.19) and (2.22).

3. The scaled model

The basic polythermal glacier flow problem posed in the last section is a nonlinear one of double free boundary type and of course too difficult to solve explicitly. In this section, we begin the process of rationally reducing the problem to a simplified form which can be profitably analysed by scaling it and identifying the important dimensionless parameters which occur naturally within it.

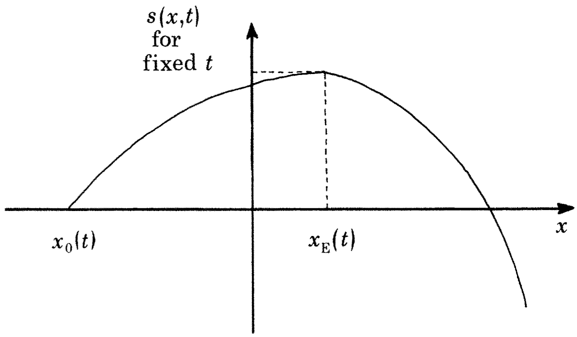

Figure 7. A typical flux function.

The non-dimensionalization of the basic problem involves the finding of typical dimensional quantities to serve as scaling factors for the various dimensional variables in the problem. There are at least two natural choices for such scale factors, namely, V as a typical y velocity value, where V is the maximum (or a typical) value of $a(x,t)$ over its domain of definition, and $T_0 = |\theta_{\text{av.}} - T_{\text{f}}|$ as a typical temperature difference value, where $\theta_{\text{av.}}$ is the time average value of T at the head of the glacier. It also seems rather natural to scale the x coordinate in such a manner that, when non-dimensionalized, x derivatives of $a(x,t)$ (and $h(x)$) are all of numerical order one, but we shall not do this here. Instead, noting from foresight that the x integral of $a(x,t)$ will play a much larger role in the ensuing analysis of our model problem than will the x derivatives, we choose to scale x in such a manner that the following 'flux' function and its first x derivative become order one under this scaling:

$$s(x,t) = \int_{x_0(t)}^{x} a(\xi, t)\,\mathrm{d}\xi. \tag{3.1}$$

From figure 2, a typical $s(x,t)$ profile for fixed t is as in figure 7. The requirement that $s_x(x,t)$ be order one is automatically satisfied when $a(x,t)$ is scaled with V as discussed above; then l, our scale for the x coordinate, is to be chosen so that $s(x,t)$ varies by an order one factor for all t over distances of size l. It will become clear later on that a typical choice for l is the total length of the glacier. We assume here that $h'(x)$ becomes order one under this scaling; for reasonably smooth landform profiles, this is not a drastic assumption.

From (2.3), (2.4), (2.6) and (2.8), the tensor $[e_{ij}]$ depends on derivatives of u and

v and $[\tau_{ij}]$ depends on $[e_{ij}]$ and either T or w. So once we choose scales for y, u and w we can use these scales in conjunction with those introduced in the last paragraph for x, v and T to define scales for all variables in our basic model except p and t. For now, let us consider flow phenomena which occur on a convective time scale and introduce the following scalings for the basic model variables:

$$
\begin{aligned}
x &= l\bar{x}, \quad y = d\bar{y}, \quad u = U\bar{u}, \quad v = V\bar{v}, \quad t = (d/V)\,\bar{t},\\
y_{\mathrm{M}}(x,t) &= d\bar{y}_{\mathrm{M}}(\bar{x},\bar{t}), \quad h(x) = d\bar{h}(\bar{x}), \quad \eta(x,t) = d\bar{\eta}(\bar{x},\bar{t}),\\
x_0(t) &= l\bar{x}_0(\bar{t}), \quad x_{\mathrm{E}}(t) = l\bar{x}_{\mathrm{E}}(\bar{t}), \quad x_{\mathrm{S}}(t) = l\bar{x}_{\mathrm{S}}(\bar{t}),\\
x_{\mathrm{T}}(t) &= l\bar{x}_{\mathrm{T}}(\bar{t}), \quad x_{\mathrm{M}}(t) = l\bar{x}_{\mathrm{M}}(\bar{t}), \quad x_{\mathrm{Z}}(t) = l\bar{x}_{\mathrm{Z}}(\bar{t}),\\
a(x,t) &= V\bar{a}(\bar{x},\bar{t}), \quad \bar{s}(\bar{x},\bar{t}) = \int_{\bar{x}_0(\bar{t})}^{\bar{x}} \bar{a}(\xi,\bar{t})\,\mathrm{d}\xi,\\
T &= T_{\mathrm{f}}+T_0\,\bar{T}, \quad w = W\bar{w}, \quad p = p_{\mathrm{A}}+\rho g' d[\bar{\eta}-\bar{y}]+P\bar{p},\\
T_{\mathrm{M}} &= T_{\mathrm{f}}+T_0\,\bar{T}_{\mathrm{M}}, \quad T_{\mathrm{Q}} = T_{\mathrm{f}}+T_0\,\bar{T}_{\mathrm{Q}}, \quad T_{\mathrm{A}} = T_{\mathrm{f}}+T_0\,\bar{T}_{\mathrm{A}},
\end{aligned} \tag{3.2}
$$

where d, U, W and P are as yet unspecified. As usual, a balance of the terms in the continuity equation requires that we choose

$$U = \left(\frac{l}{d}\right) V. \tag{3.3}$$

In any real situation, we expect that

$$\delta = \left(\frac{d}{l}\right) \ll 1 \tag{3.4}$$

(i.e. we expect the 'shallow ice' approximation to be valid); it will be shown below that in fact this is typically the case. We specify d by requiring that, under the above scalings, the x momentum equation retains an order one longitudinal gravity influence term (so that the flow is gravity driven). From (2.2), (2.3) and (2.4), we have that

