

## Thermal stability properties of a model of glacier flow

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Received 1979 December 5

**Summary.** We consider the linear stability of the steady state temperature profile in cold regions of valley glaciers. The model we use incorporates surface accumulation and ablation, free surface and melting boundaries and a non-linear temperature-dependent viscous flow law. The main simplifying assumption is that the Graetz number is small, in other words the glacier behaves essentially like a slab: although this is not a reasonable assumption, it enables a fully analytic solution of the problem to be obtained, and may point the way for future analyses which include advective heat transport. We find that the steady state is 'effectively' stable.

### 1 Introduction

Robin (1955, 1969) proposed that glacier surges (Meier & Post 1969) might be triggered by a thermal instability in the temperature profile of cold glaciers. Such an instability might lead to basal areas of temperate ice, which would thus promote sliding. The mechanism for the instability would be the non-linear stress–temperature coupling induced by Glen's (1955) flow law: an increase of temperature would lead to an increase in shear rate and hence in the viscous heating term, which would promote further increase in the temperature. This effect is liable to be offset by removal of excess heat by conductive and advective transport, and also the free surface boundary, since if the velocity increases the depth will correspondingly decrease, leading to a decrease in the shear stress and hence also in the viscous heating. One might therefore expect that if the viscous heating term is sufficiently 'large' to overcome these stabilizing influences, the steady state would be thermally unstable.

An interesting aspect of the problem is that it is well-known that boundary value problems for elliptic equations with non-linear exponential source terms can have multiple solutions (Joseph 1966), unless the magnitude of the source term exceeds a certain critical value, in which case the temperature undergoes a super-exponential rate of increase towards infinity in a finite time: of course the model breaks down before the temperature becomes too large. For ice flows, the temperature reaches the melting point. These kinds of catastrophic instabilities resulting from multiple steady states (which bear a certain

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resemblance to the onset of surges) have been examined by Clarke, Nitsan & Paterson (1977) and Yuen & Schubert (1979), and their relevance to ice sheet disintegration and glacier surges is discussed by Paterson, Nitsan & Clarke (1977) and Cary, Clarke & Peltier (1979). However, Fowler (1980) gave warning that the consideration of the top surface as a free boundary which had to be found as part of the problem might remove the multiplicity of solutions, and in a companion paper to the present one (Fowler & Larson 1980b) it was proved that a realistic model incorporating the above effect but neglecting advective heat transport had a unique solution for the temperature in the cold ice region. This appears to rule out the possibility of such a catastrophic thermal instability.

Nevertheless, it still remains to examine the linearized stability of the given steady state, since this may equally well lead to steady time-dependent behaviour. Previous work in this area is that of Fowler & Larson (1978) and Thompson (1979) who considered the (unrealistic) case of a temperature independent flow law: in both cases the basic state was found to be stable. Yuen & Schubert (1979) also investigated the linear stability of their steady state solutions, and similarly found them to be all stable. Here we extend these results by including an analysis of the free boundary surface, the lengthwise variation in the input data (so also the finite extent of the glacier) and the varying kinds of basal boundary conditions dependent on the temperature regime there. Our conclusion is also that the temperature profile is effectively stable, but we find that the notion of 'effective stability' requires some subtlety in discussing regions of 'sub-temperate' basal sliding, and that a future non-linear finite amplitude stability analysis may qualify the conclusions presented here.

## 2 Derivation of the stability equations

The model we use is essentially that derived by Fowler & Larson (1978) with the additional assumption that  $\mu = 0$  (i.e. the surface slope varies relatively little from the mean bedrock slope). Although this model is derived for consideration of polythermal glaciers, Fowler & Larson (1980b) show that when the Graetz number (Pearson 1978) is small, so that advective heat transport can be neglected ( $1/\beta_2 \rightarrow 0$  in their notation), then the equations governing the temperature of the cold part of the glacier uncouple from the determination of the flow properties in the temperate part. It is found that the temperature can be found together with the unknown bottom boundary (or melting boundary) from a prescription of three boundary conditions for the second-order temperature equation.

The same is true for time-dependent motions. We define the conductive time-scale

$$\tau = \beta_2 t, \quad (1)$$

and use the same notation as Fowler & Larson (1980b): thus  $(x, \xi)$  are Cartesian coordinates along and perpendicular to the mean bedrock slope,  $\xi$  being measured downwards from the top surface;  $\theta$  is the dimensionless temperature, and  $\Psi$  is a stream function. The energy equation may then be written in the form

$$\theta_\tau + \frac{1}{\beta_2} [\Psi_x \theta_\xi - \Psi_\xi \theta_x] = \alpha \xi^{n+1} \exp(\theta) + \theta_{\xi\xi} + \delta^2 [\theta_{xx} + \eta_{xx} \theta_\xi + \eta_x \theta_{x\xi} + \eta_x^2 \theta_{\xi\xi}]. \quad (2)$$

