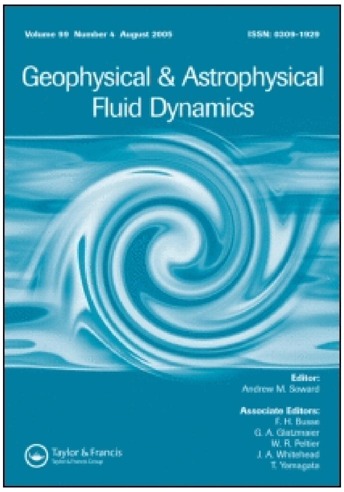

Geophysical & Astrophysical Fluid Dynamics
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713642804

On the transport of moisture in polythermal glaciers

A. C. Fowler [a]
[a] Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA

Online Publication Date: 01 March 1984

To cite this Article Fowler, A. C.(1984)'On the transport of moisture in polythermal glaciers',Geophysical & Astrophysical Fluid Dynamics,28:2,99 — 140

To link to this Article: DOI: 10.1080/03091928408222846

URL: http://dx.doi.org/10.1080/03091928408222846

Geophys. Astrophys. Fluid Dynamics, 1984, Vol. 28, 99–140
0309-1929/84/2802-0099 $18.50/0

On the Transport of Moisture in Polythermal Glaciers†

A. C. FOWLER

Room 2-336, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

(*Received January 11, 1983; in final form November 2, 1983*)

We reconsider the problem of formulation of a model for polythermal glaciers, focussing attention in particular on the temperate zone where ice and water can coexist at the melting temperature. The energy equation for the ice-water mixture in this zone introduces a *moisture flux*, and a constitutive law for this flux is required. By analogy with the flow through a porous medium, we use Darcy's law (i.e. the second momentum equation of a two-phase flow model with "porous" geometry), and then require a mechanical constitutive relation relating the water pressure p_w to the average ice pressure p_I. Experience in two phase flows suggests that $p_w = p_I$ may be problematical, and experience in soil mechanics suggests it is inaccurate. A constitutive relation is therefore presented based on work of Nye (1976), and its effect on the well-posedness of the model is examined. Considerations of the sort presented here have clear relevance in the formulation of similar problems in other geophysical situations, notably mantle convection.

1. INTRODUCTION

The problem of the flow of a melt through its own solid phase is one which has a wide variety of applications, particularly in the geophysical sciences. We might mention subglacial drainage systems and moisture transport in temperate glaciers (Nye, 1976; Nye and Frank, 1973; Spring and Hutter, 1981; Hutter, 1982), melt water runoff from snow (e.g. Colbeck, 1977) and magma percolation through

†This work was reported at the Euromech symposium 172 on Mechanics of Glaciers, Interlaken, Switzerland, during September 1983.

solid rock, with relevance to volcanism and emplacement of magma chambers (as well as mantle convection) (Frank, 1968; Turcotte and Ahern, 1978; Stevenson, 1982a, b). Other applications suggest themselves in metallurgy and crystal growth, if the density difference associated with the phase change is assigned the importance which, in principle, it is due (Rubinstein, 1979).

In common with other two-phase flows (Drew, 1983), a correct formulation of these problems is not an easy matter; consequently, this paper will be largely concerned with *formulation*: clearly, this is a necessary preliminary to the process of calculation. In particular, we will focus attention on the particular case of the transport of moisture through ice, when such moisture is created by frictional heating in a *polythermal* glacier, i.e. one which is partly cold and partly temperate. The reason for this emphasis is that a realistic model of a fully temperate glacier must take account of larger scale water transport as well, which may not necessarily be describable in terms of the model developed here.

The nature and dynamics of moisture within temperate ice has been of interest for some time (Lliboutry, 1971, 1976; Nye and Mae, 1972; Nye and Frank, 1973), and Lliboutry (1976) explicitly wrote an energy equation for temperate ice, which involves the *moisture content* w. In their model of polythermal glaciers, Fowler and Larson (1978) also had such an equation, together with certain continuity conditions across the cold-temperate interface. Their treatment was somewhat abrupt, although they did claim to have produced a well-posed problem. More recently, Hutter (1982) reconsidered the formulation, and concluded that Fowler and Larson's (1978) conditions at the cold-temperate transition were both physically and mathematically in error.

In this paper, we reconsider the problem of moisture transport, and in particular examine the possible constitution of the moisture flux, which will lead to a constitutive law for the water pressure. As a consequence, we seem to find a well-posed problem, which in some respects is akin to that proposed by Fowler and Larson (1978).

In addition, we will present a model equation describing the transport of salts within the water phase of temperate ice. It is often held (e.g. Glen *et al.*, 1977) that salts have an important influence on the behavior of temperate ice. Our point of view is that while this may be true at the bedrock, for example, the dependence of melting temperature on salt concentration is so weak (Lliboutry, 1976), that the salt migration problem is uncoupled from the interior flow, and consequently is not of *primary* interest: it is conceivable to have a pure glacier.

In studying the hydrology of glaciers, it is not clearly feasible to construct a detailed continuum mechanical model for situations in which water-filled crevasses, moulins, etc. dominate the nature of the flow; in such cases, a more *ad hoc* approach might be more useful. On the other hand, many "Arctic" type glaciers which maintain an average surface temperature below 0°C. have been found to have temperate zones adjoining the base (Clarke and Goodman, 1975; Jarvis and Clarke, 1975), and the dynamic nature of such temperate zones should be well represented by the *kind* of model considered here. In particular, this is of interest in the examination of possible thermal runaway type instabilities in *cold* glaciers (Clarke *et al.*, 1977), which if relevant, could have important consequences for ice age dynamics (Schubert and Yuen, 1982).

For these reasons, we will primarily focus on the situation shown in Figure 1. We will consider an (Arctic) glacier whose annual mean

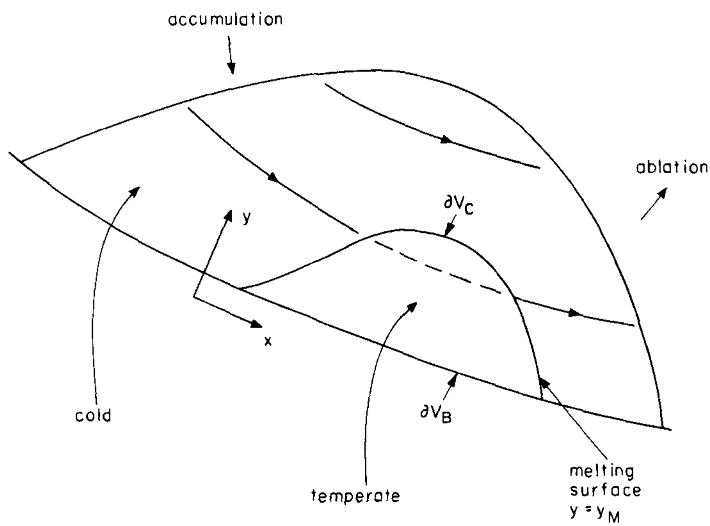

FIGURE 1 Schematic representation of an Arctic-type polythermal glacier. Streamlines are indicated by arrows. In this figure, boundaries of temperate ice are ∂V_B at the bedrock, ∂V_C at cold ice. If the temperate surface extended to the surface, the boundary at the atmosphere is denoted by ∂V_A.

surface temperature is less than zero centigrade: the near surface ice is therefore cold. Ice accumulates upstream from packed fallen snow, flows downhill under gravity, and ablates downstream near the *snout* (due to summer insolation). Although cold near the surface, heat is supplied to the deeper ice by a geothermal heat flux at the base, and by viscous dissipation within the ice. It is quite possible that this heating will be sufficient to raise the temperature T to “the” melting temperature $T=T_M$, so that a temperate ice zone appears, as shown in the figure. The problem which now presents itself is to formulate the field equations for the temperate zone. In this zone $T=T_M$, and the energy equation now describes either the moisture content (if ice and water coexist at $T=T_M$), or the water temperature (if water alone is present). It is this problem which we will address in the following sections, although for completeness, a discussion of the fully temperate case is also given.

2. GOVERNING FIELD EQUATIONS

(a) Cold ice

Models for the motion for cold ice ($T<T_M$, where T is temperature and T_M is melting temperature) have been given by Fowler and Larson (1978), Grigoryan *et al.* (1976) and Hutter (1982). In essence, we can write

$$\operatorname{div}\mathbf{u}=0, \quad \nabla\cdot\boldsymbol{\sigma}+\rho_I\mathbf{g}=\mathbf{0},$$

$$\rho_I c_p[T_t+\mathbf{u}\cdot\nabla T]=k\nabla^2 T+\sigma_{ij}e_{ij}, \tag{2.1}$$

for an incompressible ice mass, where $\mathbf{u}$ is velocity, σ_{ij} is the stress tensor, e_{ij} is the strain rate tensor, and

$$\sigma_{ij}=-p\delta_{ij}+\tau_{ij},\ \tau_{ij}=\eta\, e_{ij}, \tag{2.2}$$

where the effective viscosity η is generally assumed to be a function of T and the second stress invariant $\tau_{ij}\tau_{ij}$ $(=2\tau^2)$. Also, T is temperature, ρ_I is ice density, $\mathbf{g}$ is gravity, c_p is specific heat, and k is thermal conductivity. The Eqs. (2.1) must be supplemented with the usual stress, velocity and temperature boundary conditions.

(b) Temperate ice

Before proceeding, we discuss the nature of the heating terms in (2.1). The term $\sigma_{ij} e_{ij}$ in $(2.1)_3$ represents a heat source within the ice due to viscous frictional heating. In addition, the thermal boundary condition at the base of the ice is usually considered to be one of a prescribed geothermal heat flux. A scaling of the equations (Fowler and Larson, 1978) reveals that typically both these heating terms are non-negligible (and thus can lead to T reaching T_M), with viscous heating being typically a factor of two to three larger.

The relevance of this is that one can expect physically different situations to occur, depending on which heating term is the dominant one. It is now well-known (Atthey, 1974, 1975; Ockendon, 1975; Tayler, 1975; Elliott and Ockendon, 1982) that in heat flow problems involving a change of phase, in which the heat is supplied from an *internal* source, a "mushy" zone will develop when melting is initiated, in which the temperature remains at the melting point, and both phases co-exist. On the other hand, if heat is supplied externally from a prescribed heat flux, then a layer of melt forms at the boundary, and the solid temperature is the solution of a classical Stefan problem.

The upshot of this is that we can consequently expect internal temperate zones to form when viscous heating is significant, and that these will consist of an ice-water mixture, in which one can define a moisture content w (mass fraction of water). In situations where geothermal heat is significant, one would expect the appearance of sub-glacial lakes, as at Grimsvötn in Iceland, and underneath the Antarctic ice sheet. The appearance of such lakes reduces shear stress (and hence, essentially, viscous heating) to zero, and so it seems reasonable to consider these two phenomena separately. On the other hand, when a temperate zone appears, then sliding occurs at the base, and it is well-known that the study of this process involves the transport of heat in the bedrock: in this case the interaction of a temperate zone (mushy zone) with a basal water film (or layer or cavity) is properly a subject for study in connection with the theory of sliding (for reviews, see Weertman, 1979; Lliboutry, 1979), or subglacial drainage (Spring and Hutter, 1982).

We shall therefore suppose that a temperate zone exists, as indicated in Figure 1, in which ice and water co-exist at the melting temperature. The field equations are then properly those describing two-phase flow (Drew, 1983), in which case one has two mass and two momentum balances; one can alternatively use mixture theory (Hutter, 1982), in which case the *diffusion velocity* of water through the ice requires a constitutive relation to specify it (rather than being found from a solution of the appropriate phasic momentum equation). In the general case, inclusion of an energy equation is extraordinarily complicated (Ishii, 1975), and indeed the whole field of two-phase flow is fraught with problems of appropriate boundary conditions, well-posedness, etc. In view of this, our philosophy will be to try and establish a simple, yet reasonable, model by which one might compute the dynamical state of a temperate ice zone, bearing in mind the difficulties that have transpired in other related studies.