$$
\begin{aligned}
\tau_1 &= A^{-1/n} u_x\, e^{-(n-1)/n},\\
\tau_2 &= A^{-1/n}\left(\frac{u_y+v_x}{2}\right) e^{-(n-1)/n}\\
e &= \tfrac{1}{2}[4u_x^2+(u_y+v_x)^2]^{\frac{1}{2}},
\end{aligned} \tag{3.5}
$$

and hence from (3.2) and (3.3)

$$
\begin{aligned}
\tau_1 &= \delta\left[\frac{lV}{2d^2A_0}\right]^{1/n}[2\bar{A}^{-1/n}\bar{u}_{\bar{x}}\bar{e}^{-(n-1)/n}] \equiv \delta[\tau]\,\bar{\tau}_1,\\
\tau_2 &= \left[\frac{lV}{2d^2A_0}\right]^{1/n}[\bar{A}^{-1/n}(\bar{u}_{\bar{y}}+\delta^2\bar{v}_{\bar{x}})\,\bar{e}^{-(n-1)/n}] \equiv [\tau]\,\bar{\tau}_2,\\
e &= \left[\frac{lV}{2d^2}\right][(\bar{u}_{\bar{y}}+\delta^2\bar{v}_{\bar{x}})^2+4\delta^2\bar{u}_{\bar{x}}^2]^{\frac{1}{2}} \equiv [e]\,\bar{e},\\
A &= A_0\,\bar{A},
\end{aligned} \tag{3.6}
$$

where, as indicated, A_0 is to be chosen (in a manner described below) so that $\bar{A}$ becomes an order one function of its argument. In this way, $\bar{\tau}_1$, $\bar{\tau}_2$ and $\bar{e}$ also represent order one functions of their arguments, and so

$$[\tau] = \left[\frac{lV}{2d^2A_0}\right]^{1/n} \quad \text{and} \quad [e] = \left[\frac{lV}{2d^2}\right]$$

represent natural stress and strain rate scales for the problem. We then find that the scaled version of the x momentum equation in (2.1) is the following:

$$\bar{\tau}_{2\bar{y}} + \delta^2\bar{\tau}_{1\bar{x}} + \left[\frac{\rho g' d\epsilon}{[\tau]}\right] - \delta\left[\frac{P}{[\tau]}\right]\bar{p}_{\bar{x}} - \left[\frac{\rho g' d\delta}{[\tau]}\right]\bar{\eta}_{\bar{x}} = \left[\frac{\rho U^2 d}{[\tau]l}\right][\bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}}]. \tag{3.7}$$

For real glaciers, observations typically reveal that $l \sim 10$ km, $d \sim 100$ m, $U \sim 100$ m/a and $[\tau] \sim 1$ bar (Paterson 1969). Using the value of ρ given in table 1 one then finds that a typical glacier flow Reynolds number is given by

$$Re = \left[\frac{\rho U^2 d}{[\tau]l}\right] \sim 10^{-13} \ll 1. \tag{3.8}$$

If we then assume that $\delta \lesssim \epsilon$ and $P \lesssim [\tau]$, we see from (3.7) (and the definition of $[\tau]$ in (3.6)) that gravity will play the desired non-trivial role in the x momentum equation as long as we choose to set

$$d = \left(\frac{[\tau]}{\rho g'\epsilon}\right) = \left(\frac{lV}{2A_0(\rho g'\epsilon)^n}\right)^{1/(n+2)} \tag{3.9}$$

We show below that $\delta \lesssim \epsilon$ is a realistic assumption and that for 'shallow' ice, where the pressure field is almost hydrostatic, $P \lesssim [\tau]$ holds. In fact, we shall simply scale P by writing

$$P = \delta[\tau], \tag{3.10}$$

and we shall verify *a posteriori* that such a choice leads to a correctly scaled problem for $\bar{p}$.

It remains to specify A_0 and W. From (2.5) and (2.8), there are two basic ways in which we can choose A_0. As shown in tables 1 and 2, it is typically true that

$$Z = \left(\frac{T_0}{T_f}\right) \ll 1, \tag{3.11}$$

and so, from (2.5), the natural choice for A_0 in the cold ice zone is the following one:

$$A_{\text{cold}} = B\exp(-Q/RT_f). \tag{3.12}$$

On the other hand, we can formally make (2.8) non-dimensional in a general way by writing

$$A = A_{\text{temp}}\,\bar{r}\left(\frac{w}{W}\right) \equiv A_{\text{temp}}\,\bar{r}(\bar{w}), \tag{3.13}$$

where A_{temp} is chosen so that $\bar{r}$ is an order one function of its (order one) argument. In this way, A_{temp} is the natural choice for A_0 in the temperate ice zone, and, as can be inferred from the data in table 2, it is probably the case that A_{temp} is somewhat larger than A_{cold}. Nevertheless, in this paper we shall be more interested in interactions between the temperature and velocity fields than between the moisture content and velocity fields, and so for now we shall scale A with A_{cold} rather than with A_{temp}. In any case, it is not expected that $A_{\mathrm{cold}}^{1/(n+2)}$ and $A_{\mathrm{temp}}^{1/(n+2)}$ will differ drastically, and so, as can be seen from (3.9), a change of A_0 from A_{temp} to A_{cold} (or vice versa) will hardly affect the other model scales (e.g. d) which depend upon A_0. To define W, let us consider the case of a completely temperate glacier. Then $A_0 = A_{\mathrm{temp}}$ and we require W to be such that the terms in the temperate energy equation balance. That is, from (2.7) and (3.2),

$$W = \left(\frac{2A_0[\tau]^{n+1}d}{\rho L V}\right) = \frac{lg'\epsilon}{L}. \tag{3.14}$$

Since W is independent of A_0, we define it by (3.14) in the general polythermal case also.