Here,  $\alpha$  is a measure of the viscous heating term,  $n$  is the exponent in Glen's flow law and  $\delta < 1$  is a measure of the shallowness of the flow. The last term in equation (2) requires some comment. This term represents longitudinal heat conduction, and is considered to be small in the steady state solution. Its neglect is indeed valid for, though apparently a singular approximation near the head and snout, no boundary layer analysis is necessary since the

depth  $H \rightarrow 0$  at these points. It is more likely that the method of strained coordinates (Van Dyke 1975) would be relevant. In the time-dependent case, however, we anticipate that instability eigenmodes may be of a singular rather than a regular perturbation nature (Lin 1955) and thus we may expect solutions  $\theta$  of equation (2) in which  $\theta_x \sim 1/\delta$ ,  $\theta_{xx} \sim 1/\delta^2$ . We therefore retain such terms so that a meaningful stability theory can be developed. In equation (2),  $\eta(x, t)$  was the position of the top surface in a different set of coordinates (Fowler & Larson 1978), thus we may consider  $\eta_x, \eta_{xx} \sim 1$ . It then follows that the only longitudinal heat conduction term of relevance in what follows will be the  $\theta_{xx}$  term: retaining this term, but otherwise considering  $\delta$  as small, and letting  $1/\beta_2 \rightarrow 0$ , we obtain the time-dependent temperature equation

$$\theta_\tau = \alpha \xi^{n+1} \exp(\theta) + \theta_{\xi\xi} + \delta^2 \theta_{xx}. \tag{3}$$

Some (not all) of the boundary conditions that are to be prescribed for  $\theta$  and  $\Psi$  are: on  $\xi = 0$  (the top surface):

$$\theta = \theta_A(x); \tag{4}$$

on  $\xi = H(x, \tau)$  (the bottom boundary):

$$\frac{\partial}{\partial \tau} \left[ \int_{x_0}^x H dx \right] = \frac{1}{\beta_2} \Psi; \tag{5}$$

$$\text{(cold)} \quad \theta_\xi = \lambda^*, \quad x \in B_Q,$$

$$\text{(sub-temperate)} \quad \theta_\xi = \lambda^* \phi(\theta) + \alpha HF(H, \theta), \quad x \in B_Z, \tag{6a}$$

$$\text{(temperate)} \quad \theta = 0, \quad x \in B_M;$$

on  $\xi = \xi_M(x, t)$  (the melting surface):

$$\text{(polythermal)} \quad \theta = 0, \quad \theta_\xi = 0, \quad x \in B_T. \tag{6b}$$

These boundary conditions are not sufficient to determine the general solution to equation (3), but are enough to specify the linear stability problem. The boundary conditions equation (6a) require some explanation. Equation (6a)<sub>1</sub> is valid for basal temperatures  $\theta < \theta_Q$ , that is, the ice is cold and non-sliding:  $\lambda^*$  is the dimensionless geothermal heat flux. When  $\theta_Q < \theta < 0$ , the ice slides at a temperature dependent rate  $F(H, \theta)$  and the heat flux released into the cold ice above is modified by a factor  $\phi(\theta)$  which decreases from 1 to 0 as  $\theta$  increases from  $\theta_Q$  to 0. When  $\theta$  reaches zero, the full sliding law is applicable, and  $\theta = 0$  until the heat flux  $\theta_\xi$  decreases to zero when the melting surface leaves the basal region, and equation (6b) is applicable.

We now let  $1/\beta_2 \rightarrow 0$  in equation (5). This implies that  $H = H(x)$  over time-scales  $\tau \sim 1$ , and in fact equation (5) simply illustrates that  $H$  changes over the slower kinematic time scale  $t = \tau/\beta_2$ . If the temperature field turns out to be stable, then it is known that integration of the equation for  $\Psi$  and application of equation (5) yields a kinematic wave equation for  $H$ : disturbances to the steady state profile disappear in a finite time (Fowler & Larson 1980a), and hence it is self-consistent to ascertain the entire stability of the cold zone on the basis of equations (3)–(6) only.

In addition we suppose  $\lambda^* = 0$ . Typically,  $\lambda^* \sim 10^{-1}$  for glaciers, so this is not unreasonable. If  $\lambda^* \neq 0$ , one needs to specify a functional form for  $\phi$ . It seems just as reasonable to put  $\lambda^* = 0$  as to consider an empirical form of  $\phi$ .

Having made these approximations, we now examine the linear stability of the unique steady state solution of equation (3) with  $\delta = 0$  as given by Fowler & Larson (1980b). To do so, we consider a small parameter  $\nu \ll 1$  representative of the amplitude of the perturbation from equilibrium, and put

$$\theta = \theta^{(0)} + \nu\theta^{(1)} + \nu^2\theta^{(2)} \dots \quad (7)$$

We then expand  $\theta$  in equations (3)–(6), retaining only powers of  $O(\nu)$ , and put

$$\theta^{(1)} = \chi(x, \xi) \exp(\sigma\tau) \quad (8)$$

we find that equation (3) becomes

$$\chi_{\xi\xi} + \delta^2\chi_{xx} + [\alpha\xi^{n+1} \exp(\theta) - \sigma]\chi = 0, \quad (9)$$

where we have for convenience dropped the zero superscript on  $\theta^{(0)}$ .