We define the void (volume) fraction of water to be α; the mixture density is then

$$\rho=\alpha\rho_w+(1-\alpha)\rho_I, \tag{2.3}$$

where we immediately assume that ice and water phases are separately incompressible: ρ_w and ρ_I are the (constant) specific water and ice densities respectively. The mixture water and ice densities are then

$$\rho^w=\alpha\rho_w,\ \rho^I=(1-\alpha)\rho_I, \tag{2.4}$$

and the mass fraction of water is given by

$$w=\rho^w/\rho. \tag{2.5}$$

If the (average) ice and water velocities are $\mathbf{u}^w$ and $\mathbf{u}^I$, respectively, we define the *barycentric velocity* $\mathbf{u}$ and the moisture transport velocity $\mathbf{v}$ by

$$\rho\mathbf{u}=\rho^w\mathbf{u}^w+\rho^I\mathbf{u}^I,\ \mathbf{v}=\mathbf{u}^w-\mathbf{u};$$

$$\mathbf{u}^I=\mathbf{u}-w\mathbf{v}/(1-w),\ \mathbf{u}^w-\mathbf{u}^I=\mathbf{v}/(1-w); \tag{2.6}$$

these relations are given by Hutter (1982).

Conservation of mass for the mixture gives

$$d\rho/dt+\rho\,\mathrm{div}\,\mathbf{u}=0, \tag{2.7}$$

where $d/dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$, and that for the water phase is

$$\rho \, dw/dt = -\operatorname{div} \mathbf{j} + C, \tag{2.8}$$

where

$$\mathbf{j} = \rho w \mathbf{v} \tag{2.9}$$

is the moisture flux of water through the ice. The term C in (2.8) represents volume production of water, and is liable to be non-zero in temperate ice due to the viscous heating.

For the momentum equation, averaging yields, for zero Reynolds number (Drew, 1983)

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} = \mathbf{0}, \tag{2.10}$$

where

$$\boldsymbol{\sigma} = \alpha \boldsymbol{\sigma}^w + (1-\alpha) \boldsymbol{\sigma}^I \tag{2.11}$$

is the mixture stress tensor, $\boldsymbol{\sigma}^w$ and $\boldsymbol{\sigma}^I$ being the (phasic) average water and ice stress tensors, respectively.

We now consider an energy equation for this mixture. To do so, we make a constitutive assumption that ice and water phases co-exist at the melting temperature, which we take in the form

$$T = T_M = T_0 - \beta p - A'c, \tag{2.12}$$

where p is the local pressure, and c is the salt concentration. Since we allow p and c to be different in each phase, the melting temperature may vary on a microscopic scale. Lliboutry (1976) gives values of the order of $\beta \sim 10^{-2}\,\mathrm{K\,bar^{-1}}$, $A' \sim 2\,\mathrm{K\,mol^{-1}\,kg}$. They represent small variations of the essentially constant melting temperature, which we retain in order to examine their effects on heat transport due to thermal gradients.

The specific internal heat content H per unit mass of the mixture may be written in the form

$$\rho H = (1-\alpha)\rho_I c_p T_M^I + \alpha \rho_w [c_p T_M^w + L]$$

$$\equiv \rho [c_p T_M + wL],$$

that is,

$$H = c_p T_M + Lw, \tag{2.13}$$

where L is the latent heat of melting, and T_M is the average melting temperature of the mixture: however, T_M^I and T_M^w are the *average* melting temperatures of each phase, which are not necessarily equal, since we do not assume that the average pressures are equal; (2.13) simply comes from adding the heat contents of ice and water in a volume element of the mixture, together with the definition of the latent heat. Since kinetic energy is negligible (zero Reynolds number) a balance of energy equation for the mixture is

$$\partial(\rho H)/\partial t + \operatorname{div}[(1-\alpha)\rho_I c_p T_M^I \mathbf{u}^I] + \operatorname{div}[\alpha\rho_w(c_p T_M^w + L)\mathbf{u}^w]$$

$$= \nabla \cdot [k\nabla T_M] + W + E, \tag{2.14}$$

where W represents the (bulk) viscous heating terms, and E is the interfacial energy source. The bulk thermal conductivity k can be determined by averaging (Ishii, 1975), or one can think of it as a constitutive term, but it hardly matters in any case, as it will eventually be neglected (being small). For convenience, we assume the specific heats of ice and water are essentially the same.

The bulk viscous heating term is given by Ishii (1975) as

$$W = \alpha\boldsymbol{\sigma}^w : \nabla\mathbf{u}^w + (1-\alpha)\boldsymbol{\sigma}^I : \nabla\mathbf{u}^I,$$

which after some manipulation takes the form

$$W = \boldsymbol{\sigma} : \nabla\mathbf{u} + \alpha\boldsymbol{\sigma}^w : \nabla[(1-w)^{-1}\mathbf{v}] - \boldsymbol{\sigma} : \nabla[(1-w)^{-1}w\mathbf{v}]. \tag{2.15}$$

Using (2.3)–(2.7), and taking k and c_p as constant, we can write (2.14) in the form

$$\rho dH/dt = \rho d(c_p T_M + Lw)/dt$$

$$= -\operatorname{div} L\mathbf{j} - \operatorname{div}[\rho w(T_M^w - T_M)c_p\mathbf{v}/(1-w)] + k\nabla^2 T_M + W + E, \tag{2.16}$$

i.e.

$$\rho L\, dw/dt + \operatorname{div}[L^* \mathbf{j}] = E + W + k\nabla^2 T_M - \rho c_p\, dT_M/dt,$$

$$L^* = L + c_p(T_M^w - T_M)/(1-w). \qquad (2.17)$$

The other energy equation (for the water phase) is taken care of by the assumption that water and ice temperature are locally equal, which at least seems plausible. We also require a momentum equation for the water phase, or a constitutive relation for $\mathbf{v}$ (or $\mathbf{j}$). In the present situation, if moisture exists in temperate ice in an interconnecting system of veins (Raymond and Harrison, 1975), then one can treat the ice as a porous medium (Nye, 1976), and it seems reasonable to "constitute" the "diffusion" velocity $\mathbf{v}$ of water relative to the centre of mass by Darcy's law,

$$\mathbf{v} = -\kappa w \nabla[p_w + \rho_w g y], \qquad (2.18)$$

where gravity points in the negative y-direction. The permeability κ may be written in the form (Nye and Frank, 1973)

$$\kappa = \rho_I l^2/\rho_w \chi \eta_w, \qquad (2.19)$$

(see also Roberts and Loper, 1982), where l is grain diameter, η_w is the viscosity of water, and χ is a geometrical parameter of order 10^3. It is very important to realise that the adoption of (2.18) refers to the moisture velocity, averaged in such a way that the microscopic details of the flow are unimportant. In particular, a time average is applied, so that the instantaneous velocity field may not resemble the averaged velocity field. It is useful to idealise (2.18) as representing the flow of water through a connected set of pores, but it would be unwise to assume uncritically that this may be identified at any instant with the vein system. In a temperate glacier, most of the water is contained in pockets (Lliboutry, 1976), whereas as little as $10^{-4}\%$ may be held in the veins (Raymond and Harrison, 1975). This latter figure would yield negligibly small permeability of the ice (instantaneously). However, recrystallisation and flow of ice will allow the vein system to sweep through the glacier (Glen *et al.*, 1977), and for time scales greater than a day (e.g. ice velocity $\sim 5\,\text{cm.day}^{-1}$, crystal size 10 cm) the vein system will encounter water pockets, and empty them (if they are at higher pressure, as would be expected from surface energy considerations). Thus, the effective "pore" system for longer time scales *includes* the water present in pockets, and the instantaneous permeability of a block of temperate ice would be much lower than the effective permeability.

It would be preferable in a two-phase flow theory to "derive" (2.18) from the corresponding averaged momentum equation, without needing recourse to the assumption that the average behavior is that of a porous medium. Luckily, this is easily done in the present situation of negligibly small Reynolds number. In this case, the averaged momentum equation for the water phase is (Drew, 1983; Ishii, 1975)

$$\mathbf{0}=\nabla\cdot[\alpha\boldsymbol{\sigma}^{w}]+\alpha\rho_{w}\mathbf{g}+\mathbf{M}, \tag{2.20}$$

where the interfacial source term $\mathbf{M}$ arises from interfacial transfer of momentum and stresses. We can write

$$\mathbf{M}=p^{w,i}\nabla\alpha+\mathbf{M}^{d}, \tag{2.21}$$

where $p^{w,i}$ is the average interfacial pressure, and $\mathbf{M}^d$ is the interfacial drag. One now has to choose constitutive forms for $p^{w,i}$ and $\mathbf{M}^d$. In keeping with the assumption of lubrication theory, as applied to the pore flow, we take

$$p^{w,i}=p_{w}, \tag{2.22}$$

and

$$\mathbf{M}^{d}\approx -A(\mathbf{u}^{w}-\mathbf{u}^{I}), \tag{2.23}$$

which simply uses the statement, that the drag exerted by a laminar Poiseuille flow is proportional to the flow velocity, in building a plausible constitutive law. $\mathbf{M}$ should properly include a term $(-\boldsymbol{\sigma}^{w,i}\cdot\nabla\alpha)$, rather than $p^{w,i}\nabla\alpha$, but this is not commonly done, and in any case has no effect below. Applying (2.22) and (2.23) to (2.20) and (2.21), we obtain

$$A(\mathbf{u}^{w}-\mathbf{u}^{I})=-\alpha\nabla[p_{w}+\rho_{w}gy]+\nabla\cdot(\alpha\boldsymbol{\tau}^{w}), \tag{2.24}$$

where $\boldsymbol{\tau}^w$ is the deviatoric part of the stress tensor. Inclusion of an average interfacial stress would lead to a term $(-\boldsymbol{\tau}^{w,i} \cdot \nabla\alpha)$ on the right hand side of (2.24), presumably of the same order of magnitude as $[\nabla \cdot (\alpha\boldsymbol{\tau}^w)]$. However, we claim that typically $|A(\mathbf{u}^w - \mathbf{u}^I)| \gg |\nabla \cdot (\alpha\boldsymbol{\tau}^w)|$, because the first term represents the variation of stress across the pore radius, whereas the second is due to a large scale secular change in the *average* stress. By assumption, these average variables change much more slowly than the small scale microscopic flow properties; consequently we can ignore them in the limit implicit in the averaging. Finally, from (2.6), we find

$$\mathbf{v} = -[w(1-w)/\rho_w A]\nabla[p_w + \rho_w g y], \tag{2.25}$$

identical to (2.18) if we define

$$\kappa\rho_w A = 1 - w. \tag{2.26}$$

Lastly, we consider the rôle of salts. These are of some importance in glaciology, but are less so in the mathematical model (which could equally well be formulated in the absence of impurities). We include a discussion for completeness, but will retain the maximum simplicity in the formulation. Salt concentration is a further thermodynamic quantity, whose variation in both phases is generally required, but much simplification ensues by taking diffusion in the solid phase to be zero, and by requiring the phases (on *average*) to be in equilibrium; this is not really true, since melting and freezing at grain boundary intersections will lead to a variation in the solid phase concentration. However, it is feasible that recrystallisation may serve to yield an average solid concentration in equilibrium with the pore concentration (Glen *et al.*, 1977). If we assume solidus and liquidus temperatures to have constant slopes as functions of concentration (Chalmers, 1964) [e.g. A' in (2.12) is constant], then at a given temperature, one has

$$c_S/c_L = c_I/c_w = \lambda, \tag{2.27}$$

a constant, where c_S and c_L are corresponding solidus (ice) and liquidus (water) concentrations: λ is called the distribution coefficient. The equation of conservation of salt for the mixture is then [analogously to (2.14)]

$$\lambda\frac{\partial}{\partial t}[(1-\alpha)\rho_I c]+\lambda\,\mathrm{div}\,[(1-\alpha)\rho_I c\mathbf{u}^I]$$

$$+\frac{\partial}{\partial t}[\rho wc]+\mathrm{div}\,[\rho cw(\mathbf{u}+\mathbf{v})]=-\mathrm{div}\,\mathbf{j}_c, \tag{2.28}$$

where $c_w = c$ is the salt concentration in the water, and $\mathbf{j}_c$ represents any flux of salt due to processes other than advection: for example, the Soret effect, Fickian diffusion, Taylor dispersion. We shall come to a discussion of these terms in the next section.

