A STUDY OF THE EFFECT OF MODE TRUNCATION ON AN EXACT PERIODIC SOLUTION OF AN INFINITE SET OF LORENZ EQUATIONS

M. BOOTY, J.D. GIBBON
Department of Mathematics, Imperial College, London SW7 2BZ, UK

and

A.C. FOWLER
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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A set of complex Lorenz equations with an infinite number of z-components is shown to have an exact periodic solution. Sufficient conditions for the instability of this solution have been found and the effect of truncation of the z-components is considered. It is shown that in certain cases truncation has little effect but in others the stability criterion is radically altered.

In applied mathematics, possible transitions to chaotic regimes in the underlying equations of motion are usually studied by reducing these equations to a finite set of ODEs. This procedure involves expanding the dependent variables of the underlying PDEs as a Fourier series with time dependent amplitudes, substituting and then truncating at a finite number of modes. Lorenz [1] used this method to derive his now famous equations from the Navier–Stokes equations for two-dimensional convection. The mode expansion and truncation method can be open to question if the further inclusion of higher modes produces radically different behaviour in the bifurcation sequence. In most cases only numerical integration of the higher dimensional sets of ODEs in which more modes have been included will say whether convergence to a particular set of transitions has occurred.

One set of equations, however, has been derived by various authors without using the Fourier expansion and truncation method

$$\dot{X} = -\sigma X + \sigma Y, \tag{1a}$$

$$\dot{Y} = (r - X)\sum_{n=1}^{\infty} Z_n - aY, \qquad r = r_1 + \mathrm{i}r_2, \quad a = 1 - \mathrm{i}e, \tag{1b}$$

$$\dot{Z}_n = -b_n Z_n + \tfrac{1}{2}\gamma_n(X^*Y + XY^*), \qquad b_n, \gamma_n > 0, \quad n = 1, ..., \infty. \tag{1c}$$

Eqs. (1) are a complex infinite version of the Lorenz equations with an infinite number of Z-components and complex r and a but real r_1, r_2, σ, e, b_n and γ_n.

With the validity of truncation in mind, our intention is to study eqs. (1) as model set of infinite equations from which we can derive a considerable amount of analytical information, to see whether certain stability criteria of the finitely truncated versions of (1) differ markedly from those of the infinite set. This, of course, depends strongly on the form of b_n and γ_n as functions of n which is the number of Z-components. Our study will be from the mathematical viewpoint and so we will consider various forms of b_n and γ_n. Eqs. (1) do have a physical basis however. For a single Z-component various real and complex versions have been derived using secular pertur-

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bation theory for the two-layer [2,3] and Eady models [4,5] of baroclinic instability and also in the laser [6,3]. For an infinite number of Z-components they have been derived for the two-layer model [7–9] and the Eady model [4]. The difference between the single and infinite sets derives from the use of side wall boundary conditions in the baroclinic models. The infinite set arises out of correct use of these boundary conditions but to apply them it is necessary to Fourier expand in the cross-stream variable. The equations as a whole occur out of the removal of secular terms in a multiple scales approach with X being a slowly varying wave envelope in all cases with the dot referring to a slow time variable. In the two-layer and Eady models, the forms of b_n and γ_n as functions of the number of cross-stream Fourier components are very complicated. We shall consider these specifically later as special cases. Our main concern is to study (1) as a general set of mathematical equations with restrictions on γ_n, b_n (>0) only in so far that certain series must converge.

The mathematical properties of (1) at the first bifurcation are very similar to the single Z-component case. Fowler et al. [10] studied this case both analytically and numerically. In contrast to the real Lorenz equations (single Z-component) the single Z-component complex equations do not show a transition to chaos ($\sigma > b + 1$) when r_1 is raised through a critical value. Instead, the origin undergoes a supercritical Hopf bifurcation to a limit cycle, the analytic form of which can be found exactly. This limit cycle is stable for all values of r_1 when $\sigma < b + 1$ but bifurcates subcritically to a two-torus when $\sigma > b + 1$ at some higher value of r_1. For $r_2, e \approx 1$, no further bifurcations were found but in the limit $r_2 \to 0$, the torus undergoes period doubling to chaotic motion. It is in this limit that the complex version reduces to the real equations. The main conclusion of ref. [10] was that complexification of the coefficients turns fixed points into limit cycles, limit cycles into tori and suppresses the chaos into a very small region of parameter space. In analogy with the results of ref. [10], eqs. (1) have only one fixed point: $X = Y = Z_n = 0$, which lies at the origin in phase space. No other fixed points occur unless $e + r_2 = 0$ which is a highly pathological condition. It is also the condition which is needed to scale out the imaginary parts in (1) by a phase rotation to reduce the equations to real form. Therefore we will always assume that $e + r_2 \neq 0$. A study of the stability of the origin shows that the stability matrix occurs in block diagonal form and the characteristic equation is