Since  $H$  is a function only of  $x$ , we suppose that it is the steady state depth. We define

$$\begin{aligned} H_0 &= H(x), & x \in B_T, \\ H_0 &= \xi_M^{(0)}(x), & x \in B_T, \end{aligned} \quad (10)$$

where  $\xi_M^{(0)}$  is the steady state melting boundary and is the leading order term in an expansion of  $\xi_M$  in powers of  $\nu$ .  $H_0$  thus denotes the base of the steady state cold zone, and is independent of  $\tau$ . Expanding the boundary conditions in powers of  $\nu$  to  $O(\nu)$  and using the steady state boundary conditions, we find that  $\chi$  must satisfy

$$\chi = 0 \text{ on } \xi = 0; \quad (11)$$

$$\text{on } \xi = H_0,$$

$$\chi_{\xi} = 0, \quad x \in B_Q, \quad (12)$$

$$\chi_{\xi} = \alpha H \frac{\partial F}{\partial \theta}(H, \theta)\chi, \quad x \in B_Z, \quad (13)$$

$$\chi = 0, \quad x \in B_M, \quad (14)$$

$$\chi = 0, \quad x \in B_T, \quad (15)$$

The equation (9) with boundary conditions (11)–(15) is a linear eigenvalue problem in which  $\chi$  and  $\sigma$  must be determined. If there exists an eigenvalue  $\sigma$  with  $\text{Re } \sigma > 0$ , then the steady state solution is linearly unstable.

Before proceeding to solve this set, we mention that Fowler & Larson (1980b) showed that since  $|\theta_Q| \sim 10^{-2} \ll 1$ , it was reasonable and convenient to use an *effective* boundary condition

$$\theta = 0 \text{ on } \xi = H, \quad x \in B_Z \quad (16)$$

as an alternative to condition (6a)<sub>2</sub>. If we linearize (16), we then obtain the *effective* boundary condition for  $\chi$ ,

$$\chi = 0 \text{ on } \xi = H_0, \quad x \in B_Z. \quad (17)$$

We shall consider both equations (13) and (17) in what follows. Note that  $B_Q, B_Z, B_M, B_T$  may be considered as steady state regions (i.e. independent of  $\tau$ ) in the analysis.

### 3 Effective boundary conditions

We consider first the system (9), (11), (12), (14), (15) and (17) and shall refer to this as system (a). The explicit solution of (a) seems unfeasible, but standard elliptic eigenvalue problem theory (Courant & Hilbert 1953) guarantees that (a) has non-trivial solutions only for a discrete set of real eigenvalues. Let us suppose that  $\sigma > 0$  is one of these eigenvalues and  $\chi \neq 0$  is a corresponding eigensolution. Then we multiply the equation (9) by  $\chi$ , integrate the resulting expression over the plane region, say  $D$ , that represents the steady state cold ice zone and use Green's theorem to find

$$\iint_D [-\chi_\xi^2 - \delta^2 \chi_x^2 + \alpha \xi^{n+1} \exp(\theta) \chi^2] dx d\xi = \sigma \iint_D \chi^2 dx d\xi > 0, \tag{18}$$

so that certainly

$$\iint_D [-\chi_\xi^2 + \alpha \xi^{n+1} \exp(\theta) \chi^2] dx d\xi > 0, \tag{19}$$

and hence

$$\int_0^{H_0(x)} [-\chi_\xi^2 + \alpha \xi^{n+1} \exp(\theta) \chi^2] d\xi > 0 \tag{20}$$

for some interval of  $x$  values. But from the standard variational principle for ordinary differential equation eigenvalue problems (Courant & Hilbert 1953), this means that for each fixed value of  $x$  in this interval there exists at least one eigensolution pair  $(\lambda, \chi(\xi))$  for the problem

$$\begin{aligned} \chi'' + [\alpha \xi^{n+1} \exp(\theta) - \lambda] \chi &= 0, \\ \chi &= 0 \text{ on } \xi = 0, \end{aligned} \tag{21}$$

$$\begin{cases} \chi' \\ \chi \end{cases} = 0 \text{ at } \xi = H_0(x) \text{ for } \begin{cases} x \in B_Q \\ x \notin B_Q \end{cases}$$

for which  $\lambda > 0$ . Note that the boundary  $\xi = H_0$  is explicitly dependent on  $\alpha$ . Now let us consider, for each fixed  $x$ , the eigenvalue problem

$$\begin{aligned} \chi'' + [\beta \xi^{n+1} \exp(\theta) - \lambda(\beta)] \chi &= 0, \\ \chi &= 0 \text{ at } \xi = 0, \end{aligned} \tag{22}$$

$$\begin{cases} \chi' \\ \chi \end{cases} = 0 \text{ at } \xi = H_0 \text{ for } \begin{cases} x \in B_Q \\ x \notin B_Q \end{cases}.$$

We restrict our attention to the maximal eigenvalue  $\lambda(\beta)$ : it is then known that the corresponding eigenfunction  $\chi$  is one-signed (say positive). We suppress the explicit dependence of  $H_0$  and  $\lambda$  on  $x$  for convenience. As  $\beta (> 0)$  varies (but  $x$  and  $\alpha$  remain constant),  $\lambda$  varies continuously; it is also easy to see that  $\lambda$  becomes negative and bounded away from zero as  $\beta$  decreases to zero (and this boundedness is uniform in  $x$ : specifically

$$\lambda(0) < \frac{-\pi^2}{4 [\max_x H_0^2]}. \tag{23}$$

It therefore follows that if  $\lambda(\beta) > 0$  when  $\beta = \alpha$  (i.e. there is a positive eigenvalue  $\lambda$  for equation (21)), then there exists a particular  $\beta \in (0, \alpha)$  such that  $\lambda(\beta) = 0$ . Restricting our attention to this value of  $\beta$ , we then have that