(c) Boundary conditions

Those for cold ice have been treated in detail before (Fowler and Larson, 1978; Hutter, 1982); for the present model, we have additionally to specify conditions at the cold-temperate interface, at the base, and at the surface if the average surface temperature is zero anywhere. We consider first the appropriate conditions at a cold-temperate interface.

Across an interface, we would generally expect (a) continuity of mass for each phase, (b) continuity of the stress tensor for each phase, (c) balance of heat transport for each phase (including latent heat terms if there is a jump in phase), (d) a balance of total salt concentration. One simple way of obtaining such jump conditions is from conservation laws (Ishii, 1975) and we see that indeed, bulk conservation of momentum and energy, which apply (in different forms) on both sides of the cold-temperate transition will determine (b) and (c) above, for the bulk mixture; similarly, the conservation of total mass equation determines a balance of mass statement in (a), and the salt conservation law can be used to determine (d). However, it is not immediately apparent that the moisture balance equation (2.8) or the momentum equation (2.25) can be so used, since there are no corresponding balance laws in cold ice: in fact, w is then determined (as zero) from the thermodynamic statement (or assumption) that moisture is absent when $T < T_M$, and not from a conservation statement. Nevertheless, one can try and formulate balance laws from (2.8) and (2.25) by using these equations in cold ice, with the source terms [C and $A(1-w)\mathbf{v}$] set equal to zero. Even this gives problems, unless we realise that these source terms arise from *volumetric* (interfacial) terms in the averaged equations, and that they do not have any effect on the corresponding balance laws. If conservation of water phase is written in the form

$$(\rho w)_t + \operatorname{div}[\rho w(\mathbf{u}+\mathbf{v})] = C, \tag{2.29}$$

the corresponding balance law is

$$[\rho w(\mathbf{u}+\mathbf{v}-\mathbf{V})\cdot\mathbf{n}]_-^+ = \tilde{C}, \tag{2.30}$$

where $\tilde{C}$ is a surface production term, and $\mathbf{V}$ is the velocity of the cold-temperate transition surface. This relation is given by Hutter (1982). The surface term $\tilde{C}$ must be constituted.

If we apply a similar procedure to (2.24) [where the left hand side is a volumetric (interfacial drag) term], then [neglecting viscous stresses, as in (2.25)] the corresponding jump condition would be

$$wp_w = \tilde{p}, \tag{2.30a}$$

where $\tilde{p}$ (a surface term) must be chosen constitutively. We return to a discussion of $\tilde{C}$ and $\tilde{p}$ in Section 4.

Finally, if we neglect τ^w in comparison with p_w (following (2.25)), we can write, using (2.15),

$$W = \operatorname{div}\mathbf{F} + \text{volumetric terms}, \tag{2.31}$$

where

$$\mathbf{F} = \boldsymbol{\sigma}\cdot[\mathbf{u}-(1-w)^{-1}w\mathbf{v}] - (1-w)^{-1}\alpha p_w\mathbf{v}, \tag{2.32}$$

and it seems reasonable to use (2.31) in writing a balance law for the bulk energy equation.

At this point, we summarise the six conservation laws to be solved in conservation form, from which the jump conditions can be read off (Hutter, 1982) as follows: if

$$\phi_t + \operatorname{div}(\mathbf{f}) = \text{volumetric terms}, \tag{2.33}$$

then

$$\mathbf{V}\cdot\mathbf{n}[\phi]_-^+=[\mathbf{f}\cdot\mathbf{n}]_-^+-\tilde{\phi} \tag{2.34}$$

in the notation of (2.30), where $\tilde{\phi}$ is a surface term. The equations are

$$(\rho w)_t+\operatorname{div}[\rho w(\mathbf{u}+\mathbf{v})]=C,$$

$$\rho_t+\operatorname{div}[\rho\mathbf{u}]=0, \operatorname{div}[\boldsymbol{\sigma}]=-\rho\mathbf{g},$$

$$(\rho H)_t+\operatorname{div}[H\mathbf{u}+\mathbf{J}]=W+E=\operatorname{div}\mathbf{F}+\ldots,$$

$$\lambda[\rho(1-w)c]_t+\lambda\operatorname{div}[\rho\{(1-w)\mathbf{u}-w\mathbf{v}\}c]$$

$$+(\rho wc)_t+\operatorname{div}[\mathbf{j}_c+\rho cw(\mathbf{u}+\mathbf{v})]=0,$$

$$\operatorname{div}[w(p_w+\rho_w gy)\delta_{ij}]=-\mathbf{v}/\kappa\ldots; \tag{2.35}$$

here

$$H=c_pT;\quad w=0;\quad \mathbf{J}=-k\nabla T;\quad \mathbf{j}_c=\mathbf{0}, \tag{2.36}$$

all for cold ice ($T<T_M$). For temperate ice ($T=T_M$),

$$T=T_M,\quad H=c_pT_M+Lw,\quad \mathbf{J}=L^*\mathbf{j}-k\nabla T_M,\quad \mathbf{j}=\rho w\mathbf{v}; \tag{2.37}$$

$\mathbf{j}_c$ and E remain to be constituted, and C is to be determined consistently with $(2.35)_4$.

A discussion of the appropriate conditions at the bedrock and at the top surface is postponed until later.

3. CONSTITUTIVE RELATIONS

We have already defined $\mathbf{j}=\rho w\mathbf{v}$ via (2.18). In addition, for temperate ice, we must prescribe $\mathbf{j}_c$, p_w, and the flow law; C is determined so that $(2.35)_1$ reduces to $(2.35)_4$.

The simplest assumption for $\mathbf{j}_c$ is simple Fickian diffusion

$$\mathbf{j}_c^F=-\rho wD\nabla c. \tag{3.1}$$

A typical value of D is about $10^{-5}\,\mathrm{cm}^2\,\mathrm{sec}^{-1} \approx 3.10^{-2}\,\mathrm{m}^2\,\mathrm{y}^{-1}$; to estimate the significance of this when the equations are appropriately scaled (Fowler and Larson, 1978), we compare D to V^*d, where V^* is a typical accumulation rate and d is a typical depth. For $V^* \sim 1\,\mathrm{m\,y}^{-1}$, $d \sim 100\,\mathrm{m}$, $D/V^*d \sim 3.10^{-4}$, and can safely be considered insignificant on glacial time scales.

A more effective spreading mechanism in Poiseuille flow at high Péclét number is that of Taylor dispersion (Taylor, 1953; Aris, 1956). We take the Taylor dispersion flux in the form

$$\mathbf{j}_c^T = -l^2 \rho w \mathbf{Q} : \nabla c / bD, \tag{3.2}$$

where $\mathbf{Q} = \mathrm{diag}(v_1^2, v_2^2, v_3^2)$, appropriate to a porous medium in which the water occurs at grain boundary intersections. Here l is grain diameter, and b is a geometrical parameter of order 50. With $l \sim 2\,\mathrm{cm}$, $D \sim 3.10^{-2}\,\mathrm{m}^2\,\mathrm{y}^{-1}$, $b = 50$, $w = 1\%$ (i.e. $= 10^{-2}$), $|\mathbf{v}| = 10^5\,\mathrm{m\,y}^{-1}$ (corresponding to the NyeFrank drainage flux), (3.2) defines a Taylor dispersion matrix $\mathbf{D}_T$ (i.e. $\mathbf{j}_c^T = -\rho \mathbf{D}_T : \nabla c$) of order $2.10^4\,\mathrm{m}^2\,\mathrm{y}^{-1}$, which is evidently quite substantial.

We assume that the interfacial energy source E is due to viscous dissipation in the (locally) Poiseuille flow in the pores. From Batchelor (1967) [or Nye, (1976)] the viscous dissipation per unit length is QG, where Q is the volume flux, and G is the magnitude of the pressure gradient. Translating this to the present geometry, we define the (dissipative) interfacial energy source

$$E = w|\mathbf{v}|\,|\nabla(p_w + \rho_w g y)| = |\mathbf{v}|^2/\kappa. \tag{3.3}$$

Now we turn to the flow law. The bulk (average) stress σ_{ij} is determined from $(2.35)_3$; we have to relate this to the average strain rate $e_{ij} = u_{i,j} + u_{j,i}$. We define the bulk pressure p, and bulk stress deviator τ_{ij}, by

$$\sigma_{kk} = -3p, \quad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}. \tag{3.4}$$

We neglect $\boldsymbol{\tau}^w$ in comparison to p_w, thus

$$p = \alpha p_w + (1-\alpha) p_I, \quad \boldsymbol{\tau} = (1-\alpha)\boldsymbol{\tau}^I. \tag{3.5}$$

The simplest flow law one could have is

$$\boldsymbol{\tau} = \eta \mathbf{e}, \tag{3.6}$$

where $\eta = \eta(w, \tau)$ to take account of the dependence of viscosity on moisture and stress (Hutter, 1982; Lliboutry, 1976). The only experimental results (Hooke, 1981) are those of Duval (1977), who showed that $1/\eta$ increased roughly linearly with w at constant τ.

More generally, one could propose the constitutive form

$$\boldsymbol{\tau} = \eta \mathbf{e} + \lambda [\operatorname{div} \mathbf{u}] \boldsymbol{\delta}, \tag{3.7}$$

where Stokes' law would give $\lambda = -2\eta/3$. Apart from a lack of experimental indication of a need for such a law, if $\operatorname{div} \mathbf{u} \approx 0$ (as will be the case for small w, see Section 4), this reduces back to (3.6). Ishii (1975) proposes, somewhat more formally,

$$\boldsymbol{\tau} = (1-\alpha)\tau^I = \eta_I [\mathbf{e} + m\{(\nabla w)\mathbf{v} + \mathbf{v}(\nabla w)\}], \tag{3.8}$$

(where $m \approx 1$) for small w (Drew, 1983) and $\eta_I(\tau)$ is the viscosity of ice. It would seem difficult to provide any evidence for (3.8).

We also need a constitutive statement relating p_w and p. This is often taken as $p = p_w$ (Drew, 1983) in two-phase flows, despite the fact that this leads to certain ill-posedness problems in the formulation (Drew, 1983; Stuhmiller, 1977; Klebanov *et al.*, 1982). The more realistic $p_w - p = \sigma\bar{\kappa}$, where σ is surface tension, and $\bar{\kappa}$ is essentially the average mean curvature of the water channels, does not offset the problem, and in fact appropriate values $\bar{\kappa} \sim 1/lw^{1/2}$, $w \sim 10^{-2}$, $l \sim 2\,\text{cm}$, $\sigma \sim 3 \cdot 4.10^{-2}\,\text{J}\,\text{m}^{-2}$, give $\sigma\bar{\kappa} \sim 10^{-4}$ bar, which is insignificant. Of course $\bar{\kappa} \to \infty$ as $w \to 0$, but only for values for which the assumed connected pore geometry becomes disconnected.

