Secular Cooling in Convection

By A. C. Fowler

We show how long-term cooling effects in convection can be analyzed in terms of the underlying boundary-layer structure, by a method analogous to averaging in nonlinear oscillations.

1. Introduction

There are a number of problems which motivate the study of slow (secular) changes in the structure of convective cells. In the laboratory, it is well known that the approach to equilibrium of a convective pattern is a long-term one, which occurs over a conductive time scale rather than a convective one [3]. It is possible that some (weak) forms of spatial chaos may be partially described in terms of such long-term evolutions [1].

In the earth's mantle, convection is plausibly largely driven by secular cooling of the earth's core [11,5], and thus the convective state of the mantle is subject to slowly varying thermal boundary conditions (as well as slowly decaying radioactive heat sources).

In the asymptotic study of the structure of convection at high Rayleigh number [10], one assumes that the core temperature is isothermal and equal to the average of the upper and lower boundary temperatures. In a symmetric case, one can deduce this from the symmetry of the problem. In more general cases (where, for example, there are asymmetric boundary conditions, some internal heating, etc.) it is not clear how one would choose the internal temperature. The method adopted here is to incorporate the small effects of conduction in the interior, which will serve (over a conductive time scale) to determine an appropriate internal temperature. This method is analogous to that of Batchelor [2] and Proctor [8] to determine vorticity in circulating flows, away from shear layers, and can be

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considered as a generalization of the method of averaging, with the closed streamlines taking the place of periodic orbits. The details of the calculations are very similar to those of Rhines and Young [9], and can be taken as confirming their analysis, although they were more concerned with passive scalar transport. Numerical studies of the problem considered here have been presented by Daly [4].

2. Governing equation

We will focus our attention on the simplest possible case: two-dimensional convection of a very viscous Newtonian fluid. Additionally, let us suppose that all boundaries are free (have zero shear stress). Then dimensionless equations which describe the flow are [10, 12]

$$\begin{aligned} \theta_t + \psi_x\theta_y - \psi_y\theta_x &= \nabla^2\theta, \\ \omega &= -\nabla^2\psi, \\ \mathrm{Ra}\,\theta_x &= \nabla^2\omega; \end{aligned} \tag{2.1}$$

here θ, ψ, and ω are dimensionless temperature, stream function, and vorticity, respectively. Ra is the Rayleigh number,

$$\mathrm{Ra} = \frac{\alpha\Delta T\rho g d^3}{\eta\kappa}, \tag{2.2}$$

where α is the thermal expansion coefficient, ΔT the temperature difference, ρ the density, g gravity, d the fluid layer depth, η the viscosity, and κ the thermal diffusivity. (x, y) are Cartesian coordinates (y vertically up), and the velocity is $\mathbf{u} = (\kappa/d)(-\psi_y, \psi_x)$; (x, y) are scaled with d, θ with ΔT, ω with κ/d^2. We have assumed that the Prandtl number $\eta/\rho\kappa$ is very large (infinite), mainly because this is the case for which boundary layer theory is well developed: one could plausibly extend the present analysis to the case of finite Prandtl number if the corresponding stationary boundary layer theory existed. According to Roberts [10] this is in preparation.

The boundary conditions for (2.1) are

$$\begin{aligned} &\theta = \pm\tfrac{1}{2} && \text{on } y = 0, 1, \\ &\theta_x = 0 && \text{on } x = 0, a, \\ &\psi = \omega = 0 && \text{on } x = 0, a, \quad y = 0, 1. \end{aligned} \tag{2.3}$$

Here a is the dimensionless cell width. We assume $\mathrm{Ra} \gg 1$, and seek to study the structure of the flow in the asymptotic limit $\mathrm{Ra} \to \infty$. The appropriate rescaling of the equations is

$$\psi, \omega \sim \frac{1}{\delta^2}, \quad t \sim \delta^2, \quad \delta^3 = \mathrm{Ra}^{-1}, \tag{2.4}$$

and when the equations (2.1) are scaled in this fashion, they are (using the same notation for the rescaled variables)

$$\begin{aligned} \theta_t + \psi_x\theta_y - \psi_y\theta_x &= \delta^2\nabla^2\theta, \\ \omega &= -\nabla^2\psi, \\ \theta_x &= \delta\nabla^2\omega, \end{aligned} \tag{2.5}$$

with the same boundary conditions as before.