$$\chi'' + \beta \xi^{n+1} \exp(\theta) \chi = 0, \quad (24)$$

and  $\chi$  is required to satisfy  $\chi > 0$  for  $\xi \in (0, H)$  and

$$\chi(0) = 0,$$

$$\chi'(0) = 1 \quad (\text{with no loss of generality}),$$

and either

$$\chi'(H_0) = 0 \quad \text{if } x \in B_Q, \quad (25)$$

or

$$\chi(H_0) = 0 \quad \text{if } x \notin B_Q.$$

Now let us reconsider the steady state solutions  $\theta$  of

$$\theta_{\xi\xi} + \alpha \xi^{n+1} \exp(\theta) = 0, \quad (26)$$

$$\theta(0) = \theta_A, \quad \theta_{\xi}(H_0) = 0 \quad (\text{i}) \quad \text{or} \quad \theta(H_0) = 0 \quad (\text{ii}).$$

If we define

$$\theta_{\xi}(0) = g, \quad (27)$$

we can consider solutions of the initial value problem of equations (26) and (27) as being dependent on  $g$ ,  $\theta = \theta(\xi; g)$ . Differentiation of equations (26) and (27) with respect to  $g$  implies that  $\theta_g$  satisfies

$$\theta_{g\xi\xi} + \alpha \xi^{n+1} \exp(\theta) \theta_g = 0 \quad (28)$$

$$\theta_g(0) = 0, \quad \theta_{g\xi}(0) = 1.$$

It follows from equation (25) that

$$\chi = \left(\frac{\alpha}{\beta}\right)^{1/(n+3)} \theta_g \left[ \left(\frac{\beta}{\alpha}\right)^{1/(n+3)} \xi; g \right]. \quad (29)$$

As regards the first boundary condition (i),  $\chi'(H_0) = 0$  if  $x \in B_Q$ , we note that the steady state solution space of  $g$  versus  $H_0$  for equation (26) (i) must have the form shown in Fig. 1. This is because for sufficiently low  $H_0$  there are two possible solutions  $g$  (Joseph 1966) but for each  $g$  there is only one value of  $H_0$  such that  $\theta'(H_0) = 0$ , since  $\theta'$  decreases monotonely with  $H_0$ . Also  $g \rightarrow 0$  or  $g \rightarrow \infty$  as  $H_0 \rightarrow 0$ .

Once again defining  $\theta(\xi; g)$  as the solution of the corresponding initial value problem, the implicit definition of the curve in Fig. 1 is then

$$\theta_{\xi}(H_0; g) = 0. \quad (30)$$

Differentiating with respect to  $g$ , we have

$$\theta_{g\xi} = -\theta_{\xi\xi} \frac{\partial H_0}{\partial g} = \alpha H^{n+1} \exp(\theta) \frac{\partial H_0}{\partial g}, \quad (31)$$

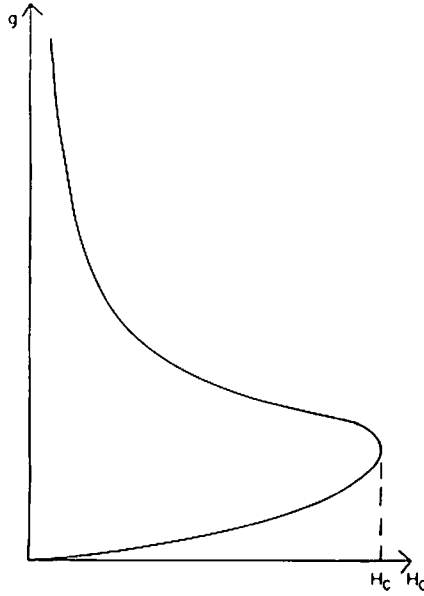


Figure 1. Solution space of  $g$  versus  $H_0$  for (26i).

which is positive for subcritical  $g$  and negative for supercritical  $g$ . The only way of satisfying (i), that is  $\chi_\xi(H) = 0$ , is if, from equation (29),

$$H_0 = \left(\frac{\alpha}{\beta}\right)^{1/(n+3)} \quad H_c > H_c \tag{32}$$

where  $H_c$  is the critical value of  $H_0$  in Fig. 1: but  $H_c$  is precisely the upper limit on the attainable values of  $H_0$ ; therefore equation (32) is impossible. Similar reasoning is valid for case (ii). Thus  $\chi$  defined by equations (24) and (25) cannot exist, and our supposition that  $\lambda(\alpha) > 0$  for some interval of  $x$  values is wrong. We therefore conclude that a positive value of  $\sigma$  cannot exist, and hence with the effective boundary conditions the steady state is linearly stable.