With $p = p_w$ and p approximately hydrostatic (through the ice, $\nabla p + \rho \mathbf{g} \approx \mathbf{0}$), (2.18) yields a net drainage flux per unit area which for $w \sim 1\%$ [a typical value, (Lliboutry, 1976)], Nye and Frank (1973) would calculate as $90\,\text{m}\,\text{y}^{-1}$. This excessive value led Lliboutry (1971, 1976) to query Nye and Frank's assumed geometry. Frank (1968) and later Turcotte and Ahern (1978) and Turcotte (1981) employed the same constitutive relation ($p_w = p$) on the basis that the solid matrix could collapse by creep processes on the appropriate (geological) time scales (Ahern and Turcotte, 1979). This notion relates to an old result of Nye (1953), that a borehole (tunnel, vein) in ice of radius a has a relative closure rate given by

$$a^{-1}\,da/dt = -A[|p-p_w|/n]^{n-1}[(p-p_w)/n], \tag{3.9}$$

where

$$e = A\tau^n \tag{3.10}$$

is Glen's flow law (Paterson, 1981) (notice A here is different to that in Section 2); in our notation, the viscosity is given by

$$1/\eta = A\tau^{n-1}. \tag{3.11}$$

Implicit in Ahern and Turcotte's analysis is the assumption that vein closure can occur on a time scale short compared to the long term (solid) viscous flow time scale. However, (3.9) states that the strain rate of closure is comparable to the bulk flow strain rate, at least when $p-p_w$ is comparable to the bulk shear stress: thus, the time scale of closure is the same as that of the bulk flow: consequently, *this effect must be included in the large scale flow dynamics.*

This point of view was partially adopted by Shreve (1972), who considered (3.9) as a constitutive law for p_w. One can do this in terms of macroscopic variables by following Nye (1976). We define S^* to be the (average) cross sectional pore area per unit area. Assuming that surface averaging is equivalent to volume averaging (Nigmatulin, 1979) yields

$$S^* = \alpha. \tag{3.12}$$

Now consider an averaging surface S through the two phase region. This will intersect numerous pores at various angles χ (to their axes) in elliptical cross-sections. We suppose $\chi \neq 0$, that is no (or very few) pores lie along S, as is reasonable. We suppose also that a typical pore radius a is much less than grain diameter l, so that pores are "very far apart", and consequently we suppose that each pore individually contracts according to (3.9). Now notice that if $\Delta = \pi a^2$ is the cross-sectional area of a pore, $a^{-1}\,da/dt = (1/2\Delta)\,d\Delta/dt$. Further, for a pore intersecting S at angle χ, the intersection of S

with the pore has an area $\Delta^* = \Delta \sec \chi$; consequently $a^{-1}\, da/dt = (1/2\Delta^*)\, d\Delta^*/dt$. Now we simply add all the contributions to closure in S; the result is

$$dS^*/dt_I = -2A[|p-p_w|/n]^{n-1}[(p-p_w)/n]S^*, \qquad (3.12a)$$

where d/dt_I is the material time derivative following the ice. To obtain a constitutive law for p_w, we simply use Nye's (1976) condition on "geometry and flow of ice" by appending to (3.12a) the rate of creation of pore space by melting: this is C/ρ_I; this condition for single channel flow was rigorously derived by Spring and Hutter (1982). With (3.12), we have

$$d\alpha/dt_I = (C/\rho_I) - (p-p_w)\alpha/\eta_b, \qquad (3.13)$$

where

$$1/\eta_b = 2A|p-p_w|^{n-1}/n^n; \qquad (3.13a)$$

η_b is a viscosity computed from (3.10) using $p-p_w$ as the stress, and multiplied by a factor $\frac{1}{2}n^n \approx 13$ for $n=3$. By multiplying by ρ_w, and using (2.4), (2.5), $d/dt_I = \partial/\partial t + \mathbf{u}^I \cdot \nabla$, and (2.6), (3.13) can be written more compactly in the form

$$\partial(\rho w)/\partial t + [\mathbf{u} - (1-w)^{-1} w\mathbf{v}] \cdot \nabla(\rho w) = (\rho_w C/\rho_I) - \rho w(p-p_w)/\eta_b. \qquad (3.14)$$

In what follows, we will concentrate on (3.14) and its mathematical consequences. However, since the microscopic pore contraction is a consequence of *viscous* behaviour, it seems reasonable that this phenomenon might be included in the flow law rather than as a constitutive relation between p and p_w. That is, whereas (3.14) really determines an extra relation for dw/dt in terms of $p-p_w$, one would try and incorporate the microscopic (3.9) into a macroscopic flow law. It is not very obvious how to do this, and we will not try. Further study of this problem is of interest in view of Duval's (1977) experimental results, which show an approximately linear increase of e_{ij} with w at constant τ_{ij}. Duval did not discuss any possible effect of pore pressure on his experiment. In the analogous case of crystalline rock viscosity in the presence of partial melt, Shankland *et al.* (1981) suggest the effect of w on the viscosity will be very small. We shall suppose the flow law is given by (3.6), and note that in spite of Duval's (1977) result, the cold viscosity (3.11) may still be relevant. For concreteness, we shall take the constitutive relation for p_w as (3.14); the case $\eta_b \to 0$ then corresponds to $p_w = p$. Note that (3.14) represents an equation expressing conservation of pore *space*, or equivalently, conservation of interfacial surface area.

4. APPROXIMATE ANALYSIS: $w \ll 1$

Our purpose in this section will be to give some preliminary idea as to the nature of the equations presented in Section 3, and in particular to study the structure of the equations in the temperate zone. To facilitate this, we first suppose $w \ll 1$, as is generally held to be the case (e.g. $w \sim 1\%$). Then, since

$$\rho = \rho_I (1 - \delta w)^{-1}, \quad \delta = (\rho_w - \rho_I)/\rho_w \sim 0.1, \tag{4.1}$$

ρ is sensibly constant for most purposes, and continuity and momentum equations give, approximately,

$$\operatorname{div} \mathbf{u} = 0, \quad \operatorname{div} \boldsymbol{\sigma} = -\rho \mathbf{g}, \qquad (4.2), (4.3)$$

which are to be solved together with the constitutive law (3.4) and (3.6)

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta} + \boldsymbol{\tau}, \quad \boldsymbol{\tau} = \eta \mathbf{e}. \tag{4.4}$$

If we assume η is (effectively) *independent* of w, then the flow problem essentially uncouples from that of determining w. Particularly, in view of (2.34) and $(2.35)_{1,2}$, $\sigma_{ij} n_j$ is exactly continuous, and $u_i n_i$ is approximately continuous across a cold-temperate interface. The point of this is that the determination of w can now be carried out, on the basis that the flow field $\mathbf{u}$ has been calculated. It is our purpose to study the structure of these moisture field equations. From $(2.35)_{1,4}$, using (2.15), (2.17), and (3.3), and neglecting terms of relative order w, we have

$$(\rho \dot{w})_t + \operatorname{div}[\rho w(\mathbf{u}+\mathbf{v})] = C, \tag{4.5}$$

$$\begin{aligned} L\{(\rho w)_t + \operatorname{div}[\rho w(\mathbf{u}+\mathbf{v})]\} \\ = \boldsymbol{\sigma}{:}\,\nabla\{\mathbf{u} - w\mathbf{v}] - \rho_w^{-1} p_w \operatorname{div}\mathbf{j} + \kappa^{-1}|\mathbf{v}|^2 - \operatorname{div}[c_p(T_M^w - T_M)\mathbf{j}] \\ + k\nabla^2 T_M - \rho c_p\, dT_M/dt \\ \equiv \rho L S \text{ (defining } S). \end{aligned} \tag{4.6}$$

A typical distribution coefficient (Hallet, 1977) is $\lambda \sim 0.02$. If we assume $w \sim \lambda \ll 1$, then (2.28) can be written, with (3.2), and the approximations already introduced in this section, as

$$(d/dt)[(\lambda + w)c] + \operatorname{div}[c\dot{w}\mathbf{v}] = \nabla \cdot [\mathbf{D}_T{:}\nabla c]. \tag{4.7}$$

Lliboutry's (1976) assumption $cw = n$, a constant, appears in the limit $\lambda, |\mathbf{v}|, D_T \to 0$, but not generally otherwise.

The final two equations are Darcy's law (2.18),

$$\mathbf{v} = -\kappa w \nabla(p_w + \rho_w g y), \tag{4.8}$$

and the constitutive law (3.14), which can be written, using (4.5) and neglecting terms of relative $O(w)$, as

$$-\rho w \operatorname{div}\mathbf{u} - \operatorname{div}[\rho w \mathbf{v}] = (\rho_I^{-1}\rho_w - 1)C - \eta_b^{-1}\rho w(p - p_w). \tag{4.9}$$

It is reasonable to assume that p is hydrostatic (through the ice, i.e. cryostatic), that is

$$\nabla[p + \rho_I g y] \approx \mathbf{0} \tag{4.10}$$

(since $\rho \approx \rho_I$), where y is vertical. Further, it is apparent that (4.5) and (4.6) are essentially identical in form, and thus determine C: however, we retain (4.5) for the purpose of extracting a balance law (2.30). If $y = h_s$ is the top surface of the glacier, then

$$p = \rho_I g(h_s - y). \tag{4.11}$$

Define

$$\phi = -\kappa(p_w + \rho_w g y); \tag{4.12}$$

then

$$\mathbf{v} = w\nabla\phi, \tag{4.13}$$

and (4.9) can be written (with $\rho \approx \rho_I$),

$$w \operatorname{div} \mathbf{u} + \operatorname{div}[w^2 \nabla\phi] + (\rho_I^{-1}\rho_w - 1)S - \eta_b^{-1} w(p - p_w) = 0, \tag{4.14}$$

where S is defined in (4.6). The moisture equation (4.6) can be written [using the "Boussinesq" approximation (4.2)]

$$w_t + \operatorname{div}[w(\mathbf{u} + \mathbf{v})] = S. \tag{4.15}$$

With $w \sim 10^{-2}$, $\lambda \sim 10^{-2}$, and $c \sim n/\lambda$, $n \sim 10^{-5}\,\mathrm{mol\,kg^{-1}}$, $A' \sim 2\,\mathrm{K\,mol^{-1}\,kg}$, $\beta \sim 10^{-2}\,\mathrm{K\,bar^{-1}}$ (Lliboutry, 1976), then ($p \sim 10$ bar, assume $|p - p_w| \lesssim 10$ bar)

$$\beta p \sim 10^{-1}\,\mathrm{K},\ \beta|p - p_w| \sim 10^{-1}\,\mathrm{K},\ A'c \sim 10^{-3}\,\mathrm{K}, \tag{4.16}$$

and consequently (2.12) suggests

$$T_M^w - T_M \approx \beta(p - p_w),\quad T_M \approx T_0 - \beta p, \tag{4.17}$$

whence the terms in T_M^w and T_M in (4.6) can be written in terms of w and ϕ.

Let us now discuss the question of boundary conditions, with the present approximations. The equation for w, (4.15), looks hyperbolic; consequently we should expect a single boundary condition for w at those parts of the temperate boundary where $\mathbf{u} + \mathbf{v} - \mathbf{V}$ points into the temperate zone (where $\mathbf{V}$ is the temperate boundary velocity). Equation (4.14) for ϕ looks elliptic, and we should expect boundary conditions for ϕ over all the temperate boundary, except possibly on those parts where $w = 0$, since the equation is degenerate there (and consequently boundary data may not be required) (Radkevič, 1967).

We first consider jump conditions at the cold-temperate interface, ∂V_C. Having (implicitly) solved the flow problem, the conditions which remain are the moisture flux condition (2.30), which in the present scheme of approximation can be written

$$\mathbf{V}\cdot\mathbf{n}[w]_-^+=[w(\mathbf{u}+\mathbf{v})\cdot\mathbf{n}]_-^+-\tilde{S}, \tag{4.18}$$

where $\mathbf{V}$ is the velocity of the interface, $\mathbf{n}$ points from the $-$ side to the $+$ side, the energy balance condition, from (2.35),

$$\mathbf{V}\cdot\mathbf{n}[\rho H]_-^+=[(\rho H\mathbf{u}+\mathbf{J}-\mathbf{F})\cdot\mathbf{n}]_-^+, \tag{4.19}$$

and the pressure jump condition (2.30a),

$$w^+p_w=\tilde{p}. \tag{4.20}$$

(We treat the solutal jump condition later.) Let us for convenience denote by $+$ *the temperate side and by* $-$ *the cold side*. We assume $[T]_-^+=0$ (bulk temperature is continuous): then, using the constitutive definitions (2.32), (2.36) and (2.37), and the approximations introduced in this section, we find $\mathbf{F}\cdot\mathbf{n}$ is effectively continuous, and (4.19) is then

$$(\mathbf{V}\cdot\mathbf{n})\rho Lw^+=[\rho Lw^+(\mathbf{u}+\mathbf{v})-k\nabla T_M^+ +k\nabla T^-]\cdot\mathbf{n}. \tag{4.21}$$

Since $w^-=0$, (4.18) is just

$$w^+[\mathbf{u}+\mathbf{v}-\mathbf{V}]\cdot\mathbf{n}=\tilde{S}, \tag{4.22}$$

whereas (4.21) is (neglecting $\partial T_M^+/\partial n$, as will be justified in Section 5)

$$\rho Lw^+[\mathbf{u}+\mathbf{v}-\mathbf{V}]\cdot\mathbf{n}+k\,\partial T^-/\partial n=0. \tag{4.23}$$

If $\tilde{S}=0$, then the condition (4.22) represents the classical jump condition across a kinematic shock (Kynch, 1952).