TABLE 1. VALUES OF PHYSICAL CONSTANTS IN THE MODEL

constant	approximate value
T_t	273.15 K
ρ	900 kg m^{-3}
g	9.8 m s^{-2}
C_v	2×10^3 J kg^{-1} K^{-1}
k	7×10^7 J K^{-1} m^{-1} a^{-1}
L	3.3×10^5 J kg^{-1}
R	8.3 J mol^{-1} K^{-1}
θ	0.74×10^{-2} K bar^{-1}

TABLE 2. ESTIMATES FOR OTHER MODEL CONSTANTS

constant	approximate value	source
A_{cold}	0.17 bar^{-n} a^{-1}	Glen (1955)
A_{temp}	an order of magnitude greater than A_{cold}	Lliboutry (1976)
n	3.17 (cold)	Glen (1955)
	3 (temperate)	Lliboutry (1976)
Q	6×10^4 J mol^{-1}	Raraty & Tabor (1958)

A number of the parameters that have appeared in the discussion thus far are known physical constants which have well established values. These values can be found in Paterson (1969) and are collected in table 1. Estimates for various other physical constants appearing in the model are presented in table 2. Values for these constants are known with much less certainty than are those in table 1, but the estimates given here are considered to be of the correct order of magnitude (Glen 1955; Raraty & Tabor 1958; Barnes *et al.* 1971). The dimensional inputs to the model are V, T_0, l, ϵ and G. These may vary considerably from glacier to glacier, typical values being as given in table 3. The values in tables 1–3 may now be used to compute values for the remaining model parameters by means of the various formulae developed in this section. These values are presented in table 4.

As mentioned above, these values do appear to be typical; this indicates that our scalings are indeed appropriate ones. Furthermore, it is clear from tables 3 and 4 that $\delta = (d/l)$ is indeed $\ll 1$ and

$$\mu = \left(\frac{\delta}{\epsilon}\right) \sim 10^{-1} \lesssim O(1). \tag{3.15}$$

These order of magnitude statements justify *a posteriori* the various assumptions made above. It should be remembered, though, that for any particular glacier a number of values in tables 2 and 3 may be out by as much as a factor of 10, and so in the general discussion below we use as few specific numbers as possible.

TABLE 3. TYPICAL VALUES OF INPUT PARAMETERS

parameter	value	source
V	1 m a^{-1}	Paterson (1969)
T_0	20 K	Paterson (1969)
l	10 km	Paterson (1969)
ϵ	10^{-1}	Paterson (1969)
G	1.6×10^6 J m^{-2} a^{-1}	Paterson (1969)

TABLE 4. COMPUTED TYPICAL VALUES OF SEVERAL MODEL PARAMETERS

parameter	computed nominal value
d	100 m
U	100 m a^{-1}
$[\tau]$	1 bar
W	3 %

Using the scaling system developed in this section, we may now rewrite the complete nonlinear free boundary problem posed in § 1 as follows:

$$\begin{aligned} \bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} &= 0, \\ \bar{\tau}_{2\bar{y}} + 1 &= \mu\bar{\eta}_{\bar{x}} + \delta^2(\bar{p}_{\bar{x}} - \bar{\tau}_{1\bar{x}}) + Re[\mathrm{D}\bar{u}/\mathrm{D}\bar{t}], \\ \bar{p}_{\bar{y}} &= \bar{\tau}_{2\bar{x}} - \bar{\tau}_{1\bar{y}} - Re[\mathrm{D}\bar{v}/\mathrm{D}\bar{t}], \\ \frac{\mathrm{D}}{\mathrm{D}\bar{t}} &\equiv \frac{\partial}{\partial \bar{t}} + \bar{u}\frac{\partial}{\partial \bar{x}} + \bar{v}\frac{\partial}{\partial \bar{y}}, \end{aligned} \tag{3.16}$$

with

$$\begin{aligned} \bar{\tau}_1 &= 2\bar{u}_{\bar{x}}[\bar{A}^{-1/n}\bar{e}^{-(n-1)/n}], \\ \bar{\tau}_2 &= [\bar{u}_{\bar{y}} + \delta^2\bar{v}_{\bar{x}}]\,[\bar{A}^{-1/n}\bar{e}^{-(n-1)/n}], \\ \bar{e} &= [(\bar{u}_{\bar{y}} + \delta^2\bar{v}_{\bar{x}})^2 + 4\delta^2\bar{u}_{\bar{x}}^2]^{\frac{1}{2}}, \end{aligned} \tag{3.17}$$

where, for $\bar{T} < \bar{T}_{\mathrm{M}}$,

$$\begin{aligned} &\bar{T}_{\bar{t}} + [\bar{u}\bar{T}_{\bar{x}} + \bar{v}\bar{T}_{\bar{y}}] = \beta_2[\bar{T}_{\bar{y}\bar{y}} + \delta^2\bar{T}_{\bar{x}\bar{x}}] + \beta_1[\bar{A}^{-1/n}\bar{e}^{(n+1)/n}], \\ &\beta_2 = \left[\frac{k}{\rho C_v V d}\right], \quad \beta_1 = \left[\frac{[\tau]}{\rho C_v T_0 \delta}\right] = \left[\frac{\epsilon g' l}{C_v T_0}\right], \\ &\bar{A} = \exp[\kappa(\bar{T} + O(Z))] \quad \text{as} \quad Z \to 0, \quad \kappa = [QT_0/RT_{\mathrm{f}}^2], \end{aligned} \tag{3.18}$$

while for $\bar{T} = \bar{T}_{\mathrm{M}}$,

$$\bar{w}_{\bar{t}} + \bar{u}\bar{w}_{\bar{x}} + \bar{v}\bar{w}_{\bar{y}} = [\bar{A}^{-1/n}\bar{e}^{(n+1)/n}],$$

$$\bar{A} = \gamma\bar{r}(\bar{w}), \quad \gamma = (A_{\mathrm{temp}}/A_{\mathrm{cold}}), \tag{3.19}$$

and we have the following sets of boundary conditions:

1. Along $\bar{y} = \bar{\eta}(\bar{x}, \bar{t})$,

$$\bar{\tau}_2 + \delta^2(\bar{p} - \bar{\tau}_1)\,\bar{\eta}_{\bar{x}} = 0,$$
$$\bar{\tau}_1 + \bar{p} + \bar{\tau}_2\,\bar{\eta}_{\bar{x}} = 0,$$
$$\bar{\eta}_{\bar{t}} + \bar{u}\bar{\eta}_{\bar{x}} - \bar{v} = \bar{a}(\bar{x}, \bar{t}) = \bar{s}_{\bar{x}}(\bar{x}, \bar{t}),$$
$$\bar{T} = \bar{T}_{\mathrm{A}}(\bar{x}, \bar{t}) \quad \text{for} \quad \bar{x} < \bar{x}_{\mathrm{T}}(\bar{t}). \tag{3.20}$$

2. Along $\bar{y} = \bar{h}(\bar{x})$,

$$\bar{u}\bar{h}_{\bar{x}} - \bar{v} = 0,$$
$$\bar{q}_{\mathrm{T}} = \bar{F}(\bar{\tau}_{\mathrm{NT}}, \bar{T}), \quad \bar{F} \text{ being a dimensionless version of } F,$$
$$\bar{\tau}_{\mathrm{NT}} = \left[\frac{(1-\delta^2\bar{h}_{\bar{x}}^2)\,\bar{\tau}_2 - 2\delta^2\bar{h}_{\bar{x}}\bar{\tau}_1}{1+\delta^2\bar{h}_{\bar{x}}^2}\right],$$
$$\bar{q}_{\mathrm{T}} = \left[\frac{\bar{u} + \delta^2\bar{v}\bar{h}_{\bar{x}}}{(1+\delta^2\bar{h}_{\bar{x}}^2)^{\frac{1}{2}}}\right],$$
$$\beta_1\,\bar{\tau}_{\mathrm{NT}}\bar{q}_{\mathrm{T}} + \beta_2\left[\frac{\bar{T}_{\bar{y}} - \delta^2\bar{h}_{\bar{x}}\bar{T}_{\bar{x}}}{(1+\delta^2\bar{h}_{\bar{x}}^2)^{\frac{1}{2}}}\right] = -\lambda\bar{\Phi}(\bar{T}) \quad \text{for} \quad \bar{x} < \bar{x}_{\mathrm{Z}}(\bar{t}),$$
$$\bar{\Phi} = \frac{\Phi}{G}, \quad \lambda = \frac{\beta_2 G d}{k T_0},$$
$$\bar{T} = 0 \quad \text{for} \quad \bar{x}_{\mathrm{Z}}(\bar{t}) \leqslant \bar{x} \leqslant \bar{x}_{\mathrm{M}}(\bar{t}). \tag{3.21}$$

3. Along $\bar{y} = \bar{y}_{\mathrm{M}}(\bar{x}, \bar{t})$,

all variables and $[\bar{T}_{\bar{y}} - \delta^2\bar{y}_{\mathrm{M}\bar{x}}\bar{T}_{\bar{x}}]$ are continuous across this line,

$$\bar{T} = \bar{T}_{\mathrm{M}} = \hat{\theta}[(\bar{\eta} - \bar{y}) + \delta\epsilon\bar{p}], \quad \hat{\theta} = \left[\frac{\rho g' d\theta}{T_0}\right].$$

$$\bar{w} = 0 \quad \text{when} \quad \bar{v} - \bar{y}_{\mathrm{M}\bar{t}} - \bar{u}\bar{y}_{\mathrm{M}\bar{x}} \lessgtr 0 \quad \text{if} \quad \bar{T} < \bar{T}_{\mathrm{M}} \quad \text{in} \quad \bar{y}_{\mathrm{M}}\pm. \tag{3.22}$$

Hereafter, the problem posed by equations (3.16) to (3.22) will be referred to as the 'scaled model'. It should be remembered in ensuing discussions that, from (3.8), (3.9), (3.11), (3.15), (3.18), (3.19), (3.21), (3.22) and tables 1–4, it is typically the case that

$$Re, \delta, \hat{\theta}, Z \ll 1, \tag{3.23}$$

and

$$\beta_1 \sim \tfrac{1}{4}, \quad \kappa \sim 2,$$
$$\beta_2 \sim \tfrac{1}{3}, \quad \mu \sim 10^{-1},$$
$$\gamma \gtrsim 10, \quad \lambda \sim 10^{-1}\beta_2 \sim 3\times 10^{-2}. \tag{3.24}$$

In the remaining sections of this paper we shall consider various limiting cases of the scaled model in view of (3.23) and (3.24). In these cases, we shall reduce one or more of the parameters here to zero, but from the discussion following (3.13) we shall always consider γ to be of order one and not large.

4. The reduced model

In this section we use the results from §3 to develop a glacier flow model which is a vast simplification of the full model presented in §2 and yet is expected to be a very good approximation to it. It is this simplified model that will be analysed in some detail in Part II of this paper.

We begin here by making the following approximations concerning the scaled model of §3:

$$Re = \delta = Z = \hat{\theta} = 0. \tag{4.1}$$

These approximations are motivated by the results in (3.23) and hence are expected to be reasonable ones. They admit the following physical interpretation: The first two indicate that the ice flows like a slow, shallow fluid. The third approximation implies that the dependence of the viscosity on temperature can be adequately represented by an exponential function, and the last means that the deviation of the pressure melting temperature from the atmospheric melting temperature is negligible. It should be noticed, however, that the simplification of certain equations in the scaled model, notably those containing terms multiplied by δ^2, leads to the neglect of highest order derivatives in these equations; thus we might expect problems of singular perturbation type to occur. In later analysis such problems will be dealt with as they arise. For now, we simply use (4.1) to reduce the scaled model to the following problem, overbars in §3 being dropped here for convenience:

$$\begin{aligned} u_x + v_y &= 0, \\ \tau_{2y} &= -1 + \mu\eta_x, \\ p_y &= \tau_{2x} - \tau_{1y}, \\ \tau_1 &= 2u_x A^{-1/n} e^{-(n-1)/n}, \\ \tau_2 &= u_y A^{-1/n} e^{-(n-1)/n}, \\ e &= |u_y|, \end{aligned} \tag{4.2}$$

where, for $T < 0$,

$$T_t + uT_x + vT_y = \beta_2 T_{yy} + \beta_1[A^{-1/n} e^{(n+1)/n}], \tag{4.3}$$

$$A = \exp[\kappa T],$$

while, for $T = 0$,

$$w_t + uw_x + vw_y = [A^{-1/n} e^{(n+1)/n}],$$

$$A = \gamma r(w) \equiv R_1(w), \tag{4.4}$$

say, and we have the following sets of boundary conditions:

1. Along $y = \eta(x, t)$,

$$\begin{aligned} &\tau_2 = 0, \\ &\tau_1 + p + \tau_2 \eta_x = 0, \\ &\eta_t + u\eta_x - v = a(x,t) = s_x(x,t), \\ &T = T_A(x,t) \quad \text{for} \quad x < x_T(t). \end{aligned} \tag{4.5}$$

2. Along $y = h(x)$,

$$\begin{aligned} &uh_x - v = 0, \\ &u = F(\tau_2, T), \\ &\beta_1 \tau_2 u + \beta_2 T_y = -\lambda \Phi(T) \quad \text{for} \quad x < x_Z(t), \\ &T = 0 \quad \text{for} \quad x_Z(t) \leqslant x \leqslant x_M(t). \end{aligned} \tag{4.6}$$

3. Along $y = y_M(x,t)$,

$$\begin{aligned} &T = T_y = 0, \quad \text{and} \\ &w = 0 \quad \text{when} \quad v - y_{Mt} - uy_{Mx} \lesseqgtr 0 \quad \text{if} \quad T < 0 \quad \text{in} \quad y_M \pm. \end{aligned} \tag{4.7}$$

We can now further simplify the problem posed by equations (4.2) to (4.7) in the following analytical manner. We integrate the second equation in (4.2) and use (4.5) to find

$$\tau_2(x,y) = (1 - \mu\eta_x)(\eta - y); \tag{4.8}$$

then from the fifth and sixth equations there

$$e = |u_y| = u_y = A\tau_2^n = A(1 - \mu\eta_x)^n (\eta - y)^n, \tag{4.9}$$

assuming that $1 - \mu\eta_x \geqslant 0$ for all relevant x. Since it is typically expected that $\mu \ll 1$ (cf. (3.24)), we make this assumption in what follows in order to keep ensuing equations free from cumbersome absolute value signs; in cases where $1 - \mu\eta_x < 0$ for some x, one can reintroduce absolute value signs and modify our analysis with very little effort. We define the stream function for the flow by

$$u = \psi_y, \quad v = -\psi_x, \quad \psi|_{x = x_0(t)} = 0. \tag{4.10}$$

In this way the first equation in (4.2) is automatically satisfied, and it is then clear from the third and fourth equations there and the second equation in (4.5) that p and τ_1 uncouple from the other equations. These two variables therefore play no role at all in the determination of ψ, A and η; in ensuing discussions we simply omit them from consideration. From (4.8) and (4.9), we can do the same with respect to τ_2 and e provided we use their equivalents there to replace these variables in (4.3), (4.4) and (4.6).

For convenience, we now introduce the following changes of variable:

$$\begin{aligned} \xi = \eta(x,t) - y, \quad \Psi = \psi + \frac{\partial}{\partial t}\int_{x_0(t)}^{x} H(\sigma, t)\,\mathrm{d}\sigma, \\ H(x,t) = \eta(x,t) - h(x). \end{aligned} \tag{4.11}$$

The problem posed by equations (4.2) to (4.7) then reduces to the following, much simplified one:

$$\Psi_{\xi\xi} = \xi^n[1-\mu(H_x+h_x)]^n A, \tag{4.12}$$

$$A = \begin{cases} \exp(\kappa T) & \text{for} \quad T < 0 \\ R_1(w) & \text{for} \quad T = 0 \end{cases}, \tag{4.13}$$

$$T_t + \Psi_x T_\xi - \Psi_\xi T_x = \beta_2 T_{\xi\xi} + \beta_1 \xi^{n+1}[1-\mu(H_x+h_x)]^{n+1}\exp(\kappa T) \quad \text{for} \quad T < 0, \tag{4.14}$$

$$w_t + \Psi_x w_\xi - \Psi_\xi w_x = \xi^{n+1}[1-\mu(H_x+h_x)]^{n+1} R_1(w) \quad \text{for} \quad T = 0, \tag{4.15}$$

subject to the following boundary conditions:

1. Along $\xi = 0$,

$$\Psi = s(x,t), \quad T = T_{\mathrm{A}}(x,t) \quad \text{for} \quad x < x_{\mathrm{T}}(t). \tag{4.16}$$

2. Along $\xi = H(x,t)$,

$$\begin{aligned} &\Psi = \frac{\partial}{\partial t}\int_{x_0(t)}^{x} H(\sigma,t)\,\mathrm{d}\sigma, \\ &\Psi_\xi = -F(H[1-\mu(H_x+h_x)], T), \\ &\beta_1 H[1-\mu(H_x+h_x)]\,\Psi_\xi + \beta_2 T_\xi = \lambda\Phi(T) \quad \text{for} \quad x < x_{\mathrm{Z}}(t), \\ &T = 0 \quad \text{for} \quad x_{\mathrm{Z}}(t) \leqslant x \leqslant x_{\mathrm{M}}(t). \end{aligned} \tag{4.17}$$

3. Along $\xi = [\eta(x,t) - y_{\mathrm{M}}(x,t)] \equiv \xi_{\mathrm{M}}(x,t)$,

$$\begin{aligned} &\Psi \text{ and } \Psi_\xi \text{ are continuous across this line,} \\ &T = T_\xi = 0, \quad \text{and} \\ &w = 0 \quad \text{when} \quad \Psi_x + \Psi_\xi \xi_{\mathrm{M}x} - \xi_{\mathrm{M}t} \gtrless 0 \quad \text{if} \quad T < 0 \quad \text{in} \quad \xi_{\mathrm{M}}\pm. \end{aligned} \tag{4.18}$$

We shall hereafter refer to the problem posed by the equations in (4.12) to (4.18) as the 'reduced model'. The five dependent variables that must be found in any solution of this problem are H and ξ_{M}, the ice thickness and melting surface profile functions, and Ψ, T and w.