#### 4 Actual boundary conditions

Let us now return to a consideration of the linear stability problem posed in Section 2, in which the actual form of the sub-temperate sliding law is used to derive the perturbed boundary condition (13) rather than (17). We thus consider the equation (9) with boundary conditions (11)–(15). It is well-known (Courant & Hilbert 1953) that the largest (real) eigenvalue  $\sigma_L$  for this system satisfies the variational principle

$$\sigma_L = \max_{\phi} \frac{\iint_D [\alpha \xi^{n+1} \exp(\theta) \phi^2 - \phi_\xi^2 - \delta^2 \phi_x^2] dx d\xi + \alpha \int_{Bz} H_0 \left(\frac{\partial F}{\partial \theta}\right)_0 \phi^2(x, H_0) dx}{\iint_D \phi^2 dx d\xi} + O(\delta^2), \tag{33}$$

where  $D$  is the plane region representing the steady state cold ice zone,  $\phi = \phi(x, \xi)$  is allowed to be any piecewise continuously differentiable function defined on  $D$  which satisfies the  $\chi = 0$  boundary conditions (11), (14) and (15), and  $(\partial F/\partial \theta)_0$  is used here to mean  $\partial F/\partial \theta(H_0(x), \theta(x, H_0(x)))$ . The error term of  $O(\delta^2)$  is included in equation (33) for the following reason. Strictly speaking, the standard variational formulation is not applicable to the system (9), (11–15), but rather to the scaled model of Fowler & Larson (1978), where the normal derivative  $\partial \chi/\partial n$  is given on the boundary  $\xi = H_0(x)$ ,  $x \in B_Q \cup B_Z$ , rather than  $\partial \chi/\partial \xi$ . However, by applying this strict variational principle to a linearly perturbed version of the original equations (in which the normal derivative is indeed given), one can easily see from (3.21) of that paper that the error in writing  $\chi_n \equiv \chi_\xi$  as determined by equation (13) is  $O(\delta^2)$ , as is also the error in writing the line element  $ds = [1 + \delta^2 h'^2]^{1/2} dx \approx dx$ . Also, the neglect of terms involving  $\eta_x, \eta_{xx}$  in the longitudinal heat conduction term leads to an error in equation (33) of precisely  $O(\delta^2)$  under fairly mild restrictions on  $\phi$ , these restrictions always being satisfied in this paper, and therefore the variational principle (33) is accurate to  $O(\delta^2)$ , as stated. By constructing a suitable trial function  $\phi$ , we show below that  $\sigma_L$  must be positive for  $\theta_Q$  sufficiently near zero, and this demonstrates the linear instability of the steady state solution in this case.

If we neglect the  $\delta^2 \chi_{xx}$  term in equation (9), then for each fixed  $x \notin B_Z$ , we obtain the ordinary differential equation eigenvalue problem

$$\chi_{\xi\xi} + [\alpha \xi^{n+1} \exp(\theta) - \sigma(x)] \chi = 0,$$

$$\chi = 0 \text{ on } \xi = 0, \tag{34}$$

$$\chi_\xi = \alpha H_0 \left( \frac{\partial F}{\partial \theta} \right)_0 \chi \text{ on } \xi = H_0,$$

the  $\sigma$  in equation (9) being replaced here by  $\sigma(x)$  in order that the problem's explicit dependence on  $x$  be clearly represented. For each such  $x$ , the variational principle for the largest eigenvalue  $\sigma_L(x)$  of this problem is then

$$\sigma_L(x) = \max_{\phi} \frac{\int_0^{H_0(x)} [\alpha \xi^{n+1} \exp(\theta) \phi^2 - \phi'^2] d\xi + \alpha H_0 \left( \frac{\partial F}{\partial \theta} \right)_0 \phi^2 [H_0(x)]}{\int_0^{H_0(x)} \phi^2 d\xi}, \tag{35}$$

where  $\phi$  is now to range over (piecewise continuously differentiable) functions of  $\xi$  which satisfy the  $\chi = 0$  boundary condition in equation (34). Now it follows from the definition of  $\theta_g(\xi)$  in equation (28) that when  $\phi = \theta_g$ ,

$$\begin{aligned} \int_0^{H_0} [\alpha \xi^{n+1} \exp(\theta) \phi^2 - \phi'^2] d\xi &= \int_0^{H_0} [\alpha \xi^{n+1} \exp(\theta) \phi^2 + \phi \phi'' - (\phi \phi')'] d\xi \\ &= -\theta_g [H_0(x)] \theta_{g\xi} [H_0(x)]. \end{aligned} \tag{36}$$

By making the choice  $\phi = \theta_g$  in equation (35), using equation (36) and letting  $\sigma_*$  be an arbitrary positive constant (introduced for later convenience), we therefore see from



equation (35) that for any specific  $x \in B_Z$ ,

$$\sigma_L(x) \geq \sigma_* > 0 \text{ if}$$

$$\alpha H_0(x) \left( \frac{\partial F}{\partial \theta} \right)_0 \geq \frac{\theta_{g\xi} [H_0(x)]}{\theta_g [H_0(x)]} + \frac{\sigma_* \int_0^{H_0(x)} \theta_g^2(\xi) d\xi}{\theta_g^2 [H_0(x)]}. \tag{37}$$