The question now arises, how best to constitute the surface terms $\tilde{S}$ and $\tilde{p}$. [Notice that we assume that no surface terms occur in *bulk* balance laws, e.g. (4.19).] Firstly, observe that (since $T\leqq T_M$ in cold ice) $\partial T^-/\partial n\geqq 0$, thus if $(\mathbf{u}+\mathbf{v}-\mathbf{V})\cdot\mathbf{n}>0$, then (4.23) implies

$$\partial T^-/\partial n=0,\ \ \tilde{S}=\tilde{p}=0,\ \ w=0. \tag{4.24}$$

The first of these, together with $T^- = T_M$, determines the unknown cold-temperate boundary $y = y_M$ (by solving the cold ice temperature problem). No other condition arises, or is necessary.

However, if $(\mathbf{u}+\mathbf{v}-\mathbf{V}) \cdot \mathbf{n} < 0$, then (4.23) may still be thought of as determining y_M, but a condition on p_w, i.e. on ϕ, is required at y_M. Candidates are (4.20) and (4.22), and a prescription of either $\tilde{p}$ or $\tilde{S}$ is necessary. We will take the following point of view: (4.22) serves to define $\tilde{S}$ [from (4.23)] in exactly the same way that C is determined by the energy equation. Then (4.22) is redundant, and we choose $\tilde{p}$ in (4.20). The only sensible choice is

$$w^+ p_w = \tilde{p} = w^+ p_I, \tag{4.25}$$

that is ice and water pressures are locally equal at the interface. In terms of ϕ, this can be written as

$$\phi = -\kappa[\rho_I g(h_s - y) + \rho_w g y], \quad w \neq 0. \tag{4.26}$$

It is completely unclear why one should be able to prescribe continuity of stress for both phases at an interface, and one can argue strenuously that a reasonable procedure is to choose $\tilde{S} = 0$ in (4.22), and neglect (4.20). None of the various arguments that one can think of seem to be very sound, and adoption of (4.25) was finally made on the basis of a related investigation, which amounts to this. The conductive term $k\nabla^2 T_M$ is strictly $k\nabla \cdot [\alpha \nabla T^w + (1-\alpha)\nabla T^I]$ (assuming equal ice and water conductivities). Use of (2.12) for each phase (and a hydrostatic bulk pressure) implies that this conductive term is proportional to $\nabla \cdot [(p_w - p_I)\nabla w]$, and such a (small) term should probably appear in (4.15). Strictly, this renders (4.15) elliptic, requiring a boundary condition for w, *except where* $p_w = p_I$ (Radkevič, 1967). It is easy to see that (4.25) satisfies this requirement naturally, whereas (4.22) cannot. This is the empirical reason for choosing (4.25), but a logical explanation is not available. (Since the coefficient of this conductive term is small, the w equation is effectively hyperbolic anyway, and the previous discussion is still appropriate.)

The boundary condition on (4.7) is obtained by writing it in the proper conservation form (2.33), and applying (2.34). Using (4.22), assuming zero flux in cold ice, and that the ice concentration $\lambda c^- = c_I$, we find c satisfies

$$[\lambda c - c_I](\mathbf{u} - \mathbf{V}) \cdot \mathbf{n} = j_T^n - c\tilde{S}, \tag{4.27}$$

where $j_T^n = [\mathbf{D}_T : \nabla c]_n$.

On parts of ∂V_C where $w = 0$, i.e. $(\mathbf{u} + \mathbf{v} - \mathbf{V}) \cdot \mathbf{n} > 0$, we have

$$j_T^n = \tilde{S} = 0, \tag{4.28}$$

since

$$\mathbf{D}_T = w\mathbf{Q}(l^2/bD), \tag{4.29}$$

and so (4.27) gives

$$c = c_I/\lambda, \tag{4.30}$$

where c_I is the (known) salt composition from the firn composition (Lliboutry's n). When $w \neq 0$ on ∂V_C, then we *assume* that the cold ice concentration is determined by the solidus-liquidus distribution, that is

$$c_I = \lambda c^+; \tag{4.31}$$

then (4.27) implies

$$D_T \partial c/\partial n = c\tilde{S}, \quad w \neq 0. \tag{4.32}$$

In summary, the boundary conditions for c can be written as

$$c = c_0, \quad w = 0, \quad j_T^n = c\tilde{S}, \quad w \neq 0. \tag{4.33}$$

Typically, $c_0 \sim 10^{-3}\,\text{mol}\,\text{kg}^{-1}$ (Lliboutry, 1976). Using (4.22) and (4.23), we have $|c\tilde{S}/j_T^n| \sim (\kappa_T/D_T)(c_p\Delta T/L)$, where κ_T is the thermal diffusivity, and ΔT a temperature scale. With $D_T \sim 2.10^4\,\text{m}^2\,\text{y}^{-1}$ [following (3.2)], $\kappa_T \sim 38\,\text{m}^2\,\text{y}^{-1}$ (Fowler and Larson, 1978), and $c_p\Delta T/L \sim 0.2$ [with $\Delta T = 20\,\text{K}$, and c_p/L from (5.1)], this is $|c\tilde{S}/j_T^n| \lesssim 10^{-3}$. In this case, (4.33) is effectively approximated by

$$c = c_0, \quad w = 0, \quad \partial c/\partial n = 0, \quad w \neq 0. \tag{4.34}$$

Boundary conditions at the bedrock really require a discussion of basal sliding in the presence of moisture. We can ask, what the analogues of (4.22) and (4.23) for the *large-scale* flow might be. Conservation of moisture at the bedrock, in which there is subglacial water flow/drainage through films/channels, would suggest an (*average*) moisture flux at the base given by

$$w^{+}[\mathbf{u}+\mathbf{v}]\cdot\mathbf{n}=-\Gamma \quad \text{on} \quad \partial V_B, \tag{4.35}$$

where $\mathbf{V}\cdot\mathbf{n}=0$, and $\Gamma>0$ represents positive net drainage.

On the other hand, an energy balance such as (4.23) is probably not directly relevant, since the incoming energy fluxes due to moisture and geothermal heat both contribute to the determination of the subglacial regelation film thickness, and hence the basal water pressure. Considerations of the sort given by Fowler (1981) suggest that the sliding theory for wet ice will serve to determine (as well as the sliding law itself) the actual basal water pressure p_b and hence Γ in (4.35) as a functional of p_w and w. Here we distinguish between the water pressure at the bed, p_b, and the *effective* water pressure p_w "felt" by the large scale flow (Fowler, 1981) (*not* Terzaghi's effective pressure). We therefore tentatively suggest that sliding theory will provide a single boundary condition for the large scale flow of the form (4.35), in which $\Gamma=\Gamma(\phi, w, \mathbf{x})$ is determined in a manner analogous to the sliding law $\tau_b=f(u_b)$. This one condition at the base is sufficient to help determine p_w and w in the temperate zone, for the same reasons as discussed before.

Finally, we mention the appropriate pair of energy and moisture conditions at the upper free surface of a temperate glacier. These are given by Hutter (1982) and in our notation are

$$\rho w(\mathbf{u}+\mathbf{v}-\mathbf{V})\cdot\mathbf{n}=M, \tag{4.36}$$

$$\rho L w(\mathbf{u}+\mathbf{v}-\mathbf{V})\cdot\mathbf{n}=E_A \quad \text{on} \quad \partial V_A, \tag{4.37}$$

where M is surface moisture flux and E_A the net atmospheric energy budget transferred to the ice. Evidently E_A and M are related in such a way that (4.36) and (4.37) are the same; we suppose

$$M=\rho w v_A, \tag{4.38}$$

if $\mathbf{n}$ is the unit *inward* normal (pointing towards the ice), and $v_A > 0$ is the net *percolating* velocity of water (as opposed to surface run-off). To relate (4.36) and (4.38) to the kinematic condition, observe that if $F(\mathbf{x}, t) = 0$ is the top surface, then

$$F_t + \mathbf{V} \cdot \nabla F = F_t + \mathbf{V} \cdot \mathbf{n}|\nabla F| = 0 \tag{4.39}$$

by definition of $\mathbf{V}$. Also

$$F_t + \mathbf{u} \cdot \nabla F = F_t + \mathbf{u} \cdot \mathbf{n}|\nabla F| = dF/dt, \tag{4.40}$$

consequently

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|\nabla F| = dF/dt, \tag{4.41}$$

and thus (4.36) and (4.38) can be written

$$dF/dt = [v_A - \mathbf{v} \cdot \mathbf{n}]|\nabla F|. \tag{4.42}$$

Additionally, mass balance of the mixture (or approximately, of the ice phase) yields

$$\rho(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n} = -\rho u_A, \tag{4.43}$$

where the source term $\rho u_A > 0$ represents the part of the ice surface which is melted by the incoming atmospheric energy budget. One cannot easily relate u_A to v_A, as run-off, rainfall, etc. complicate matters. Equation (4.43) can be written

$$dF/dt = -u_A|\nabla F| = +a(\mathbf{x}, t), \tag{4.44}$$

and is the kinematic condition at the free surface ($-a$ is ablation, or a is accumulation rate): then (4.42) gives the moisture flux condition

$$w\mathbf{v} \cdot \mathbf{n} = w(u_A + v_A) \equiv f_A \tag{4.45}$$

(remember, $\mathbf{n}$ points *towards* temperate ice).

Finally, the stress jump condition for the water phase at the surface is essentially

$$p_w = p_A, \tag{4.46}$$

where p_A is atmospheric pressure. Consequently, if this surface is given by $y = h_s$ ($F \equiv h_s - y$), (4.46) is

$$\phi = -\kappa \rho_w g h_s \quad \text{on} \quad S, \tag{4.47}$$

where we subtract the ambient p_A. If f_A is considered known, then (4.45) determines the seepage velocity v_A. The condition (4.46) tacitly assumes that the *water table* (Shreve and Sharp, 1970; Nye and Frank, 1973) extends to the surface of the glacier, whereas this is not necessarily the case. If we suppose the water table is at $y = h_w \leqq h_s$, then (4.46) should be amended to read

$$p_w = p_A \quad \text{at} \quad y = h_w, \tag{4.48}$$

and (4.47) is then

$$\phi = -\kappa \rho_w g h_w \quad \text{at} \quad y = h_w; \tag{4.49}$$

h_w is not known *a priori*, but can be determined by assuming that in this case, the surface percolation (4.45) is a given quantity: the extra condition then fixes the free boundary h_w. If h_w (as thus calculated) *is* less than h_s, it makes physical sense to take f_A as given, whereas if $h_w = h_s$, then the glacier only admits "as much as it can", and the rest runs off. Extra physical constraints are that $0 < p_w < p$, i.e. that

$$\phi + \kappa \rho_w g y < 0, \quad \text{and} \quad \phi + \kappa \rho_I g (h_s - y) + \kappa \rho_w g y > 0; \tag{4.50}$$

the first of these is always required: if the second is violated, particularly at the base, then glacier flotation becomes possible, and the assumptions at the bedrock need reconsidering: this may well occur during glacier surges.

To complete the boundary conditions, we need analogues of (4.45) and (4.35) for salt concentration c at ∂V_A and ∂V_B. The flux of salt at an interface is, from (4.7),

$$c(\lambda + w)(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n} + c w \mathbf{v} \cdot \mathbf{n} - j_T^n = J_c. \tag{4.51}$$

If we assume that drainage concentration c is equal to that of the pores, then (with $\mathbf{V}\cdot\mathbf{n}=0$), $J_c=-c\Gamma$ at ∂V_B, whence (4.35) gives

$$j_T^n+\lambda c\mathbf{u}\cdot\mathbf{n}=0 \quad \text{on} \quad \partial V_B. \tag{4.52}$$

A condition analogous to (4.45) on ∂V_A is

$$c(\lambda+w)(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}+cw\mathbf{v}\cdot\mathbf{n}-j_T^n=c_A f_A, \tag{4.53}$$

where the effective surface concentration is not necessarily the same as c_I in (4.30).