5. Analysis of a cold glacier with temperature-independent viscosity

In this section we consider the form and stability of steady-state solutions to a particularly simple asymptotic limit of the reduced model developed in §4. The analysis presented here is primarily intended to serve as a preliminary example of the more detailed work that will appear in part II; however, since very little analytical work has been done in this area, the analysis is of interest in its own right.

We simplify the reduced model in three ways here. Firstly, we consider the flow of glaciers which are completely cold by requiring that $T < 0$ throughout. In fact, we shall make the slightly stronger assumption that

$$T < T_{\mathrm{Q}}, \tag{5.1}$$

so that no basal sliding occurs. This will be satisfied if the surface temperature T_A is less than T_Q everywhere and the viscous dissipation parameter β_1 is sufficiently small. Secondly, we put

$$\mu = 0, \tag{5.2}$$

which means that we neglect the effect of variations of the surface slope of the glacier from the mean bedrock slope. This is motivated by (3.24) and is a simplifying but non-essential approximation in the present analysis. Thirdly, we uncouple the temperature field from the flow field in the reduced model by setting

$$\kappa = 0, \tag{5.3}$$

i.e. we assume the viscosity is independent of temperature. From (3.24), we see that (5.3) is *not* expected to be a reasonable approximation in any real glacier. It is simply made here in order to gain some initial understanding of the model.

We now use the approximations in (5.1) to (5.3) to reduce the model in §4 to the simpler one given shortly below. In this case, w and ξ_M are irrelevant variables and are omitted from the simpler model. Also, in anticipation of the steady-state form and stability studies that are made later in this section, we require now that the data functions $s(x,t)$ and $T_A(x,t)$ be time independent and therefore they are replaced by $s(x)$ and $T_A(x)$, respectively. The simpler model is then

$$\begin{gathered} \Psi_{\xi\xi} = \xi^n, \\ T_t + \Psi_x T_\xi - \Psi_\xi T_x = \beta_2 T_{\xi\xi} + \beta_1 \xi^{n+1}, \end{gathered} \tag{5.4}$$

subject to the following conditions:

1. Along $\xi = 0$,

$$\Psi = s(x), \quad T = T_A(x). \tag{5.5}$$

2. Along $\xi = H(x,t)$,

$$\Psi = \frac{\partial}{\partial t}\int_{x_0(t)}^{x} H(\sigma,t)\,\mathrm{d}\sigma, \quad \Psi_\xi = 0, \quad \beta_2 T_\xi = \lambda. \tag{5.6}$$

We now consider steady-state solutions to the problem posed by (5.4) to (5.6). The first equation in (5.4) may be immediately integrated subject to the first condition in (5.5) and the first two conditions in (5.6) to show that

$$\Psi(x,\xi) = s(x) - \left[\frac{H(x)^{n+1}}{n+1}\right]\xi + \left[\frac{\xi^{n+2}}{(n+1)(n+2)}\right], \tag{5.7}$$

where $H(x)$, the steady-state version of $H(x, t)$, is given by

$$H(x) = [(n+2)\,s(x)]^{1/(n+2)}. \tag{5.8}$$

Since $H(x)$ represents the steady state ice thickness at x, it is then clear from figure 6 that the steady-state positions of the glacier head and snout are precisely at the first and second zeros, respectively, of the (assumed known) function $s(x)$. Physically, this means that the head lies at the point where the uphill ice accumulation rate first becomes positive, and the snout lies at the point where the ice supplied by the uphill accumulation rate has been completely depleted by ablation. Between the head and the snout, the steady-state ice flow pattern is completely determined by (5.7) (and (4.10) and (4.11)) and the temperature field is to be found as the solution to the following problem:

$$\Psi_x(x,\xi)\,T_\xi - \Psi_\xi(x,\xi)\,T_x = \beta_2 T_{\xi\xi} + \beta_1 \xi^{n+1},$$
$$T(x,0) = T_A(x), \quad \beta_2 T_\xi(x, H(x)) = \lambda. \tag{5.9}$$

This problem is linear but has variable coefficients which, from (5.6), 'degenerate' (i.e. equal zero) along the curve where $\xi = H(x)$. As discussed by Friedman (1964), a study of its solutions is therefore a non-elementary task. On the other hand, if $T_{\xi\xi}$ in (5.9) is replaced by $[T_{\xi\xi} + \delta^2 T_{xx}]$ with $\delta > 0$, as was indeed the case in the scaled model of § 3, then it is shown by Friedman that there does in fact exist a unique solution to the resulting modified version of (5.9). Motivated by this result, we shall simply assume that the same is true for (5.9) as it stands, and turn our attention to the linear stability problem for the (assumed unique) solution to the full set of equations in (5.4) to (5.6).