As discussed above, we are interested here in the physically realistic case where  $\theta_Q \approx 0$ . In this case, it is suggested from Fig. 4 of Fowler & Larson (1978) that over some closed sub-interval of  $B_Z$ ,  $(\partial F/\partial \theta)_0$  might be large enough such that the condition in equation (37) is satisfied and  $\sigma_L(x) \geq \sigma_*$ . Mathematically, this clearly occurs if  $\alpha$  and the functions  $s(\cdot)$  and  $T_A(\cdot)$  are held fixed and  $\theta_Q$  is then chosen sufficiently near zero; in what follows, we suppose  $\theta_Q$  is chosen in this manner and denote by  $I \equiv [x_1, x_2]$  the largest connected sub-interval of  $B_Z$  over which  $\sigma_L(x) \geq \sigma_*$ . We now construct an instability trial function  $\phi(x, \xi)$  for the variational problem (33) by setting

$$\phi(x, \xi) = \Phi(x) V(x, \xi), \tag{38}$$

where  $V(x, \xi) = 0$  for  $x \notin B_Z$ , and for each  $x \in B_Z$ ,  $V(x, \xi)$  is the eigensolution of equation (34) which corresponds to the eigenvalue  $\sigma_L(x)$  and which satisfies the normalizing condition

$$\int_0^{H_0(x)} V^2(x, \xi) d\xi = 1, \tag{39}$$

and  $\Phi$  is the piecewise smooth function defined by

$$\begin{aligned} \Phi &= 0, & x &\leq x_1, \\ \Phi &= \frac{x - x_1}{\delta}, & x_1 < x &\leq x_1 + \delta, \\ \Phi &= 1, & x_1 + \delta < x &\leq x_2 - \delta, \\ \Phi &= \frac{x_2 - x}{\delta}, & x_2 - \delta < x &\leq x_2, \\ \Phi &= 0, & x &> x_2, \end{aligned} \tag{40}$$

it being assumed here that  $\delta < \frac{1}{2}(x_2 - x_1)$  since we are studying the  $\delta \rightarrow 0$  limit. It can easily be checked that for this  $\Phi$ ,

$$\begin{aligned} \int_{x_1}^{x_2} \Phi^2 dx &= x_2 - x_1 - \frac{2}{3}\delta, \\ \iint_D \delta^2 \Phi'^2 d\xi dx &= O(\delta) \text{ as } \delta \rightarrow 0, \end{aligned} \tag{41}$$

where here and subsequently  $O(\delta)$  denotes positive terms of order  $\delta$ . Applying this (admissible) trial function of equation (38) to equation (33) and using equations (35), (37)

and (41), we then have that as  $\delta \rightarrow 0$ ,

$$\begin{aligned}
 \sigma_L &> \frac{\int_I \Phi^2 \int_0^{H_0(x)} [\alpha \xi^{n+1} \exp(\theta) V^2 - V_\xi^2] d\xi dx - O(\delta) + \alpha \int_I H_0 \left( \frac{\partial F}{\partial \theta} \right)_0 \Phi^2 V^2(x, H_0) dx}{\int_I \Phi^2 \int_0^{H_0(x)} V^2 d\xi dx} \\
 &+ O(\delta^2) \\
 &= \frac{\int_I \Phi^2 \left[ \int_0^{H_0(x)} \{ \alpha \xi^{n+1} \exp(\theta) V^2 - V_\xi^2 \} d\xi + \alpha H_0 \left( \frac{\partial F}{\partial \theta} \right)_0 V^2(x, H_0) \right] - O(\delta)}{\int_I \Phi^2 \int_0^{H_0(x)} V^2(x, \xi) dx d\xi} + O(\delta^2) \\
 &= \frac{\int_{x_1}^{x_2} \Phi^2 \sigma_L(x) dx - O(\delta)}{\int_{x_1}^{x_2} \Phi^2 dx} + O(\delta^2) \\
 &\geq \sigma_* - \frac{O(\delta)}{x_2 - x_1 - \frac{2}{3}\delta} + O(\delta^2), \tag{42}
 \end{aligned}$$

and hence for any given  $\sigma_* > 0$  (and therefore  $(x_2 - x_1)$ ),  $\sigma_L > 0$  for  $\delta$  sufficiently small. (Note that this reasoning is invalid when  $x_2 - x_1$  is of numerical order  $\delta$ , which will generally occur when  $F(H, O) \sim |\theta_Q| \sim \delta$ . For the purposes of this paragraph, we assume that this is not the case.) As claimed, the original steady state solution is therefore linearly *unstable* as long as  $\theta_Q$  is sufficiently near zero (and  $\delta$  is correspondingly near zero, so that this solution is a meaningful one); as discussed above, this is expected to be the case in reality.

### 5 Discussion

In summary up to this point, we have paradoxically found that in the physically interesting limit where  $\theta_Q \rightarrow 0$ , the solution is linearly stable or unstable according to whether  $\theta_Q$  is *a priori* set equal to zero and the stability problem then considered or the stability problem is considered first for  $\theta_Q < 0$  and the limiting result as  $\theta_Q \rightarrow 0$  then taken. In order to understand how this paradox arises, one must recall that the formal linear stability analysis carried out here involves the substitution of  $\theta(x, \xi, \tau) = \theta^{(0)}(x, \xi) + \nu \theta^{(1)}(x, \xi, \tau) + \dots$  into the time-dependent reduced model (and similar substitutions for the other dependent variables there) and the subsequent study of the resulting equation for  $\theta^{(1)}$  in the limit where  $\nu \rightarrow 0$ . In deriving the stability/instability results above, we have in essence studied the equation for the two distinct double limits symbolically described by

$$\lim_{\nu \rightarrow 0} \lim_{\theta_Q \rightarrow 0} \text{ and } \lim_{\theta_Q \rightarrow 0} \lim_{\nu \rightarrow 0},$$