$$\prod_{n=1}^{\infty} (\lambda + b_n)[(\lambda + \sigma)(\lambda + a) - \sigma r] = 0. \tag{2}$$

The roots of eqs. (2) are $\lambda = -b_n$ and

$$\lambda = \tfrac{1}{2}\{-(\sigma + a) \pm [(\sigma + a)^2 + 4\sigma(r - a)]^{1/2}\}. \tag{3}$$

The value of $r_1 (r_{1c})$ at which the origin becomes unstable is given by $\mathrm{Re}(\lambda) = 0$. We find

$$r_{1c} = 1 + (e - \sigma r_2)(e + r_2)/(\sigma + 1)^2 \tag{4}$$

and the frequency of the critically stable eigenmodes is ($\omega = \mathrm{Im}\,\lambda$ at $r_1 = r_{1c}$) given by

$$\omega = \sigma(e + r_2)(\sigma + 1)^{-1}. \tag{5}$$

When either $e = r_2 = 0$ or $e + r_2 = 0$, then $\omega = 0$ and $r_{1c} = 1$ which is the result found by Lorenz [1] for his real equations. Since $\omega = \mathrm{Im}(\lambda) = 0$ in that case, no complex conjugate eigenvalues occur and a Hopf bifurcation is not possible. However, in the complex case, $\omega \neq 0$ and a Hopf bifurcation is possible yielding a small amplitude limit cycle of frequency ω about the origin. It is usually very rare for a limit cycle solution to be found exactly, but in ref. [10] an exact periodic solution for the complex single Z-component version of (1) was found. Following ref. [10] but excluding the algebra, an exact periodic solution of (1) for all Z-components is given by

$$X = A \exp(\mathrm{i}\omega t); \quad Y = \sigma^{-1}(\sigma + \mathrm{i}\omega)A \exp(\mathrm{i}\omega t), \quad Z_n = \gamma_n b_n^{-1} |A|^2, \tag{6a}$$

$$|A|^2 = (r_1 - r_{1c})\left(\sum_{n=1}^{\infty} \gamma_n b_n^{-1}\right)^{-1}, \tag{6b}$$

where ω and r_{1c} are given by (4) and (5). Firstly, this is only a valid solution if the sum in (6b) converges. Hence, if $b_n \sim n^B$ and $\gamma_n \sim n^G$ as $n \to \infty$, then we require $G < B - 1$. Secondly, this transition from the origin to the limit cycle solution given in (6) is a supercritical Hopf bifurcation as (6b) shows. Thirdly, when $e = r_2 = 0$ or $e + r_2 = 0$, then $\omega = 0$; $r_{1c} = 1$ and the solution (6) becomes a continuum of fixed points. This continuum of fixed points replaces the two fixed points of the real Lorenz equations [1] because the complex nature of X and Y (even when $e = r_2 = 0$) allows these two points to be rotated to form a circle of fixed points. The fortunate occurrence of an exact periodic solution in the form given in (6) allows us to investigate the stability of this solution by transforming to a frame rotating with frequency ω:

$$X = x\exp(\mathrm{i}\omega t), \quad Y = y\exp(\mathrm{i}\omega t), \quad z_n = Z_n. \tag{7}$$

The new equations are

$$\dot{x} = -(\sigma + \mathrm{i}\omega)x + \sigma y, \quad \dot{y} = \left(r - \sum_{n=1}^{\infty} z_n\right)x_n - (a + \mathrm{i}\omega)y, \quad \dot{z}_n = -b_n z_n + \tfrac{1}{2}\gamma_n(x^* y + x y^*). \tag{8}$$