5. APPROXIMATE ANALYSIS

There is now some purpose to evaluating the expected orders of magnitude of the various terms in (4.14) and (4.15). The terms $-\boldsymbol{\sigma}$: $\nabla(w\mathbf{v})-(p_w/\rho_w)\operatorname{div}\dot{\mathbf{j}}$ in (4.6) are of order $O(\tau, p-\rho_I p_w/\rho_w)\operatorname{div}[w\mathbf{v}]$. Since $\rho L\sim 3000$ bars (Lliboutry, 1976) $p\leqq 10$ bars (typically) it is plausible to neglect these terms in ρLS in comparison to $\rho L\operatorname{div}(w\mathbf{v})$, both in (4.6) and in (4.14). To estimate the remaining terms, we use the following approximate scales (Fowler and Larson, 1978; Lliboutry, 1976):

$$w\sim 10^{-2}, \quad |\nabla|\sim d^{-1}\sim 10^{-2}\,\text{m}^{-1},$$

$\eta_b\sim 1$ bar y (approximate viscosity of ice), $\quad t\sim 10^2$ y,

$$p\sim 10 \text{ bars}, \quad |\mathbf{u}|\sim 10^2\,\text{m}\,\text{y}^{-1}, \quad \tau\sim 1 \text{ bar},$$

$$\rho_w, \rho_I\sim 10^3, 0\cdot 9.10^3\,\text{kg}\,\text{m}^{-3},$$

$$g\sim 10\,\text{m}\,\text{s}^{-2}, \quad \rho c_p\sim 30\,\text{bars}, \quad \rho L\sim 3.10^3\,\text{bars},$$

$$h_s, y\sim 10^2\,\text{m}, \quad \kappa\sim 10^9\,\text{m}^2\,\text{bar}^{-1}\,\text{y}^{-1} \text{ [from (2.19)]}. \tag{5.1}$$

It should be emphasised that the scale of w should be *deduced* from the equations: the estimate of $\sim 10^{-2}$ follows observation, and should (we hope) be consistent. Then, using (4.1),

$$|w\operatorname{div}\mathbf{u}|\approx|\delta w(dw/dt)|\sim 10^{-7}\,\text{y}^{-1}, \tag{5.2}$$

$$[\boldsymbol{\sigma}:\nabla\mathbf{u}]/\rho L \sim 3|w \operatorname{div}\mathbf{u}| + 3.10^{-4}\,\mathrm{y}^{-1} \sim 3.10^{-4}\,\mathrm{y}^{-1}; \tag{5.3}$$

if we suppose

$$\phi \sim [\phi]\, m^2 y^{-1} \quad \text{(to be chosen)}, \tag{5.4}$$

and that

$$p - p_w \sim 1\,\text{bar} \quad (\text{if } p \sim p_I), \tag{5.5}$$

then

$$|\mathbf{v}|^2/\rho_I \kappa L \sim 3.10^{-21}[\phi]^2\,\mathrm{y}^{-1}, \tag{5.6}$$

$$w(p-p_w)/\eta_b \sim 10^{-2}\,\mathrm{y}^{-1}, \tag{5.7}$$

$$|\operatorname{div}(w\mathbf{v})| \sim 10^{-8}[\phi]\, y^{-1}. \tag{5.8}$$

From (4.17),

$$|T_M - T_M^w| \sim 10^{-2}\,\mathrm{K}, \quad dT_M \sim 10^{-1}\,\mathrm{K}; \tag{5.9}$$

then

$$|\operatorname{div}[c_p(T_M^w - T_M)\mathbf{j}]/\rho L| \sim 10^{-4} \operatorname{div}(w\mathbf{v}), \tag{5.10}$$

$$|k\nabla^2 T_M/\rho L| \sim 10^{-5}\,\mathrm{y}^{-1}, \tag{5.11}$$

$$|\rho c_p(dT_M/dt)/\rho L| \sim 10^{-5}\,\mathrm{y}^{-1}. \tag{5.12}$$

Accepting the implications of these scaling, then of the various terms in the definition of S in (4.6), (5.10) is negligible in comparison to div$(w\mathbf{v})$ [on the left hand side of (4.6)], (5.11) and (5.12) are negligible by comparison to (5.3). This implies that only the interfacial energy source term E (3.3), and the bulk viscous heating term (5.3), are of significance in S. Thus

$$S \approx S_W + S_E, \tag{5.13}$$

where S_W is given by (5.3), S_E by (5.6) [or see (5.19) below]. It is plausible to choose $[\phi]$ to balance the net downward pressure gradient in $(p-p_w)$ in (4.9): thus, from (4.12) and (5.1), we choose

$$[\phi]\sim\kappa(\rho_w-\rho_I)gy\sim 10^9\,\mathrm{m}^2\,\mathrm{y}^{-1}. \tag{5.14}$$

With this choice of $[\phi]$, and with $w\sim 10^{-2}$, we find

$$|\mathbf{v}|\sim 10^5\,\mathrm{m\,y}^{-1}, \tag{5.15}$$

hence

$$S_E\sim 3.10^{-3}\,y^{-1}, \tag{5.16}$$

and is thus (perhaps) an order of magnitude greater than bulk viscous heating. This is consistent with the balance suggested by Nye (1976) for the intergranular vein system. Examination of (4.14) now yields

$$\begin{aligned} &w\,\mathrm{div}\,\mathbf{u}\sim 10^{-7}\,\mathrm{y}^{-1}, \quad [\text{from } (5.2)],\\ &\mathrm{div}\,[w\mathbf{v}]\sim 10\,\mathrm{y}^{-1}, \quad [\text{from } (5.8)],\\ &[\rho_w/\rho_I-1]S\sim 10^{-4}\,\mathrm{y}^{-1},\\ &w(p-p_w)/\eta_b\sim 10^{-2}\,y^{-1}, \quad [\text{from } (5.7)], \end{aligned} \tag{5.17}$$

which suggests that only the term $\mathrm{div}(w\mathbf{v})$ is of any significance (with these scales). Furthermore, $|\mathbf{u}\cdot\nabla|/|\mathbf{v}\cdot\nabla|\lesssim 10^{-3}$, so that $dw/dt\approx\partial w/\partial t$ on time scales of interest here. We now gather the equations for w and ϕ, retaining terms in (5.17) except $w\,\mathrm{div}\,\mathbf{u}$ for a moment.

$$\begin{aligned} &w_t+\mathrm{div}\,(w\mathbf{v})=S_E+S_W=S,\\ &\mathrm{div}\,(w\mathbf{v})=-[(\rho_w-\rho_I)/\rho_I]S+w(p-p_w)/\eta_b, \end{aligned} \tag{5.18}$$

where

$$S_W=[\boldsymbol{\sigma}:\nabla\mathbf{u}]/\rho_I L, \quad S_E=|\mathbf{v}|^2/\rho_I\kappa L, \quad \mathbf{v}=w\nabla\phi, \tag{5.19}$$

and where [in terms of axes (x',y') inclined at an angle θ downwards from horizontal] we can write

$$p-p_w=\rho_I g(\bar{h}_s-y')\cos\theta+\rho_w g\{y'\cos\theta-x'\sin\theta\}+\phi/\kappa, \tag{5.20}$$

where $y' = \bar{h}_s(x')$ is the top surface $(y = h_s)$: (x', y') are natural coordinates for a valley glacier whose mean bedrock slope is $\tan\theta$. We assume $\mathbf{u}$ is (in principle) known. The temperate domain V_T is determined [in view of the extra boundary condition (4.23) on T], and we suppose its boundary may consist of three parts: ∂V_C (cold-temperate), at which

$$w(p - p_w) = 0; \tag{5.21}$$

∂V_B (bedrock), at which (with $\mathbf{u} \cdot \mathbf{n} = 0$)

$$w^2\, \partial\phi/\partial n + \Gamma(\phi, w, x) = 0; \tag{5.22}$$

and piezometric free surface ∂V_P, at which pressure and flux are given:

$$\phi = -\kappa\rho_w g\{\bar{h}_w \cos\theta - x' \sin\theta\}, \quad w^2\, \partial\phi/\partial n = f_A. \tag{5.23}$$

If the calculated $\bar{h}_w > \bar{h}_s$, we take $\bar{h}_w = \bar{h}_s$, and omit the second condition in (5.23).

We now choose explicit scales for w and ϕ based on the rough balance of terms in (5.18) indicated earlier. We first rewrite (5.18) as

$$\operatorname{div}(w\mathbf{v}) = -[(\rho_w - \rho_I)/\rho_I]S + w(p - p_w)/\eta_b,$$

$$w_t = (\rho_w/\rho_I)S - w(p - p_w)/\eta_b. \tag{5.24}$$

From (5.20), as already suggested, we define

$$[\phi] = \kappa(\rho_w - \rho_I)gd\cos\theta \sim 10^9\ \mathrm{m^2\,y^{-1}}; \tag{5.25}$$

then w is scaled to balance the two terms on the right hand side of $(5.24)_2$. Thus with $S \sim S_E \sim w^2[\phi]^2/\rho_I\kappa L d^2$, this suggests $w \sim [w]$, where

$$[w] = \rho_I^2 L d^2/\rho_w \eta_b[\phi]$$

$$= \rho_I^2 L d/\rho_w(\rho_w - \rho_I)\eta_b \kappa g \cos\theta. \tag{5.26}$$

Roughly, $[w] \sim 10\, Ld/\eta_b \kappa g$. With the values given in (5.1), we compute $[w] \sim 3.10^{-2}$, i.e. 3%. Obviously, this result is only an order of magnitude, but it is heartening that it seems to correspond to observation. It is perhaps of interest that $[w]$ decreases as κ increases, i.e. as grain size increases, corresponding to observations of Vallon *et al.* (1976). From (5.17), we neglect the right hand side of $(5.24)_1$, and thus we propose the following reduced (still dimensional) set of equations for w and ϕ (at least for the present glacial scales):

$$\nabla \cdot [w^2 \nabla \phi] = 0, \tag{5.27}$$

$$w_t = (\rho_w/\rho_I)[w^2 |\nabla \phi|^2/\rho_I \kappa L + S_W] \\ -(w/\eta_b \kappa)[\phi + \kappa g\{(\rho_w - \rho_I) y' \cos\theta - \rho_w x' \sin\theta + \rho_I \bar{h}_s \cos\theta\}]. \tag{5.28}$$

This can be written in dimensionless form, by defining

$$w = [w] w^*, \quad \phi = [\phi]\phi^*, \quad t = [t] t^*, \quad (x', y') = d(x^*, y^*), \tag{5.29}$$

where $[w]$ and $[\phi]$ are given by (5.25) and (5.26), and

$$[t] = \eta_b \kappa / [\phi] \sim 1 \text{ year}. \tag{5.30}$$

We find

$$\nabla \cdot [w^{*2} \nabla \phi^*] = 0, \tag{5.31}$$

$$w^*_{t^*} = w^{*2} |\nabla \phi^*|^2 + s_W - w^*[\phi^* + G(x^*, y^*)], \tag{5.32}$$

where

$$s_W = \rho_I \kappa L d^2 S_W / [w]^2 [\phi]^2 \lesssim 0.1, \tag{5.33}$$

$$G(x^*, y^*) = y^* - \beta x^* + R h^*_s(x^*), \tag{5.34}$$

with

$$\beta = \rho_w \tan\theta / (\rho_w - \rho_I) \sim O(1), \tag{5.35}$$

$$h^*_s(x^*) = \bar{h}_s(x')/d, \quad R = \rho_I/(\rho_w - \rho_I). \tag{5.36}$$

For reasonably smooth bedrocks, one can expect $\bar{h}_s$ to vary on a length scale $\sim 10^2 d$, consequently $h_s^* \sim O(1)$, but is a slowly varying function ($h_s^{*\prime} \sim 10^{-2}$). Boundary conditions for ϕ^* are

$$\text{on} \quad \partial V_A: \phi^* = -rh_w^* + \beta x^*, \quad w^{*2}\,\partial\phi^*/\partial n^* = f_A^*;$$

$$\text{on} \quad \partial V_B: w^{*2}\,\partial\phi^*/\partial n^* = -\Gamma^*(\phi^*, w^*, x^*);$$

$$\text{on} \quad \partial V_C: w^*(\phi^* + G^*) = 0. \tag{5.37}$$

Here, we define

$$h_w^*(x^*) = \bar{h}_w/d, \quad r = \rho_w/(\rho_w - \rho_I). \tag{5.38}$$

The positivity and w non-flotation conditions in (4.50) can be written in the form

$$\phi^* + [\rho_w/(\rho_w - \rho_I)]y^* - \beta x^* < 0 \quad \text{(positivity)},$$

$$\phi^* + y^* - \beta x^* + Rh_s^* > 0 \quad \text{(non-flotation)}. \tag{5.39}$$

Equations (5.31) and (5.32), and boundary conditions (5.37) provide a reduced set of equations for w^* and ϕ^*. We shall discuss some of their properties in the following section.