In a steady state, the structure of the flow is this, as $\delta \to 0$. Away from boundaries, $\theta \approx 0$ (isothermal core) and $\nabla^4\psi = 0$, with $\psi = 0$ on boundaries, $\omega = 0$ on top and bottom, but $\omega \neq 0$ on the sides. This is because a vorticity layer of thickness $\sim\delta$ exists at each side wall, in which θ and ω change by $O(1)$, and $\psi \sim \delta$. Thus $\theta_x \sim \delta\omega_{xx}$, $\psi_{xx} \sim 0$, and $\psi_x\theta_y - \psi_y\theta_x \sim \delta^2\theta_{xx}$. The vorticity generated by these thermal plumes drives the biharmonic interior flow. At top and bottom, there are thermal boundary layers, across which θ changes by $O(1)$, and $\psi \sim \omega \sim y$ (or $1 - y$) $\sim \delta$.

Let us now study the time dependent problem (2.5), with the same scaling. The same boundary-layer structure will apply, but we shall find that the core temperature (assumed zero in the steady state) can evolve on a long, conductive time scale.

In the core, we put

$$\begin{aligned} \theta &\sim \delta\theta_1 + \delta^2\theta_2 + \cdots, \\ \psi &\sim \psi_0 + \delta\psi_1 + \cdots, \\ \omega &\sim \omega_0 + \delta\omega_1 + \cdots. \end{aligned} \tag{2.6}$$

We suppose $\theta \sim \delta$, since otherwise (if $\theta \sim 1$) (2.5)$_3$ implies $\omega \sim 1/\delta$, which would necessitate a rescaling of the equations; for simplicity we avoid such complications. It seems plausible that an initial value $\theta \sim 1$ will relax to $O(\delta)$ on a time scale $t \gtrsim 1$, and this is confirmed by Daly [4]. Bearing this in mind, (2.6) has a solution given by

$$\begin{aligned} \theta_1 &= f(\psi_0, \tau), \\ \nabla^4\psi_0 + f_\psi\psi_{0x} &= 0, \end{aligned} \tag{2.7}$$

with ψ_0 and $\nabla^2\psi_0$ prescribed on the boundary. The core temperature evolves on the slow time scale

$$\tau = \delta^2 t, \tag{2.8}$$

which reflects the occurrence of the conductive term in the equation at $O(\delta^3)$. However, the boundary layers react to changes on a time scale $t \sim 1$; therefore the boundary conditions for the core flow may be considered to be quasistatic.

Now the core temperature equation contains no forcing term at $O(\delta^2)$, and thus it seems we may put

$$\theta_2 = 0; \tag{2.9}$$

possibly, this is not quite correct, if matching to the boundary layer at $O(\delta)$ requires the existence of nonzero θ_2. But even if this is the case, we can telescope the $O(\delta)$ term into θ_1 (as this implicitly allows θ_1 to depend on δ, for possible matching purposes). Since detailed matching is not our concern, it is an equivalent statement to assume (2.9): then, also, $\omega_1 = \psi_1 = 0$. Then at $O(\delta^3)$, $(2.5)_1$ is (assuming $\partial/\partial t = 0$)

$$\frac{\partial(\psi_2,\theta_1)}{\partial(x,y)} + \frac{\partial(\psi_0,\theta_3)}{\partial(x,y)} = \nabla^2 f - f_\tau. \tag{2.10}$$