We do this by writing

$$\Psi = \Psi_0 + \Psi_1 e^{\sigma t}, \quad H = H_0 + H_1 e^{\sigma t}, \quad T = T_0 + T_1 e^{\sigma t}, \tag{5.10}$$

where the suffix zero refers to the (assumed known) steady-state quantities and the suffix one refers to small perturbations from these quantities. Substituting (5.10) into (5.4) to (5.6), expanding the boundary conditions for Ψ in (5.6) as Taylor series, and neglecting second and higher order terms throughout, we obtain the following linear stability equations:

$$\Psi_{\xi\xi} = 0,$$
$$\sigma T + [\Psi_{0x} T_\xi + T_{0\xi} \Psi_x - \Psi_{0\xi} T_x - T_{0x} \Psi_\xi] = \beta_2 T_{\xi\xi}, \tag{5.11}$$

subject to the following conditions:

1. Along $\xi = 0$,

$$\Psi = 0, \quad T = 0. \tag{5.12}$$

2. Along $\xi = H_0(x)$,

$$\Psi = \sigma \int_{x_0}^{x} H(\omega)\,d\omega,$$
$$\Psi_\xi + \Psi_{0\xi\xi} H = 0,$$
$$T_\xi + T_{0\xi\xi} H = 0. \tag{5.13}$$

Here, x_0 is the time-independent value of $x_0(t)$, and for convenience we have dropped suffixes on perturbed quantities. An immediate integration shows that

$$\Psi(x,\xi) = -\Psi_{0\xi\xi}(x, H_0(x))\, H(x)\cdot\xi$$
$$= -H_0^n(x)\, H(x)\cdot\xi, \tag{5.14}$$

where, from a differentiated form of the first condition in (5.13),

$$[H_0^{n+1}(x)\cdot H(x)]_x+\sigma H(x) = 0. \tag{5.15}$$

The general solution of this first order equation is easily shown to be

$$H(x) = \left[\frac{E}{H_0^{n+1}(x)}\right]\exp\left(-\int_c^x \frac{\sigma\,\mathrm{d}\omega}{H_0^{n+1}(\omega)}\right), \tag{5.16}$$

where E and c ($> x_0$) are arbitrary constants. Now $H_0 > 0$ for $x > x_0$, and so

$$\int_c^x \frac{\mathrm{d}\omega}{H_0^{n+1}(\omega)} < 0 \quad \text{for} \quad x_0 < x < c;$$

hence, if $\mathrm{Re}\,(\sigma) > 0$, H in (5.16) is unbounded as $x\to x_0$ unless $E = 0$. In other words, if $\mathrm{Re}\,(\sigma) > 0$ in (5.11) to (5.13), then $H(x)$ and hence $\Psi(x, \xi)$ must be identically zero. The temperature perturbation problem is then the following one:

$$\begin{gathered}\beta_2 T_{\xi\xi}+[\boldsymbol{q}_0\cdot\boldsymbol{\nabla}T]-\sigma T = 0, \quad \boldsymbol{q}_0 = (-\Psi_{0\xi}, \Psi_{0x}),\\ T(x,0) = 0, \quad T_\xi(x, H_0(x)) = 0.\end{gathered} \tag{5.17}$$

Multiplying the first equation here by T^* (i.e. the complex conjugate of T) and integrating the result over $\mathscr{D}$, the region occupied by the steady-state ice mass, we have

$$\beta_2\int_{\mathscr{D}} T^*T_{\xi\xi}\,\mathrm{d}\omega+\int_{\mathscr{D}} T^*(\boldsymbol{q}_0\cdot\boldsymbol{\nabla}T)\,\mathrm{d}\omega-\sigma\int_{\mathscr{D}}|T|^2\,\mathrm{d}\omega = 0, \tag{5.18}$$

and also its conjugate

$$\beta_2\int_{\mathscr{D}} TT^*_{\xi\xi}\,\mathrm{d}\omega+\int_{\mathscr{D}} T(\boldsymbol{q}_0\cdot\boldsymbol{\nabla}T^*)\,\mathrm{d}\omega-\sigma^*\int_{\mathscr{D}}|T|^2\,\mathrm{d}\omega = 0. \tag{5.19}$$

Adding these two equations and using Green's theorem and the identity

$$\mathrm{div}\,(T\boldsymbol{Q}) = T\,\mathrm{div}\,\boldsymbol{Q}+\boldsymbol{Q}\cdot\boldsymbol{\nabla}T,$$

which is valid for any scalar T and vector $\boldsymbol{Q}$, we find, putting $\boldsymbol{Q} = (0, -T_\xi)$,

$$\begin{aligned}0 &= 2\,\mathrm{Re}\,(\sigma)\int_{\mathscr{D}}|T|^2\,\mathrm{d}\omega-\int_{\mathscr{D}}\boldsymbol{q}_0\cdot(T^*\boldsymbol{\nabla}T+T\boldsymbol{\nabla}T^*)\,\mathrm{d}\omega-\beta_2\int_{\mathscr{D}}[T^*T_{\xi\xi}+TT^*_{\xi\xi}]\,\mathrm{d}\omega\\ &= 2\,\mathrm{Re}\,(\sigma)\int_{\mathscr{D}}|T|^2\,\mathrm{d}\omega-\int_{\mathscr{D}}\mathrm{div}\,(|T|^2\boldsymbol{q}_0)\,\mathrm{d}\omega\\ &\quad+\beta_2\int_{\mathscr{D}}[\mathrm{div}\,(T^*\boldsymbol{Q})+\mathrm{div}\,(T\boldsymbol{Q}^*)-\boldsymbol{Q}\cdot\boldsymbol{\nabla}T^*-\boldsymbol{Q}^*\cdot\boldsymbol{\nabla}T]\,\mathrm{d}\omega\\ &= 2\,\mathrm{Re}\,(\sigma)\int_{\mathscr{D}}|T|^2\,\mathrm{d}\omega-\int_{\partial\mathscr{D}}|T|^2\,\boldsymbol{q}_0\cdot\boldsymbol{n}\,\mathrm{d}s\\ &\quad+2\beta_2\int_{\mathscr{D}}|T_\xi|^2\,\mathrm{d}\omega+2\beta_2\int_{\partial\mathscr{D}}[\mathrm{Re}\,(T^*\boldsymbol{Q}\cdot\boldsymbol{n})]\,\mathrm{d}s,\end{aligned} \tag{5.20}$$

where $\partial\mathscr{D}$ represents the boundary of $\mathscr{D}$ and $\boldsymbol{n}$ (as usual) represents the unit outward normal to $\partial\mathscr{D}$. From (5.17), T vanishes when $\xi = 0$ and, from (5.6), (5.17) and the definition of $\boldsymbol{Q}$, $\boldsymbol{q}_0$ and $\boldsymbol{Q}$ vanish when $\xi = H_0(x)$. It then follows that, for Re $(\sigma) > 0$, (5.20) can only be satisfied if T is identically zero, and therefore we have succeeded in demonstrating the linear stability of the assumed steady-state solution to the simplified problem posed by equations (5.4) to (5.6).