It is an easy exercise to scale the solutal equation and boundary conditions similarly. Since $D_T \sim 10^4\,\mathrm{m^2\,y^{-1}}$ (with $w \sim 10^{-2}$, $|\mathbf{v}| \sim 10^5\,\mathrm{m\,y^{-1}}$), it dominates the boundary conditions, which are essentially

$$\text{on} \quad \partial V_A: \partial c/\partial n = 0;$$

$$\text{on} \quad \partial V_B: \partial c/\partial n = 0;$$

$$\text{on} \quad \partial V_C: w = 0, \quad c = c_0;$$
$$w \neq 0, \quad \partial c/\partial n = 0. \tag{5.40}$$

The approximate (dimensional) form of (4.7) is then [using (5.31)]

$$\partial[(\lambda + w)c]/\partial t + w\mathbf{v} \cdot \nabla c = \nabla \cdot [\mathbf{D}_T : \nabla c]. \tag{5.41}$$

Since the calculation of salt concentration uncouples from the determination of w and ϕ, we shall not consider it further here. Unless additional approximations are made, (5.41) is a difficult equation to solve.

6. DISCUSSION

Let us first summarise the physical meaning of (5.31) and (5.32). The equation for ϕ^*, (5.31), can be thought of as a statement of conservation of water phase ($\equiv$conservation of moisture), as given by Lliboutry (1976), except that (with $w \sim 10^{-2}$), the driving differential pressure gradient [G in (5.32)] will be such that typically $|\mathbf{v}| \sim 10^5\,\mathrm{m\,y^{-1}}$, and is so large that other source terms are negligible in comparison with it, including source heating due to viscous heating. The Eq. (5.32) is an averaged version of Nye's (1976) postulated balance equations for intergranular vein closure, $w^{*2}|\nabla\phi^*|^2$ is the interfacial energy source due to viscous heating in the pores, s_W is the bulk viscous stress heating, and $-w^*[\phi^*+G]$ is proportional to the pressure excess $(p-p_w)$ tending to close the pores. Scaling the terms indicates that s_W is "small", and also that interfacial viscous heating can balance pore closure ($w^*_{t^*}=0$), as suggested by Nye (1976). In a steady state, and neglecting s_W,

$$w=(\phi+G)/|\nabla\phi|^2, \tag{6.1}$$

where we now drop asterisks on w^*, ϕ^*, and t^*. The equation for ϕ is then the nonlinear elliptic equation

$$\nabla\cdot[|\nabla\phi|^{-4}(\phi+G)^2\nabla\phi]=0, \tag{6.2}$$

with the various boundary conditions indicated by (5.37). Equation (6.1) is invalid if $|\nabla\phi|\to 0$, in which case (5.32) shows that $w=0$ is the appropriate value in a steady state, provided $\phi+G>0$ (i.e. $p>p_w$), as is physically plausible (this is just the non-flotation condition).

In fact, (5.32) with $s_W\neq 0$ but small has two non-trivial steady states, approximately given by (6.1) and

$$w\approx s_W/[\phi+G] \tag{6.3}$$

(if $\phi + G > 0$), and the general question of stability of either solution arises. For constant ϕ, (6.1) would be unstable [as shown by Nye (1976)], but the general stability problem when ϕ also can change is more difficult. Nye suggested that his analogue of (6.1) could be realistically stable. If the equations presented here are of relevance, then stability of the (at least) two steady state solutions is of some interest.

The derivation of the model presented here relies fundamentally on the ability of temperate ice to behave like a porous medium. Provided we accept the concept of a two-phase flow, *some* relation such as (2.18) can be deduced, provided the water phase is mobile. The fact that calorimetric measurements of temperate alpine glacier ice yield values $w \sim 1\%$ (Lliboutry, 1976), as opposed to direct microscopic observations of veins (Raymond and Harrison, 1975; Nye and Mae, 1972) (which suggest $w \sim 10^{-6}$) is not in contradiction to this concept, provided we assume that continuing strain and recrystallisation (Lliboutry, 1971) will provide access to the vein system for all the water contained in isolated pockets on a time scale of a day (or less). Whether the resultant effective permeability κw will be realistically computable on the assumption of a steady state porosity is debatable, but it certainly seems that the effective porosity will be larger than that due to instantaneous transport through the actual veins: perhaps w in (2.18) should be replaced by w^{α}, where $0 < \alpha < 1$. This will, of course, affect the precision of the numerical values of the parameters, but not the *concept* of the model. Ideally, a more precise study of the interaction of pockets and veins (on a local, microscopic scale) would lead to a more realistic assumption than (2.18), but a "porous" medium provides a plausible first effort.

Some of the problems which this theory may be useful in addressing are the following:

i) on a small scale, what are the effects of rapid changes in pressure on temperate ice cores: what do field experiments tell us? (Harrison, 1972, 1975) [we can expect some dynamic effect, based on analogous effects in soil mechanics (Rice and Cleary, 1976)];

ii) can we explain Carol's (1947) observations at the bedrock, and Robin's heat-pump mechanism (Robin, 1976; Goodman *et al.*, 1979)?

iii) can we give a theory for a wet sliding law, and relate this to other sub-glacial hydrological studies (Weertman, 1972;

Röthlisberger, 1972), and ultimately to the study of surges (Robin and Weertman, 1973)?

iv) can we give a quantitative explanation of short-term (diurnal) variation in sliding velocity (Collins, 1979; Hodge, 1974) and basal water pressure (Hodge, 1976, 1979)?

To conclude, we will present two simple solutions of the equations, to illustrate the kind of problems one might solve analytically. A general solution would inevitably require a numerical solution. The equations under consideration are

$$\nabla \cdot [w^2 \nabla \phi] = 0, \quad w_t = w^2 |\nabla \phi|^2 - w[\phi + y - \beta x + Rh_s] + s, \qquad (6.4)$$

($s = s_W$), with boundary conditions

$$\text{on} \quad \partial V_A(y = h_w): \quad \phi = -rh_w + \beta x, \quad w^2\, \partial\phi/\partial n = f,$$

$$\text{on} \quad \partial V_B(y = h_b): \quad w^2\, \partial\phi/\partial n = -\Gamma,$$

$$\text{on} \quad \partial V_C(y = y_M): \quad w(\phi + y - \beta x + Rh_s) = 0. \qquad (6.5)$$

For a nearly parallel flow, $\partial/\partial y \gg \partial/\partial x$, and there are approximate solutions of the form

$$w = w(y, t), \quad \phi = \beta x + \psi(y, t), \qquad (6.6)$$

where w and ψ satisfy

$$\frac{\partial}{\partial y}\left[w^2 \frac{\partial \psi}{\partial y} \right] = 0, \quad w_t = [\beta^2 + \psi_y^2] w^2 - w[\psi + y + Rh_s] + s, \qquad (6.7)$$

and related (x-independent) boundary conditions from (6.5). For a temperate zone adjoining the base,

$$w^2\, \partial\psi/\partial y = -\Gamma, \qquad (6.8)$$

where Γ is the basal drainage (and may depend on t, and *slowly* on x). Consider first a nearly parallel flow of temperate ice beneath cold ice, i.e. $h_b < y < y_M$. Take $h_b = 0$, and suppose $\Gamma = 0$ (as is plausible, if the drainage system is plugged by cold ice at the snout). Then

$$\psi_y = 0, \quad w(\psi + y + Rh_s) = 0 \quad \text{on} \quad y_M, \tag{6.9}$$

so that

$$\psi = -[y_M + Rh_s]. \tag{6.10}$$

Then

$$\psi + y + Rh_s = -[y_M - y],$$

and

$$w_t = \beta^2 w^2 + [y_M - y]w + s; \tag{6.11}$$

evidently w increases indefinitely with time (in fact $w \to \infty$ at finite time). This is a highly unstable situation (notice, also, that the non-flotation condition $(5.39)_2$ is violated). It is possible that such a régime may have a bearing on the build-up of water at the glacier bed which (possibly) heralds surging behavior. However, notice also that (4.23) can be written (dimensionlessly) as

$$\partial T^- / \partial n = -\lambda w^2 \, \partial\phi/\partial n = -\lambda\Gamma, \tag{6.12}$$

where

$$\lambda = \rho L[\phi][w]^2 / k[T] \sim 2.10^5, \tag{6.13}$$

taking $[T] = 20\,\text{K}$, k (thermal conductivity) $= 700\,\text{m}^2\,\text{bar}\,\text{y}^{-1}\,\text{K}^{-1}$, and (cold) ice temperature is scaled with $[T]$. Even for very small drainage, $\lambda\Gamma$ can be large, which may affect the location of y_M.

Secondly, consider a purely temperate parallel glacier flow. From (6.8), we have (provided $\Gamma = f$)

$$\psi = -rh + \Gamma \int_y^h w^{-2}\, dw, \quad h = h_w, \tag{6.14}$$

using $(6.5)_1$. Hence w satisfies

$$w_t = \beta^2 w^2 + w^{-2}\Gamma^2 - \left\{\Delta + y + \Gamma \int_y^h w^{-2}\, dy\right\} w + s, \tag{6.15}$$

where

$$\Delta = Rh_s - rh \geqq 0. \tag{6.16}$$

If the drainage Γ is $\Gamma(\phi, x)$, then (since f is given) the consistency condition $\Gamma = f$ determines h implicitly by the condition

$$\Gamma\left[\beta x - rh + \Gamma\int_y^h w^{-2}\, dy, x\right] = f. \tag{6.17}$$

It is not particularly clear what kind of function Γ should be. It is not necessarily just the drainage determined by flow in Röthlisberger's (1972) network of channels through the ice. One can visualise basal water as occurring in (regelation) sheets, cavities, etc., whose pressure is determined by the sliding process, together with a network of R-channels, whose drainage is determined by the induced pressure: some drainage could take place through the sheet system. A Röthlisberger type calculation (Paterson, 1981) yields a drainage flux of the form

$$\Gamma \propto [\phi + G]^{m_1}/[\phi_x]^{m_2}\big|_{y=0}, \tag{6.18}$$

where (e.g.) $m_1 \approx 12$, $m_2 \simeq 5.5$. In the present instance this would be

$$\Gamma = \gamma\left\{\Delta + \Gamma\int_0^h w^{-2}\, dy\right\}^m, \tag{6.19}$$

where γ is some constant. If a typical melt rate at the surface is taken as $10\,\mathrm{m\,y^{-1}}$, then the scales introduced in Section 5 ($w \sim 10^{-2}$, $|\mathbf{v}| \sim 10^5\,\mathrm{m\,y^{-1}}$) would suggest $f \sim 10^{-2}$, and thus $\Gamma \sim 10^{-2}$ in (6.19). With $\Delta \sim 1$, this means that a reasonable value of γ is $\gamma \sim 10^{-2}$, and a first approximation of (6.19) is

$$\Gamma \sim \gamma\Delta^m, \tag{6.20}$$

where $m \approx 12$ for Röthlisberger drainage. Consequently, (6.17) determines Δ as

$$\Delta \sim (f/\gamma)^{1/m} \approx 1, \tag{6.21}$$

if $f \sim \gamma$. If we take $\Gamma \sim 10^{-2}$, then (6.15) is approximately

$$w_t = \beta^2 w^2 + w^{-2}\Gamma^2 - \{\Delta + y\}w + s. \tag{6.22}$$

For sufficiently small Γ (and each y), there are two steady states, the higher value $[w \sim (\Delta + y)/\beta^2]$ being unstable. If $\Gamma = 0$, the lower one is $w \sim s/(\Delta + y) \sim 10^{-2}$ if $s \sim 10^{-2}$ (corresponding to a moisture level of 10^{-4}—remember w is here a *scaled* variable). But if $\Gamma \neq 0$, the term Γ^2/w^2 becomes comparable to s if $w \sim \Gamma/s^{1/2}$. Thus, the lower steady state $w \sim s/(\Delta + y)$ can be increased by non-zero Γ to a higher level, for example

$$w \approx \Gamma^{2/3}/(\Delta + y)^{1/3} \tag{6.23}$$

if $\Gamma \gg s^{3/2}$, as for example if $\Gamma \sim s \sim 10^{-2}$. In this range, this stable solution continues to exist up to some critical Γ (at which the right hand side of (6.22) is always positive). If β is small (very gentle valley slope) then (6.23) continues to hold as long as $\Gamma \ll 1/\beta^3$, so that moderately large moisture levels can be attained in this way.