With $\mathbf{q}_0 = (-\psi_{0y}, \psi_{0x})$, $\mathrm{div}\,\mathbf{q}_0 = 0$, one finds

$$\frac{\partial(\psi_2,\theta_1)}{\partial(x,y)} = \psi_2 \mathbf{q}_0\cdot\nabla f_\psi - \mathrm{div}\left[f_\psi \psi_2 \mathbf{q}_0\right], \tag{2.11}$$

$$\frac{\partial(\psi_0,\theta_3)}{\partial(x,y)} = \mathrm{div}\left[\theta_3 \mathbf{q}_0\right]. \tag{2.12}$$

Now the contours $\psi_0 =$ constant form closed streamlines in the convecting cell, with particle motion being clockwise round them. Let C be a streamline, $\psi_0 = \psi$, and A be the interior of C. On C, $\mathbf{q}_0\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the unit outward normal. Also

$$\mathbf{q}_0\cdot\nabla f_\psi[\psi_0,\tau] = q_0\frac{\partial}{\partial s}\left[f_\psi(\psi_0,\tau)\right],$$

where s is arc length along $\psi_0 =$ constant in the direction of $\mathbf{q}_0$. Obviously $\partial\psi_0/\partial s = 0$, so that $\mathbf{q}_0\cdot\nabla f_\psi = 0$ also. Then integrating (2.11) and (2.12) over A and using Stokes' theorem gives, from (2.10),

$$\iint_A \left[\nabla^2 f - f_\tau\right] dA = 0. \tag{2.13}$$

Now let σ be the Lagrangian time following a particle; that is,

$$\sigma = \int_{\psi_0 = \text{const}} \frac{ds}{|\mathbf{q}_0|}. \tag{2.14}$$

It follows that

$$\frac{\partial(\psi_0,\sigma)}{\partial(x,y)} = 1. \tag{2.15}$$

Let R be a thin annulus between C and $C + \delta C$: then

$$\iint_R f_\tau\, dA = \iint_R f_\tau[\psi_0,\tau]\, d\psi_0\, d\sigma$$

$$\approx \delta\psi \oint_C f_\tau\, d\sigma \tag{2.16}$$

(notice that this integral traverses C clockwise, in the direction of increasing σ). Since f_τ is constant on C, (2.16) is

$$\iint_R f_\tau\, dA \approx T(\psi) f_\tau \delta\psi, \tag{2.17}$$

where

$$T(\psi) = \int_C d\sigma \tag{2.18}$$

is the time for a particle to circulate around C.

Also, (going anti-clockwise around C),

$$\iint_R \nabla^2 f\, dA = \oint_{C|_{\psi+\delta\psi}} f_\psi \psi_n\, ds - \int_{C|_\psi} f_\psi \psi_n\, ds$$

$$= \delta\left[\oint_C f_\psi \psi_n\, ds\right]$$

$$= \delta\left[f_\psi \oint_C \psi_n\, ds\right]$$

$$= \delta\left[\Gamma(\psi) f_\psi\right], \tag{2.19}$$

where

$$\Gamma(\psi) = \oint_C \psi_n\, ds = \iint_A \omega\, dA \tag{2.20}$$

is the circulation around C. Applying $d/d\psi$ to (2.13), and using (2.17) and (2.20), we find that $f(\psi, \tau)$ satisfies

$$T(\psi) f_\tau = \left[\Gamma(\psi) f_\psi\right]_\psi, \tag{2.21}$$

that is, a diffusion equation. By inspection, ω and ψ are positive in the cell, and thus $\Gamma(\psi)$ [and $T(\psi)$] are positive. Notice that Γ and T also depend on f_ψ, through (2.7), so that (2.21) is decidedly nonlinear.