respectively, and the interchange of limits in these studies has been responsible for the noted paradox. This situation is directly comparable to that of the stability analysis for the very simple physical case of a pencil being balanced on its (sharp) tip, where we suppose that this tip has a flat circular end-face of small but positive radius. A brief study of this simple second situation provides us with an idea as to how we might determine which of the contradictory stability results found above is the physically relevant one and hence resolve the noted paradox. If the radius of the pencil tip is fixed at any positive value (no matter how

small this is), then the equilibrium state of the pencil being balanced is clearly a stable one as long as the magnitudes of all possible perturbations are required to be *sufficiently small*. That is, the balanced state is formally seen to be linearly stable for arbitrarily small positive values of the tip radius, and hence this is the limiting result as the radius tends towards zero through positive values. On the other hand, if the pencil is first sharpened so that the tip is (theoretically, at least) a point (of zero radius), then the balanced state is clearly unstable to perturbations of arbitrarily small size, and hence the stability analysis for this state involves a double limit paradox of a type analogous (but with the roles of stability and instability interchanged) to that encountered in the glacier stability problem above. This paradox can be easily resolved, however, in the case of the pencil: since the balanced state is clearly unstable to *very small* but yet *finite amplitude* perturbations (e.g. horizontal pencil top displacements of tip radius size) and perturbations of this size arise naturally in any realistic balancing experiment, the state is 'effectively' unstable in a real sense and the physically relevant stability result for this case is the instability one. That is, the physically relevant result is found not by a strict linear stability analysis, but rather by a 'finite amplitude' stability analysis where the perturbations are allowed to be *of the same order of magnitude as the radius of the tip*. Motivated by this observation, let us now study the glacier stability problem in a similar way by defining

$$\nu = |\theta_Q| (\ll 1) \tag{43}$$

and considering the effect of thermal perturbations (i.e. as represented by the  $\nu\theta^{(1)} + \dots$  terms above) of the order of magnitude of  $\nu$  on the stability of the steady state solution in the limit where  $\nu \rightarrow 0$ . In order to see whether the boundary condition (13) or (17) (or some other one entirely) should be used in the linear stability problem in this limit, let us recall that when

$$\begin{aligned} \theta &= \theta^{(0)}(x, \xi) + \nu\theta^{(1)}(x, \xi) + o(\nu), \\ H(x, t) &= H_0(x) + o(\nu) \text{ as } \nu \rightarrow 0, \end{aligned} \tag{44}$$

the boundary condition (6a)<sub>2</sub> involving  $F(H, \theta)$  becomes, with  $\lambda^* = 0$ ,

$$F[H_0, \theta^{(0)} + \nu\theta^{(1)}] = \frac{\theta_{\xi}^{(0)} + \nu\theta_{\xi}^{(1)}}{\alpha H_0} + o(\nu) = F[H_0, \theta^{(0)}] + \frac{\nu\theta_{\xi}^{(1)}}{\alpha H_0} + o(\nu) \tag{45}$$

as  $\nu \rightarrow 0$  (assuming, as is most reasonable here, that  $H_0$  is bounded away from zero and  $\partial F/\partial H$  is uniformly  $O(1)$  in this limit). If we now make the (possibly strong) assumption that for all sufficiently small  $\nu > 0$  we can implicitly solve the equation  $F(H, \theta) = F$  for all relevant  $H$  and  $F$  values to find  $\theta \in (-\nu, 0)$  as

$$\theta = \nu G[H, F], \tag{46}$$

with  $\partial G/\partial F$  being uniformly bounded over this range of  $\nu$ ,  $H$  and  $F$  values, then we can re-write equation (45) as

$$\begin{aligned} \theta^{(0)} + \nu\theta^{(1)} &= \nu G \left[ H_0, F(H_0, \theta^{(0)}) + \frac{\nu\theta_{\xi}^{(1)}}{\alpha H_0} + o(\nu) \right] = \nu G [H_0, F(H_0, \theta^{(0)})] \\ &+ \nu \left[ \frac{\partial G}{\partial F}(H_0, F(H_0, \theta^{(0)})) \right] \left( \frac{\nu\theta_{\xi}^{(1)}}{\alpha H_0} \right) + o(\nu^2) \\ &= \theta^{(0)} + \nu^2 \frac{\theta_{\xi}^{(1)}}{\alpha H_0} \frac{\partial G}{\partial F} [H_0, F(H_0, \theta^{(0)})] + o(\nu^2) \end{aligned} \tag{47}$$

as  $\nu \rightarrow 0$ , and hence

$$\theta^{(1)} = O(\nu) \quad (48)$$

in this limit. That is, as long as this assumption concerning the existence and nature of  $G(H, F)$  is valid, the thermal boundary condition that should be used along  $\xi = H_0(x)$  for  $x \in B_2$  in the linear stability problem for the solution in this case (where the perturbations are taken to be of the same order of magnitude as  $|\theta_Q|$  as  $|\theta_Q| \rightarrow 0$ ) is equation (17) (and hence as in equation 16) and *not* equation (13) (and hence equation 6a<sub>2</sub>). From our study of (a), we must therefore formally conclude that the solution is stable in some sense (that will be discussed further below) against perturbations of amplitudes of the order of  $|\theta_Q|$ , at least as long as the assumption concerning  $G(H, F)$  is valid. From equation (46), the size of  $\partial G/\partial F$  depends on the size of  $\partial F/\partial \theta$ , and since a discussion of the precise functional form of  $F$  is beyond the scope of this paper, we shall not pursue here the question of this validity. Let us simply note that if  $F(H, 0)$  is of numerical order  $\nu$  for  $\nu \ll 1$ , this being perfectly realizable if the bedrock is very rough, then  $\partial G/\partial F$  will be  $O(1/\nu)$ , so that our assumption will *not* be valid. In this case, the possibility of a genuine finite amplitude instability arises, but the study of this situation requires a fully non-linear perturbation analysis which will not be considered here.