6. Conclusions

In this paper we have shown how generally accepted views on the mechanics of ice can be incorporated into a complete fluid-dynamical description of two-dimensional glacial ice flows. This description includes some novel features. The ice is divided (by a 'melting surface') into two distinct types of region, in which the ice is cold and temperate, respectively; both the melting surface and the top surface of the glacier are unknown, and so the problem is one of double free boundary type. One of the kinematic boundary conditions on the bedrock (the 'sliding law') is assumed for physical reasons to be a continuous function of the temperature. This apparently minor change from assumptions made previously by others is justified in Part II, where it is shown to have a major effect on the resulting glacier dynamics. For the sake of completeness, we include an energy equation for the temperate zone which describes the evolution of the moisture content of the ice. It is emphasized that this equation is *not* intended to be definitive (as it neglects any description of water transport *through* the ice), but is rather put forward as a first attempt to describe the hydrology of temperate ice over long time scales, and as a necessary constituent to any complete model (such as that presented here).

Having described the model, we show how it may be scaled and non-dimensionalized in a rational and self-consistent manner without making *a priori* assumptions about the size of various terms, but using only the given input data. The results of this procedure then *imply* that glaciers have depths, velocities, stresses and moisture contents of the observed order of magnitude (and this justifies our results). This *scaled* model is then simplified by making the assumptions that the numbers Re, δ and $\hat{\theta}$ introduced in § 3 are zero. These numbers are indeed small, and can be taken to represent respectively the slowness of the flow, the shallowness of the glacier profile, and the smallness of the deviation of the pressure melting temperature from the atmospheric melting temperature.

After being simplified, the model is then partially solved and a much simpler (though still mathematically complex) *reduced* model is finally obtained. In this model, the further approximation $\mu = 0$ is also suggested by the scalings; this approximation represents the neglect of deviations of the surface slope from the mean bedrock slope and is used extensively as a simplifying assumption in Part II.

Lastly, with the assumption that $\mu = 0$, we have presented an analysis of the particular simple limiting case of a completely cold glacier with temperature-independent viscosity (for which $T < T_Q$ everywhere and $\kappa = 0$). The assumption that the viscosity is independent of temperature makes this case somewhat unrealistic, and, as will be seen from Part II, devoid of the complexity of dynamical behaviour that glacier flows can exhibit, but the case does serve as an initial example of the kind of analysis that may be done using the reduced model and provides a preview of the more detailed and relevant results to appear in Part II.

7. Nomenclature

$a(x, t)$	accumulation/ablation rate
A	rheological function in (2.3)
A_0	typical value of A
A_{cold}	typical value for cold ice
A_{temp}	typical value for temperate ice
B	constant in (2.5)
C_v	specific heat of ice
d	typical depth (3.9)
e	strain-rate invariant (3.5)
e_{ij}	strain-rate tensor (2.4)
$[e]$	typical scale for e
exp	the exponential function
f, F	sliding laws (2.12), (2.14)
g	gravity
g'	gravity component perpendicular to mean bedrock slope
G	geothermal heat flux
$h(x)$	bedrock profile
$H(x, t)$	dimensionless ice thickness (4.11)
k	thermal conductivity of ice
l	typical length scale
L	latent heat
n	exponent in stress–strain rate law (2.3)
$\boldsymbol{n}$	unit outward normal at a boundary
p	pressure
P	pressure scale (3.10)
p_{A}	atmospheric pressure
Q	activation energy in (2.5)
$\boldsymbol{q}$	velocity
$r(w)$, $R_1(w)$	rheology functions in (2.8), (4.4)
R	gas constant
Re	Reynolds number
$s(x, t)$	flux function (3.1)
t	time
T	temperature
T_{A}	atmospheric temperature

T_M	melting temperature of ice (2.18)
T_0	temperature scale
T_Q	temperature at which basal sliding begins to occur
T_f	atmospheric melting temperature of ice
u	horizontal velocity
U	scale for u
v	vertical velocity
V	scale for v
w	moisture content of temperate ice
$W_M(x, t)$	'suction' velocity at the bedrock due to melt-water run-off
W	scale for w
x	length coordinate
x_T	top melting point
x_M	bottom melting point
x_0	glacier head
x_S	glacier snout
x_E	equilibrium point
x_Q	bottom point where sliding begins
x_Z	bottom 'zero temperature' point
y	height coordinate
y_M	melting surface
Z	defined in (3.11)
β_1	viscous dissipation parameter (3.18)
β_2	heat conduction parameter (3.18)
γ	defined in (3.19)
δ	shallow ice parameter (3.4)
ϵ	mean bedrock slope
$\eta(x, t)$	top surface
θ	Clausius–Clapeyron constant
$\hat{\theta}$	dimensionless Clausius–Clapeyron constant (3.22)
κ	defined in (3.18)
λ	geothermal heat flux parameter
μ	surface slope parameter (3.15)
ν	small 'moisture diffusion' coefficient on page 226
ξ	vertical coordinate, measured downwards from the surface (4.11)
$\xi_M(x, t)$	melting surface
ρ	density of ice
σ_{ij}	stress tensor
τ	second invariant of stress deviator tensor (2.4)
τ_{ij}	stress deviator tensor
$[\tau]$	typical stress
Φ	'geothermal' heat flux function (2.16)
ψ, Ψ	stream functions (4.10), (4.11)

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