The boundary conditions for f are that it should be regular at $\psi = \psi_m = \max(\psi)$ [since from (2.20), $\Gamma = 0$ there], and a heat balance condition at $\psi = 0$ (the cell boundary). An exact heat balance for the entire cell B follows from integrating (2.5) over B:

$$\frac{\partial}{\partial t} \iint_B \theta\, dA = \delta^2 \int_{\partial B} \frac{\partial \theta}{\partial n}\, ds. \tag{2.22}$$

The left-hand side is essentially the integral of the core temperature, whereas the right-hand side is the heat loss through the boundary layers. Thus

$$\begin{aligned} \frac{\partial}{\partial t} \iint_B \theta\, dA &\approx \delta^2 \frac{\partial}{\partial \tau} \iint_B \delta f\, d\psi\, d\sigma \\ &= \delta^3 \int_0^{\psi_m} T(\psi) f_\tau\, d\psi \\ &= -\delta^3 \Gamma(0) f_\psi|_{\psi=0}; \end{aligned} \tag{2.23}$$

since the thermal boundary layers respond rapidly to changes in f, we can suppose that the heat loss through them is determined by the steady-state solution of the boundary-layer equations. We take this heat loss to be $K(\theta_\infty - \theta_0)/\delta$, where θ_∞ is the far-field core temperature, θ_0 is the boundary temperature, and K is an $O(1)$ number, determined from the boundary-layer equations ($K = \mathrm{Nu}/\mathrm{Ra}^{1/3}$ here). Then

$$\begin{aligned} \delta^2 \int_{\partial B} \frac{\partial \theta}{\partial n}\, ds &= \delta^2 \left[-\frac{K}{\delta}\left(\delta f + \tfrac{1}{2}\right) + \frac{K}{\delta}\left(\tfrac{1}{2} - \delta f\right) \right] \\ &= -2K\delta^2 f, \end{aligned} \tag{2.24}$$

and thus (2.22) is

$$2Kf(0,\tau) = \delta \Gamma(0) f_\psi(0,\tau), \tag{2.25}$$

which simply implies

$$f(0,\tau) \approx 0 \tag{2.26}$$

as a second boundary condition. For more general boundary conditions (asymmetric), the appropriate condition would be determined from equating heat loss through the top to heat gain at the bottom.

3. Basic solution

We have to solve (2.21) with boundary conditions (2.26), and regularity of f at $\psi = \psi_m$. Additionally, Γ and T depend on f through (2.7). Obviously, we are not going to be able to solve such a complicated system. What we *can* do is to illuminate the structure of the solution by choosing simple, but representative forms for Γ and T. Thus we suppose Γ and T are independent of f (e.g. if f is small enough). To see how Γ and T behave near the center of the cell ($\psi = \psi_m$, say), we solve for ψ_0 there. In local polar coordinates, we have

$$\begin{aligned} \omega &\approx \omega_m, \\ \psi_r &\approx -\tfrac{1}{2}\omega_m r, \\ \psi &\approx \psi_m - \tfrac{1}{4}\omega_m r^2. \end{aligned} \tag{3.1}$$

Thus

$$T(\psi) = \oint_C d\sigma \approx \int_0^{2\pi} \frac{-r\, d\theta}{-\frac{1}{2}\omega_m r} \approx \frac{4\pi}{\omega_m}, \tag{3.2}$$

$$\Gamma(\psi) = \iint_A \omega\, dA \approx \pi \omega_m r^2 \approx 4\pi(\psi_m - \psi). \tag{3.3}$$

We shall suppose that these asymptotic behaviors extend throughout the cell. That is, define

$$z = \psi_m - \psi, \tag{3.4}$$

and put

$$T = T_m, \qquad \Gamma = \Gamma_m' z, \tag{3.5}$$

where T_m and Γ_m' are constants.

