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COMMENT

Note on a paper by G Rowlands

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Abstract. This note discusses the possible use of perturbation methods in studying chaotic trajectories of ordinary differential equations, with particular focus on a recent paper on this topic by Rowlands.

In a recent paper, Rowlands (1983) has presented an approximate analysis of the Lorenz (1963) equations, which gives a (quantitative) description of the cusp-shaped difference map found by Lorenz, which related successive values in the sequence $\{M_n\}$ of maxima of one of the variables, Z. Since this non-monotone map provides an explanation of the chaotic (aperiodic) behaviour of numerically computed solutions of the Lorenz equations (Lorenz 1963, Li and Yorke 1975, May 1976, Collet and Eckmann 1980), its prediction would essentially 'solve' the system—at least in principle—and one could then seek to apply Rowlands' method to other chaotic systems.

The method used is none other than the standard method of multiple scales (as developed by Stuart (1960) to follow disturbances to shear flows into the finite amplitude range) applied to Hopf bifurcation from a steady state solution. This method is well known and widely used, but in the original problem (turbulence in shear flow) for which it was devised, it has failed to achieve anything resembling a description of turbulence; the basic reason for this seems to be that, although the theory is relevant to the problem at hand (there *is* a sub-critical Hopf bifurcation in plane Poiseuille flow, for example (e.g. Stewartson and Stuart 1971), it is *not* obviously or even necessarily closely related to the phenomenon of turbulence: bifurcation theory is *essentially* concerned with (asymptotically) *small* amplitude disturbances, *close* to some critical parameter value.

Despite this, the temptation to try and push nonlinear stability theory further than it can go is strong. Davey and Nguyen (1971) tried a Stuart–Watson theory for pipe flow, despite the apparent non-existence of any finite critical Reynolds number. In a different vein, Lin and Kahn (1980) in studying periodic solutions of the delayed logistic equation, achieved a numerical extension of the range of validity of their small amplitude expansion by a judicious change of variable. Similarly, Rowlands (1983) constructs a finite amplitude difference equation for M_n using just such an expansion procedure, and shows that it accurately mirrors Lorenz's (1963) results. Nevertheless, this method is only formally valid for small amplitude motions, in which case (e.g. for ordinary differential systems) a marginally stable eigenmode proportional to $\exp(i\Omega t)$ has a slowly varying complex amplitude A which satisfies the Landau–Stuart equation

$$dA/dt = k_1 A + k_2 |A|^2 A, \tag{1}$$

whatever the system under consideration. Equation (1) is valid when $|A|^2 \sim |\mathrm{Re}\, k_1| \ll 1$, and is trivially solved, with the conclusion that if $\mathrm{Re}\, k_2 < 0$, a stable supercritical Hopf limit cycle exists for $\mathrm{Re}\, k_1 > 0$, and if $\mathrm{Re}\, k_2 < 0$, then (for $\mathrm{Re}\, k_1 > 0$) infinitesimal disturbances grow super-exponentially until $|A| = O(1)$, *when the analysis is no longer valid.*

In the light of these remarks, a closer look at Rowlands' results seems worthwhile, with two main considerations in view: firstly, although the analysis is quantitatively vindicated, it is conceptually inaccurate, and it would be unwise to leave the general reader with the impression that multiple scales analyses will provide an understanding, *in themselves*†, of chaotic behaviour. Secondly, one might suppose that conceptual inaccuracy hardly matters if the analysis is quantitatively accurate, and so I wish to point out that the numerical validity of the approximation to M_{n+1} as a function of M_n is not systematic (it is equivalent to taking a two-term Taylor series of a general function), and the claim of predicting a non-monotone function (which is where the chaos comes from) is erroneous.

As already explained, Rowlands analyses the Lorenz system

$$\dot{X} = -\sigma X + \sigma Y, \qquad \dot{Y} = (r - Z)X - Y, \qquad \dot{Z} = XY - bZ, \tag{2}$$

near the critical value $r = r_c = \sigma(\sigma + b + 3)/(\sigma - b - 1)$, where the two non-trivial steady states $X = Y = \pm[b(r-1)]^{1/2}$, $Z = r - 1$ (for $r > 1$) bifurcate oscillatorily. Consequently, with solutions of the form

$$X = \pm[b(r-1)]^{1/2} + \{A \exp(\mathrm{i}\Omega t) + (^*)\} + O(A^2), \tag{3}$$

etc, $(^*)$ as usual denoting the complex conjugate, A satisfies the Landau equation (1), where $\mathrm{Re}\, k_1 \sim r - r_c \ll 1$, and $|A| \sim |r - r_c|^{1/2}$, by assumption.