Apart from the origin, which is unstable when $r_1 > r_{1c}$, (8) has fixed points at $x = A, y = (1 + \mathrm{i}\omega\sigma^{-1})A$; $z_n = \gamma_n b_n^{-1}|A|^2$. Perturbing about these latter fixed points we find that the characteristic equation takes the form

$$\left|\begin{array}{ccccc|ccccc}
-\sigma N-\lambda & \sigma & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\
P & -L-\lambda & 0 & 0 & -A & -A & \cdots & -A & \cdots \\
0 & 0 & -\sigma N^*-\lambda & \sigma & 0 & 0 & & 0 & \\
0 & 0 & P^* & -L^*-\lambda & -A & -A & \cdots & -A & \cdots \\
\hline
\frac{1}{2}\gamma_1 AN^* & \frac{1}{2}\gamma_1 A & \frac{1}{2}\gamma_1 AN & \frac{1}{2}\gamma_1 A & -b_1-\lambda & 0 & \cdots & 0 & \cdots \\
\cdot & \cdot & \cdot & \cdot & 0 & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
\frac{1}{2}\gamma_n AN^* & \frac{1}{2}\gamma_n A & \frac{1}{2}\gamma_n AN & \frac{1}{2}\gamma_n A & 0 & & & -b_n-\lambda &
\end{array}\right| = 0, \tag{9}$$

$$L = a + \mathrm{i}\omega, \quad N = 1 + \mathrm{i}\omega\sigma^{-1}, \quad P = r - A^2\left(\sum_{n=1}^{\infty} \gamma_n b_n^{-1}\right). \tag{10}$$

Using the fact that $LN = P$, we find by good fortune , that we are able to evaluate the infinite determinant in (9) to give

$$\lambda\left\{\lambda[(\lambda + \sigma + 1)^2 + q^2] + A^2(\lambda + 2\sigma)(\lambda + \sigma + 1)\sum_{n=1}^{\infty}\frac{\gamma_n}{\lambda + b_n}\right\} = 0, \tag{11}$$

where $q^2 = (2\omega - e)^2$ and we are assuming without loss of generality that A is real. We can deduce three things immediately. Firstly the $\lambda = 0$ root derives from the phase invariance of eqs. (1) which gives rise to the limit cycle and has been discussed in detail in ref. [10]. Secondly, no positive real values of λ exist since each term in (11) is positive definite when $\lambda > 0$. Thirdly, no further zero roots occur since $\Sigma_{n=1}^{\infty} \gamma_n b_n^{-1}$ is finite and positive. If the limit cycle is to become unstable, it must then be through a complex conjugate pair of roots $\lambda = \lambda_R \pm \mathrm{i}\Omega$ crossing the imaginary axis. Taking real and imaginary parts of (11) when $\lambda = \pm\mathrm{i}\Omega$ we find

$$DA^2S = Q, \quad DA^2R = P, \quad S = \sum_{n=1}^{\infty} \frac{\gamma_n}{b_n^2 \mp \Omega^2}, \quad R = \sum_{n=1}^{\infty} \frac{\gamma_n b_n}{b_n^2 + \Omega^2}, \tag{12}$$

where

$$Q = 2\Omega^2(\sigma+1)(3\sigma+1) + [2\sigma(\sigma+1) - \Omega^2][(\sigma+1)^2 + q^2 - \Omega^2], \tag{13a}$$

$$P = -\Omega^2(3\sigma+1)[(\sigma+1)^2 + q^2 - \Omega^2] + 2\Omega^2(\sigma+1)[\Omega^2 - 2\sigma(\sigma+1)], \tag{13b}$$

$$D = [2\sigma(\sigma+1) - \Omega^2]^2 + \Omega^2(3\sigma+1)^2. \tag{13c}$$

Since $A^2, \Omega^2, R, S, D > 0$ then we must have $P, Q > 0$. This is violated if $\sigma \leqslant 1$ which means that no roots $\Omega^2 > 0$ exist. The limit cycle is therefore stable for all $r_1 > r_{1c}$ when $\sigma \leqslant 1$.

When $\sigma > 1$, it is possible to find certain sufficient conditions for instability. The algebra is straightforward but long, so we shall just summarise the results. Firstly, since A^2 is proportional to r_1 we can consider solutions of (12) in terms of large and small A^2 with these limits being equivalent to large and small r_1. Without specific forms of γ_n and b_n we cannot sum the series in (12) but instead we consider solutions Ω^2 of each separately as functions of A^2 and call them $\Omega_R^2(A^2)$ and $\Omega_S^2(A^2)$. For instability, there must be a coincident solution of both equations and so we must have $\Omega_R^2 = \Omega_S^2$ for at least one value of A^2.