The equation (2.21) now has separable solutions

$$f = \exp\left[-\sigma \Gamma_m' \tau / T_m\right] \chi(z), \tag{3.6}$$

where

$$z\chi'' + \chi' + \sigma\chi = 0, \tag{3.7}$$

with

$$\chi(0) \text{ regular}, \qquad \chi(\psi_m) = 0. \tag{3.8}$$

The solutions of (3.7) and (3.8) are

$$\chi = J_0[2\sqrt{\sigma z}], \qquad \sigma = j_n^2/4\psi_m, \tag{3.9}$$

where j_n is the nth zero of J_0. Since the Bessel functions are complete, any initial condition gives rise to a solution for f which can be represented as a sum of functions,

$$f(\psi,\tau) = \sum_1^\infty A_n J_0\left[\frac{j_n(\psi_m-\psi)^{1/2}}{\psi_m^{1/2}}\right]\exp\left[-\frac{j_n^2\Gamma_m'}{4\psi_m T_m}\tau\right]. \tag{3.10}$$

The values of j_n are $j_1 = 2.4048\ldots$, $j_2 = 5.520\ldots$; thus (3.10) rapidly decays, and for moderate τ values, the first mode dominates:

$$f(\psi,\tau) \approx A_1 J_0\left[\frac{j_1(\psi_m-\psi)^{1/2}}{\psi_m^{1/2}}\right]\exp\left[-\frac{j_1^2\Gamma_m'}{4\psi_m T_m}\tau\right]. \tag{3.11}$$

Adopting (3.1), (3.2), and (3.3), and $\psi_m \approx \omega_m/4$ (so $\psi = 0$ on $r \approx 1$), (3.11) is roughly

$$f(\psi,\tau) \approx A_1 J_0(2.4r)\exp[-5.8\tau], \tag{3.12}$$

indicating a (roughly) parabolic core temperature profile which decays on a time scale $t \sim 1/6\delta^2$. Equation (3.12) is just how an initial thermal anomaly would cool by pure diffusion, *despite the fact that convective transport is much larger*. The analogy with averaging is that the advective terms are conservative, and the long-term evolution is controlled by the dissipative terms (heat conduction).

4. Variations

The method outlined above can obviously be extended to other related situations, and to conclude, we consider how the analysis is altered when either a slow variation in heating occurs, or a small amount of internal heating occurs.

We consider first an internal heating term in $(2.5)_1$. If this is $O(\delta)$, then the flow is essentially driven by the internal heating: consequently, all streamlines visit the top thermal boundary layer, and the previous analysis is inapplicable (the scalings change). We will therefore suppose internal heating $\ll O(\delta)$, and specifically of $O(\delta^3)$; thus

$$\theta_t + \psi_x\theta_y - \psi_y\theta_x = \delta^2\nabla^2\theta + \delta^3 h. \tag{4.1}$$

With this scaling, we can adopt our previous analysis, whereas $O(\delta^2)$ heating appears to render the scaling inappropriate. The analysis proceeds as before, and we find (2.13) with an extra term $\iint_A h\,dA$. Its differential is $\iint_R h\,dA \approx \delta\psi\int_C h\,d\sigma$, and thus we find (2.21) becomes

$$T(\psi)f_\tau = \left[\Gamma(\psi)f_\psi\right]_\psi + H(\psi), \tag{4.2}$$

where

$$H(\psi) = \int_C h\,d\sigma. \tag{4.3}$$

With h constant, $H = hT$. If we adopt (3.4) and (3.5), and put

$$\tau = \psi_m T_m\theta/\Gamma_m',$$
$$H = \Gamma_m' q/T_m\psi_m, \tag{4.4}$$
$$z = \psi_m\xi,$$

then

$$f_\theta = (\xi f_\xi)_\xi + q. \tag{4.5}$$

The transient solution to this equation, with boundary conditions $f(1,\tau) = 0$, $f(0,\tau)$ regular, is still (3.10). However, the steady state, to which (4.2) relaxes, satisfies

$$(\xi f')' + q = 0, \qquad f(0) \text{ regular}, \quad f(1) = 0, \tag{4.6}$$

that is,

$$f(\xi) = q(1-\xi) = q\psi/\psi_m; \tag{4.7}$$

with constant vorticity (3.1), the steady heating gives a temperature profile which is approximately parabolic away from the central stagnation point.