Rowlands next uses the solution of (1) (obtained by first writing it as

$$\mathrm{d}|A|^2/\mathrm{d}t = K_1|A|^2 + K_2|A|^4, \tag{4}$$

where $K_1 = 2\mathrm{Re}\, k_1$, $K_2 = 2\mathrm{Re}\, k_2$) to form a difference equation for $\bar{M}_n = M_n - (r-1)$ (M_n is the nth maximum of Z in a sequence, obtained (say) by numerical integration of (2)). A more direct, and asymptotically equivalent, approach, is essentially to apply the method of averaging to (4), thus replacing $\mathrm{d}|A|^2/\mathrm{d}t$ by $\{|A_{n+1}|^2 - |A_n|^2\}/\tau$, where $\tau = 2\pi/\Omega$ is the basic period of the oscillation, and A_n and A_{n+1} are values of A a time τ apart. This yields from (4)

$$|A_{n+1}|^2 = |A_n|^2(1 + \tau K_1 + \tau K_2|A_n|^2) + O(|A_n|^6), \tag{5}$$

and is valid for $|A_n|^2 \sim K_1 \ll 1$. This is equivalent to Rowland's equation (3.7). He then shows that one can take $\bar{M}_n = |A_n|$, and consequently (5) gives, as the Lorenz map $\bar{M}_n \to \bar{M}_{n+1}$,

$$\bar{M}_{n+1} = (1 + \varepsilon)\bar{M}_n(1 + K\bar{M}_n^2 + O(\bar{M}_n^4)), \tag{6}$$

where

$$\varepsilon = \tfrac{1}{2}\tau K_1, \qquad K = \tfrac{1}{2}\tau K_2; \tag{7}$$

† See the later remarks on 'co-dimension two' bifurcation when *two* parameters take on critical values.

equation (6) is asymptotically equivalent to Rowland's equation (3.9), and is based on the assumption $\bar{M}_n = O(\varepsilon^{1/2})$, where by assumption $\varepsilon \ll 1$.

There is always a requirement for interpretation when *asymptotic* (i.e. limiting) analyses are applied to actual numerical examples. If ε is a small number, then usually $\varepsilon = 0.1$ is considered 'small' enough. Depending on the situation, however, $\varepsilon = 1$ may also be 'small' (e.g. Stokes flow in fluid mechanics at Reynolds numbers of O(1)), and sometimes computer extension of asymptotic series provides results for $\varepsilon \gg 1$ (e.g. Reynolds numbers O(10), see Van Dyke 1975)!

In this case, Lorenz's parameters $r = 28$, $\sigma = 10$, $b = \frac{8}{3}$ (which are close to critical in the sense that $r - r_c \approx 3.26$ is 'small') gives an ε of 0.064, from Rowland's equation (3.9), and (6) above. Consequently, one expects his difference equation to be formally valid in the *numerical* range $0 < \bar{M}_n = O(\varepsilon^{1/2}) \approx 0.25$. Clearly the extension of the prediction up to values $\bar{M}_n = \bar{M}_c \approx 11.8$ is *a posteriori*, in the sense that it works (*in this case*); in general, there is no reason for it to do so. Actually, inspection of (5) indicates that (6) represents the first two terms in a Taylor expansion of $|A_{n+1}|^2 = f(|A_n|^2)$; the coincidence of predicted and observed results simply indicates that f is 'nearly linear', and presumably higher coefficients are numerically small. For example, $K = \tau\beta \approx 0.0025$ using Rowland's values, even though (formally) it is O(1).

Whereas one can accept small Taylor coefficients as a reason for quantitative accuracy, it is not clear why any such difference equation should apply *at all*, since uniqueness of solutions to ordinary differential equations implies the existence of a vertical structure at each $\bar{M}_n$, which is often considered to be a Cantor set, if a strange attractor is present (Ott 1981). In fact, it is known that a strong contraction rate of phase volumes will squash this structure to be 'nearly' zero-dimensional, so that a curve does ensue.