There are two ranges of q^2 to consider. When

$$0 < q^2 < (\sigma+1)^2(\sigma-1)/(3\sigma+1), \tag{14}$$

then no bounds occur on Ω^2, but when

$$q^2 > (\sigma+1)^2(\sigma-1)/(3\sigma+1), \tag{15}$$

then

$$\Omega^2 > q^2(3\sigma+1)/(\sigma-1) - (\sigma+1)^2. \tag{16}$$

However, for both ranges of q^2, we can show that $\Omega_R^2(0) > \Omega_S^2(0)$ for all q and so *a sufficient condition for instability is that $\Omega_S^2(A^2) > \Omega_R^2(A^2)$ as $A^2 \to \infty$*. If this is satisfied then the two curves must intersect at least once or an odd number of times. From (12) and (13) we see that $A^2 \to \infty$ as $\Omega^2 \to \infty$ and in this limit

$$D \sim \Omega^4, \quad P \sim (\sigma-1)\Omega^4, \quad Q \sim \Omega^4, \tag{17}$$

which reduces eqs. (12) to

$$A^2 R[\Omega_R^2(A^2)] \sim (\sigma-1), \quad A^2 S[\Omega_S^2(A^2)] \sim 1. \tag{18}$$

Any sufficient condition for instability will now ultimately depend on how R and S behave in the limit $\Omega^2 \to \infty$. Firstly we consider a finite truncation of the sums R and S. Let us assume that we truncate after N modes. R and S now take the asymptotic form

$$R \sim \Omega^{-2} \sum_{n=1}^{N} \gamma_n b_n, \quad S \sim \Omega^{-2} \sum_{n=1}^{N} \gamma_n, \quad (\Omega^2 \to \infty), \tag{19}$$

and this gives immediately that a sufficient condition for instability is

$$\sigma > \left(\sum_{n=1}^{N} \gamma_n b_n\right)\left(\sum_{n=1}^{N} \gamma_n\right)^{-1} + 1. \tag{20}$$

If only a single z-component is taken in the original equations then (20) reduces to $\sigma > b + 1$, a result already obtained by Lorenz [1] for his equations.

For the infinite set of equations it is necessary to deduce how R and S behave as functions of Ω by using the Euler–McLaurin summation formula. For any finite truncation, R and S behave as Ω^{-2} for large Ω but may not necessarily behave like this for an infinite number of modes. In particular, if R and S do not behave the same as one another for large Ω in the infinite case, then the stability criterion may be radically changed, if it exists at all. In order to study the asymptotic behaviour of R and S for an infinite number of modes, we will consider two cases in order to illustrate this point.

(i) b_n bounded as $n \to \infty$. If γ_n and b_n are such that $\Sigma\gamma_n b_n$ and $\Sigma\gamma_n$ converge then the Euler–McLaurin summation formula shows that

$$R \sim \Omega^{-2} \sum_{n=1}^{\infty} \gamma_n b_n, \quad S \sim \Omega^{-2} \sum_{n=1}^{\infty} \gamma_n \quad (\Omega^2 \to \infty), \tag{21}$$

and the instability criterion is given by (20) with the partial sums replaced by infinite sums. Thus for this case, there is no qualitative difference between truncation at large N and the infinite system. A physical example in this first category is the form of γ_n and b_n for the two-layer model for baroclinic instability. We find in this case that

$$\gamma_n = \frac{2m^2(n-\frac{1}{2})^2}{[(n-\frac{1}{2})^2 - m^2]\,[(n-\frac{1}{2})^2\pi^2 + \frac{1}{4}a^2]}, \quad b_n = \frac{2\pi^2(n-\frac{1}{2})^2}{(n-\frac{1}{2})^2\pi^2 + \frac{1}{4}a^2}. \tag{22}$$

In this case it is possible to evaluate the infinite sums for R and S in (12) by using Fourier series. The final result is most easily expressed in the form

$$\mathrm{i}\Omega = 2\mu^2(1-\mu^2)^{-1}, \quad R - \mathrm{i}\Omega S = \frac{(1-\mu^2)m^2\pi^2}{(a\mu)^2 + 4m^2\pi^2}\left[1 - \frac{4a\mu \tanh(a\mu/2)}{(a\mu)^2 + 4m^2\pi^2}\right]. \tag{23}$$