Acknowledgements

I would like to thank John Nye for his comments, particularly on viscoelasticity, Jim Rice for directing me to the soil mechanics literature, and Marius Ungarish for many useful discussions on two phase flows. Maura McNiff did an excellent job of the final manuscript production.

I am also grateful to an anonymous referee, and particularly to Kolumban Hutter, for their diligent efforts in criticizing earlier versions of this paper.

Partial support of this research was provided by an Exxon Foundation Fellowship at M.I.T. during June, 1983.

References

Ahern, J. L. and Turcotte, D. L., "Magma migration beneath an ocean ridge," *Earth & Planet. Sci. Lett.* **45**, 115–122 (1979).

Aris, R., "On the dispersion of a solute in a fluid flowing through a tube," *Proc. R. Soc. London* **A235**, 67–77 (1956).

Atthey, D. R., "A finite difference scheme for melting problems," *J. Inst. Math. & Appl.* **13**, 353–366 (1974).

Atthey, D. R., "A finite difference scheme for melting problems based on the method of weak solutions," in: *Moving Boundary Value Problems in Heat Flow and Diffusion* (ed. J. R. Ockendon and W. R. Hodgkins), Clarendon Press, Oxford, pp. 182–191 (1975).

Batchelor, G. K., *An Introduction to Fluid Dynamics*, Cambridge: C.U.P. (1967).
Carol, H., "The formation of roches moutonnées," *J. Glaciol.* **1,** 57–59 (1947).
Chalmers, B., *Principles of Solidification*, Wiley, New York (1964).
Clarke, G. K. C. and Goodman, R. H., "Radio echo soundings and ice-temperature measurements in a surge-type glacier," *J. Glaciol.* **14,** 71–78 (1975).
Clarke, G. K. C., Nitsan, U. and Paterson, W. S. B., "Strain heating and creep instability in glaciers and ice sheets," *Rev. Geophys. & Space Phys.* **15,** 235–247 (1977).
Colbeck, S. C., "Short-term forecasting of water run-off from snow and ice," *J. Glaciol.* **19,** 571–588 (1977).
Collins, D. N., "Quantitative determination of the subglacial hydrology of two Alpine glaciers," *J. Glaciol.* **23,** 347–362 (1979).
Drew, D. A., "Mathematical modeling of two-phase flow," *Annu. Rev. Fluid Mech.* **15,** 261–291 (1983).
Duval, P., "The rôle of the water content on the creep rate of polycrystalline ice," *IAHS Publ.* **118,** 29–33 (1977).
Elliott, C. M. and Ockendon, J. R., *Weak and variational methods for moving boundary problems*, Pitman Research Notes, Boston (1982).
Fowler, A. C., "A theoretical treatment of the sliding of glaciers in the absence of cavitation," *Philos. Trans. R. Soc. London* **A298,** 637–681 (1981).
Fowler, A. C. and Larson, D. A., "On the flow of polythermal glaciers—I. Model and preliminary analysis," *Proc. R. Soc. London* **A363,** 217–242 (1978).
Frank, F. C., "Two-component flow model for convection in the earth's upper mantle," *Nature* **220,** 350–352 (1968).
Glen, J. W., Homer, D. R. and Paren, J. G., "Water at grain boundaries: its rôle in the purification of temperate glacier ice," *IASH Publ.* **118,** 263–271 (1977).
Goodman, D. J., King, G. C. P., Millar, D. H. M. and Robin, G. de Q., "Pressure-melting effects in basal ice of temperate glaciers: laboratory studies and field observations under Glacier d'Argentiere," *J. Glaciol.* **23,** 259–271 (1979).
Grigoryan, S. S., Krass, M. S. and Shumskiy, P. A., "Mathematical model of a three-dimensional non-isothermal glacier," *J. Glaciol.* **17,** 401–417 (1976).
Hallet, B., "Subglacial chemical deposits and the composition of basal ice," *IASH Publ.* **118,** 289–292 (1977).
Harrison, W. D., "Temperature of a temperate glacier," *J. Glaciol.* **11,** 15–29 (1972).
Harrison, W. D., "Temperature measurements in a temperate glacier," *J. Glaciol.* **14,** 23–30 (1975).
Hodge, S. M., "Variations in the sliding of a temperate glacier," *J. Glaciol.* **13,** 349–369 (1974).
Hodge, S. M., "Direct measurement of basal water pressures: a pilot study," *J. Glaciol.* **16,** 205–218 (1976).
Hodge, S. M., "Direct measurement of basal water pressures: progress and problems," *J. Glaciol.* **23,** 309–319 (1979).
Hooke, R. Le B., "Flow law for polycrystalline ice in glaciers: comparison of theoretical predictions, laboratory data, and field measurements," *Rev. Geophys and Space Phys.* **19,** 664–672 (1981).
Hutter, K., "A mathematical model of polythermal glaciers and ice sheets," *Geophys. and Astrophys. Fluid Dyn.* **21,** 201–224 (1982).

Ishii, M., *Thermo-fluid Dynamic Theory of Two-Phase Flow*, Paris: Eyrolles, (1975).

Jarvis, G. T. and Clarke, G. K. C., "The thermal régime of Trapridge Glacier and its relevance to glacier surging," *J. Glaciol.* **14,** 235–250 (1975).

Klebanov, L. A., Kroshilin, A. E., Nigmatulin, B. I. and Nigmatulin, R. I., "On the hyperbolicity, stability and correctness of the Cauchy problem for the system of equations of two-speed motion of two-phase media," *Prikl. Mat. and Mekh.* **46,** 83–95 (1982). In translation: *J. Appl. Math. and Mech.* **46,** 66–74 (1983).

Kynch, G. J., "A theory of sedimentation," *Trans. Farad. Soc.* **48,** 166–176 (1952).

Lliboutry, L. A., "Permeability, brine content and temperature of temperate ice," *J. Glaciol.* **10,** 15–29 (1971).

Lliboutry, L. A., "Physical processes in temperate glaciers," *J. Glaciol.* **16,** 151–158 (1976).

Lliboutry, L. A., "Local friction laws for glaciers: a critical review and new openings," *J. Glaciol.* **23,** 67–95 (1979).

Nigmatulin, R. I., "Spatial averaging in the mechanics of heterogeneous and dispersed systems," *Int. J. Multiphase Flow* **5,** 353–385 (1979).

Nye, J. F., "The flow law of ice from measurements in glacier tunnels, laboratory experiments and the Jungfraufirn borehole experiment," *Proc. R. Soc. London* **A219,** 477–489 (1953).

Nye, J. F., "Water flow in glaciers: jökulhlaups, tunnels and veins," *J. Glaciol.* **17,** 181–207 (1976).

Nye, J. F. and Mae, S., "The effect of non-hydrostatic stress on intergranular water veins and lenses in ice," *J. Glaciol.* **11,** 81–101 (1972).

Nye, J. F. and Frank, F. C., "Hydrology of the intergranular veins in a temperate glacier," *Association Internationale d'Hydrologie Scientifique.* Symposium on the hydrology of glaciers, Cambridge, 1969, pp. 157–161 (1973).

Ockendon, J. R., "Techniques of analysis," in *Moving Boundary Problems in Heat Flow and Diffusion* (ed. J. R. Ockendon and W. R. Hodgkins), Clarendon Press, Oxford, pp. 138–149 (1975).

Paterson, W. S. B., *The Physics of Glaciers*, 2nd edition, Pergamon, Oxford (1981).

Radkevič, E. V., "The second boundary value problem for a second order equation with non-negative characteristic form," *Vestn. Mosk. Univ. Ser. I. Mat. Mekh.* **22,** 3–11 (1967) (in Russian); *Math. Rev.* **37,** No. 5531, 1031 (1969).

Raymond, C. F. and Harrison, W. D., "Some observations on the behaviour of the liquid and gas phases in temperate glacier ice," *J. Glaciol.* **14,** 213–233 (1975).

Rice, J. R. and Cleary, M. P., "Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents," *Rev. Geophys. and Space Phys.* **14,** 227–241 (1976).

Roberts, P. H. and Loper, D. E., "Towards a theory of the structure and evolution of a dendrite layer," preprint (1982).

Robin, G. de Q., "Is the basal ice of a temperate glacier at the pressure melting point?," *J. Glaciol.* **16,** 183–196 (1976).

Robin, G. de Q. and Weertman, J., "Cyclic surging of glaciers," *J. Glaciol.* **12,** 3–18 (1973).

Röthlisberger, H., "Water pressure in intra- and subglacial channels," *J. Glaciol.* **11,** 177–203 (1972).

Rubinstein, L., "The Stefan problem: Comments on its present state," *J. Inst. Math. and Appl.* **24,** 259–277 (1979).

Schubert, G. and Yuen, D. A., "Initiation of ice ages by creep instability and surging of the East Antarctic ice-sheet," *Nature* **296,** 127–130 (1982).

Shankland, T. J., O'Connell, R. J. and Waff, H. S., "Geophysical constraints on partial melt in the upper mantle," *Rev. Geophys. and Space Phys.* **19,** 394–406 (1981).

Shreve, R. L., "Movement of water in glaciers," *J. Glaciol.* **11,** 205–214 (1972).

Spring, U. and Hutter, K., "Numerical studies of jökulhlaups," *Cold Regions Science and Technology* **4,** 227–244 (1981).

Spring, U. and Hutter, K., "Conduit flow of a fluid through its solid phase and its application to intraglacial channel flow," *Int. J. Eng. Sci.* **20,** 327–363 (1982).

Stevenson, D. J., "Volcanism and igneous processes in small icy satellites," *Nature* **298,** 142–144 (1982a).

Stevenson, D. J., "Migration of fluid-filled cracks: applications to terrestrial and icy bodies," abstract, in *Proc. 13th Lunar Planet. Sci. Conf.*, A.G.U., Washington, D.C., 768–769 (1982b).

Stuhmiller, J. H., "The influence of interfacial pressure forces on the character of two-phase flow equations," *Int. J. Multiphase Flow* **3,** 551–560 (1977).

Tayler, A. B., "The mathematical formulation of Stefan problems," in *Moving Boundary Problems in Heat Flow and Diffusion* (ed. J. R. Ockendon and W. R. Hodgkins), Clarendon Press, Oxford, pp. 120–137, (1975).

Taylor, G. I., "Dispersion of soluble matter in solvent flowing slowly through a tube," *Proc. R. Soc. London* **A219,** 186–203 (1953).

Turcotte, D. L., "Some thermal problems associated with magma migration," *J. Volcanol. Geotherm. Res.* **10,** 267–278 (1981).

Turcotte, D. L. and Ahern, J. L., "A porous flow model for magma migration in the asthenosphere," *J. Geophys. Res.* **83,** 767–772 (1978).

Vallon, M., Petit, J.-R. and Fabre, B., "Study of an ice core to the bedrock in the accumulation zone of an alpine glacier," *J. Glaciol.* **17,** 13–28 (1976).

Weertman, J., "General theory of water flow at the base of a glacier or ice sheet," *Rev. Geophys. and Space Phys.* **10,** 287–333 (1972).

Weertman, J., "The unsolved general glacier sliding problem," *J. Glaciol.* **23,** 97–115 (1979).