Perhaps the main point I want to make is that, even if one accepts the extension of the local result beyond its expected range of validity, the notion that it will predict chaos by virtue of a non-monotone map is incorrect. Rowlands ascribes the critical value $\bar{M}_c$ of $\bar{M}_n$ to the occurrence of a zero in trajectories in X. This is actually right, but for the wrong reason. The Lorenz equations are indeed symmetric, but this does not mean that the curve for $\bar{M}_n < \bar{M}_c$ is reflected about $\bar{M}_c$ to obtain that for $\bar{M}_n > \bar{M}_c$. On the contrary, the symmetry of the equations is lost in the difference equation which takes the same form whether X is greater or less than zero. Actually, even accepting Rowlands' definition of $\bar{M}_c$, one cannot, even heuristically, extend the present analysis to the range $\bar{M}_n > \bar{M}_c$: for, when the trajectory crosses from $X > 0$ to $X < 0$, it then oscillates about $X = -[b(r-1)]^{1/2}$, and cannot be represented in the form (3) (where a plus sign applies to $X > 0$). That the map is symmetric about $\bar{M}_n = \bar{M}_c$ is simply not generally true, even though it seems so for $r = 28$, $\sigma = 10$, $b = \frac{8}{3}$.

The presence of a cusp, and hence the non-monotonicity of $\bar{M}_{n+1}(\bar{M}_n)$, is due to the occurrence of a homoclinic orbit (Robbins 1977, Kaplan and Yorke 1979) which leaves and returns to the origin. This orbit is of finite amplitude and infinite period, and thus not conceptually obtainable by the Landau–Stuart equation. It is as a consequence of this that the *form* of Yorke and Yorke's (1979) approximate analysis is valid, and it is actually possible to systematically extend their local analysis in a global manner, again approximately (Fowler and McGuinness 1982). Furthermore, the bifurcation of the homoclinic orbit takes place at a '*homoclinic explosion*' (Sparrow 1982) for a value of r which is parametrically *unrelated* to r_c. For Lorenz's parameters, this value is $r_H \approx 13.296$, as opposed to $r_c \approx 24.74$.

The circumstance in which one can find chaos by a regular multiple scale expansion is by having a 'co-dimension two' birfurcation, i.e. one at which two different instabilities occur (almost) simultaneously. This can either be due to two parameters attaining critical values (Arnéodo *et al* 1982), or by seeking some distinguished limit (Gibbon and McGuinness 1982), but in any case one does not predict chaos directly, since the generalised amplitude equations one can obtain are of third order, and again require a numerical solution. One can, in such circumstances, prove the existence of a strange invariant set of trajectories (e.g. Holmes 1980, Keener 1982), but even whether the invariant set is attracting or repelling cannot yet be stated.

The only other approximate analyses I am aware of for attempting to predict chaos are based on the analysis of weakly non-Hamiltonian systems, of which the Lorenz equations as $r \to \infty$ are an example (Robbins 1979). By use of the method of averaging (Sparrow 1982) or that of Kuzmak–Luke (Kevorkian and Cole 1981, Poyet 1980), one can construct a pair of second-order autonomous differential equations for the slowly varying amplitude functions B and D (see Robbins 1979 for notation). A fixed point solution of these equations corresponds to a periodic solution of the original equations. Unfortunately, the worst a pair of autonomous differential equations can do is have limit cycle behaviour, and even that cannot occur in the Lorenz system.

There is one other facet of the averaged equations of some importance, and that is that the line $B=D$ is degenerate, in that Lipschitz continuity of the system does not hold there. The averaged equations can have non-unique solutions as a consequence, if trajectories in (B,D) space reach this line, which one can think of as being like a cut in the plane. There is then some hope that the averaged analysis can be extended to cope with this line (it formally breaks down as $B-D \to 0$), but it is difficult. A similar investigation has been attempted by Shimizu and Ichimura (1982) for a different problem (but still with such an invariant line); they identify the cause of 'phase-mixing' chaos (i.e. crossing the invariant line), but are unable to give a predictive analysis. The point here is that the invariant line exists because the underlying periodic solutions of the system (whose amplitudes are B and D) have periods which tend to infinity as $B \to D$, and *this* is a consequence of the existence of a homoclinic orbit in the system. Again, *the homoclinicity is the fundamental cause of the* (eventually appearing) *chaotic behaviour.*

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