Notice that (4.5) takes the same form if $h = h(\psi,\tau)$. For example, convection with a (weak) exponentially slowly decaying heat source would satisfy (4.5) with

$$q = q_0 e^{-\lambda\theta} \tag{4.8}$$

(radioactive heat sources in the mantle, for example). The transients are as before, and there is a particular solution

$$f(\zeta,\tau) = \frac{q_0}{\lambda}\left[\frac{J_0\{2\sqrt{\lambda\zeta}\}}{J_0\{2\sqrt{\lambda}\}} - 1\right]e^{-\lambda\theta}, \tag{4.9}$$

which dominates the transient (3.11) if $\lambda < 2.4$, and tends (for fixed θ) to (4.7) as $\lambda \to 0$. Roughly (for small λ) (4.9) contributes a decaying parabolic (in r) temperature profile.

Now let us suppose that the boundary conditions change slowly (external cooling); for example, if the basal temperature is $\frac{1}{2} - 2\delta S(\tau)$, then the emended (2.24) leads to

$$f \approx -S(\tau) \qquad \text{on} \quad \psi = 0. \tag{4.10}$$

For example (with zero internal heating), we take $S(\tau) = -a\tau$, and solve (4.5) with $q = 0$, $f(1,\tau) = -a\tau$, $f(0,\tau)$ regular. Again, transients are given by (3.10), and a particular solution is given by

$$f = -a[\tau + 1 - \zeta], \tag{4.11}$$

which dominates the thermal profile at moderate values of τ (≥ 1). Again, the profile is approximately parabolic, and decreases linearly with the cooling.

Finally, suppose that the heating h depends on temperature in the form

$$h \propto e^{bf}, \tag{4.12}$$

as would be (approximately) appropriate for viscous heating (if viscosity depends exponentially on temperature), ohmic heating (if electrical conductivity depends exponentially on temperature), or the exothermic heat of chemical reactions. Then $h = h(\psi,\tau)$, and (4.5) is still appropriate [with the simplifications (3.5)], in the form

$$f_\theta = (\zeta f_\zeta)_\zeta + \Lambda e^f, \tag{4.13}$$

where we can take $b = 1$ by a rescaling of f. The boundary conditions are as in (4.6).

The steady-state solution of (4.13) can be determined explicitly, and is

$$f(\zeta) = \ln\left[\frac{1}{2\Lambda\zeta}\operatorname{sech}^2\{\tfrac{1}{2}\ln\zeta + B\}\right], \tag{4.14}$$

where B is given by either of

$$B = \pm\cosh^{-1}\left[\frac{1}{(2\Lambda)^{1/2}}\right]. \tag{4.15}$$

Thus there are two steady states if $\Lambda < \frac{1}{2}$, but none if $\Lambda > \frac{1}{2}$. It is well known that this is a problem of the thermal-runaway type [6, 7], and that if $\Lambda > \frac{1}{2}$, the solution f of (4.13) will tend to infinity at some ζ for a finite value of θ: consequently, the convective state would break down. The appearance of this phenomenon in an analysis of convection is quite interesting, although to obtain $\Lambda \sim 1$ may require unrealistically high values of heating (for example, it is unlikely that viscous heating would ever be so large).

5. Conclusions

We have shown how secular changes of temperature in rapidly convecting fluids can be analyzed by deriving a nonlinear diffusion equation. By using simple, but reasonable, approximations (so that the equation is rendered linear), we are able to solve the problem explicitly for a number of situations involving internal heating, or slow variation of the boundary conditions. The results are not directly applicable, but the ideas expressed may be of use in studies of laboratory convection and turbulence, studies of mantle convection, and further analyses of high-Rayleigh-number convection in differing environments.

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