In summary then, we have shown that although the solution is unstable in a strict linear sense for arbitrarily small fixed values of  $\nu = |\theta_Q| > 0$ , perturbations to this solution of amplitudes of numerical order  $\nu$  do *not* grow in time as long as the fixed value of  $\nu$  is sufficiently small. The solution is therefore '*effectively stable*' in the following physically meaningful sense: although perturbations to this solution will *not* in general decay to zero as  $\tau \rightarrow \infty$ , they will grow no larger in numerical magnitude than  $O(\nu)$  as long as they are  $O(\nu)$  (or possibly even larger) to begin with and  $\nu$  is sufficiently small, and hence for all time they will remain negligible in comparison with the steady state solution in the realistic case where  $\nu \ll 1$ . This demonstration of the effective stability of the solution is intended to be the basic result of this section, but since we have also shown that the solution is in fact 'infinitesimally' unstable (due to the occurrence of sub-temperate basal sliding when  $x \in B_2$ ), we wonder if this 'instability' might be related to the phenomenon of 'stick-slip' motion (i.e. where the basal ice is intermittently not moving at all and moving at its full ( $\theta = 0$ ) sliding velocity, the average motion being determined by the large scale ice dynamics, e.g. Robin 1976). Such a phenomenon would occur physically due to the 'periodic' build-up of a liquid interface at the bedrock, and its release in the form of a jerk forward. We realize that there are undoubtedly other mechanisms available for explaining stick-slip motion, and so we do not offer the present one as much more than speculation; we merely note that the explanation of this motion based on it is supported by the mathematics done in this section.

## References

- Cary, P. W., Clarke, G. K. C. & Peltier, W. R., 1979. A creep instability analysis of the Antarctic and Greenland ice sheets, *Can. J. Earth Sci.*, **16**, 182–188.
- Clarke, G. K. C., Nitsan, U. & Paterson, W. S. B., 1977. Strain heating and creep instability in glaciers and ice sheets, *Rev. Geophys. Space Phys.*, **15**, 235–247.
- Courant, R. & Hilbert, D., 1953. *Methods of Mathematical Physics*, vol. 1, Interscience, New York.
- Fowler, A. C., 1980. The existence of multiple steady states in the flow of large ice masses, *J. Glaciol.*, **25**, 183–184.
- Fowler, A. C. & Larson, D. A., 1978. On the flow of polythermal glaciers – I. Model and preliminary analysis, *Proc. R. Soc. Lond A*, **363**, 217–242.

- Fowler, A. C. & Larson, D. A., 1980a. On the flow of polythermal glaciers – II. Surface wave analysis, *Proc. R. Soc. Lond. A*, **370**, 155–171.
- Fowler, A. C. & Larson, D. A., 1980b. The uniqueness of steady state flows of glaciers and ice sheets, *Geophys. J. R. astr. Soc.*, **63**, 333–345.
- Glen, J. W., 1955. The creep of polycrystalline ice, *Proc. R. Soc. Lond. A*, **228**, 519–538.
- Joseph, D. D., 1966. Bounds on  $\lambda$  for positive solutions of  $\Delta\psi + \lambda f(r) \{\psi + G(\psi)\} = 0$ . *Q. appl. Math.*, **23**, 349–354.
- Lin, C. C., 1955. *The Theory of Hydrodynamic Stability*, Cambridge University Press.
- Meier, M. F. & Post, A. S., 1969. What are glacier surges? *Can. J. Earth Sci.*, **6**, 807–817.
- Paterson, W. S. B., Nitsan, U. & Clarke, G. K. C., 1977. An investigation of creep instability as a mechanism for glacier surges, *Proceedings of the International Workshop on the Dynamics of Glacier Variations and Surges*, USSR Academy of Sciences, Moscow, in press.
- Pearson, J. R. A., 1978. Polymer flows dominated by high heat generation and low heat transfer, *Polymer Engng Sci.*, **18**, 222–229.
- Robin, G. de Q., 1955. Ice movement and temperature distribution in glaciers and ice sheets, *J. Glaciol.*, **2**, 523–532.
- Robin, G. de Q., 1969. Initiation of glacier surges, *Can. J. Earth Sci.*, **6**, 919–928.
- Robin, G. de Q., 1976. Is the basal ice of a temperate glacier at the pressure melting point? *J. Glaciol.*, **16**, 183–196.
- Thompson, D. E., 1979. Stability of glaciers and ice sheets against flow perturbations, paper presented at the *Symposium on the Dynamics of Large Ice Masses*, Ottawa, Canada, August 1978, *J. Glaciol.*, **24**, 427–441.
- Van Dyke, M. D., 1975. *Perturbation Methods in Fluid Mechanics*, Second edn, Parabolic Press, Stanford.
- Yuen, D. A. & Schubert, G., 1979. The role of shear heating in the dynamics of large ice masses, *J. Glaciol.*, **24**, 195–212.