This integration was first performed by Smith [9] for the two layer problem with real variables only i.e. $e = r_2 = 0$ which implies that $\omega = 0$ and the limit cycle reduces to fixed points. He did not consider any other forms of γ_n or b_n. We can see that, as predicted, R and S do indeed behave as Ω^{-2} for large Ω. In passing, it is interesting to point out that the partial sums for the γ_n and b_n of (22) do *not* tend to the same limit for large values of a and m as the analytical form of the infinite sum expressed in (23) since R(partial) $\sim a^{-4}$ but R(infinite) $\sim a^{-3}$ as $a \to \infty$.

(ii) b_n unbounded as $n \to \infty$. There are several cases we could consider, but they can be summarised by taking the example of $b_n \sim n$ and $\gamma_n \sim n^{-p}$ $(p > 0)$ as $n \to \infty$. This establishes the convergence of the infinite sum in the periodic solution (6). If $p > 2$, then it may be verified by the summation formula that $R, S \sim \Omega^{-2}$ as $\Omega \to \infty$ and so truncation makes very little difference and the stability criterion is again given by (20) with N replaced by infinity. An example of this is the Eady model for baroclinic instability [4] in which

$$\gamma_n = \frac{(2mS_0/ah)(a^2 - m^2\pi^2)(1-\theta^{-1}\tanh\theta)}{[(n-\frac{1}{2})^2 - m^2]^2}, \quad \theta = (2n-1)\pi S_0 a^{-1}, \quad b_n = \theta\tanh\theta. \tag{24}$$

In this case $b_n \sim n$ and $\gamma_n \sim n^{-4}$ as $n \to \infty$ and so $p = 4$. Truncation is therefore established for this model as a procedure which will not materially affect the instability criterion.

If however, $0 < p \leqslant 2$ then the summation formula shows that the partial sums do not converge uniformly with Ω. We find for the $p = 1$ and 2 cases that in the $\Omega \to \infty$ limit

$$p = 2: \quad R \sim \mathrm{O}(\Omega^{-2}\log_e\Omega), \quad S \sim \mathrm{O}(\Omega^{-2}), \tag{25a}$$

$$p = 1: \quad R \sim O(\Omega^{-1}), \qquad S \sim O(\Omega^{-2}\log_e \Omega). \tag{25b}$$

In both cases, using (18), we find that for fixed $\sigma > 1$,

$$\Omega_R^2(A^2) > \Omega_S^2(A^2) \quad \text{as } A^2 \to \infty \tag{26}$$

and so no instability can occur as A^2 (or r_1) $\to \infty$. This is in contrast to the truncated case for which a stability criterion in the form of (20) always exists. We conclude in this case that it is not valid to truncate the equations.

Finally we note that there still remains the possibility of multiple intersections of the Ω_R^2 and Ω_S^2 curves. For a final instability (stability) there would need to be an odd (even) number of intersections which would produce closed "windows" of instability along the r_1-axis. A numerical check on the two-layer formulae showed no evidence of these windows.

We should like to point out that our study of the infinite complex Lorenz equations has had a two-fold motivation. Firstly we have used them as a test-case set of equations which have an exact solution to test the validity of truncation procedures. Secondly, we have shown analytically that truncation does not alter the characteristics of the Eady and two-layer models mentioned above. Pedlosky and Frenzen [7] numerically integrated the real version of the infinite Lorenz equations for up to 24 modes. Our analytical conclusions therefore confirm that truncation is a valid procedure in the neighbourhood of the instability of (in this case) the continuum of fixed points. However, this analysis predicts nothing about behaviour at other bifurcation points such as possible transitions from periodic or quasi-periodic solutions to chaos. For the specific case of the two-layer model these have been studied numerically in ref. [7] but it would be interesting to also perform a numerical investigation with various other forms of γ_n and b_n for both the uniform and non-uniform cases.

Finally we note that Manley and Trève [11] and Trève [12] using a method of Foias and Prodi [13] have shown that a lower bound on the number of modes in a Fourier expansion is needed in Bénard convection in order to obtain qualitatively correct approximate solutions of the Navier–Stokes equations.

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