Thermal Runaway in the Earth's Mantle

By *A. C. Fowler*

We examine the possibility that thermal runaway may occur locally in the earth's asthenosphere, due to a coupling between the velocity and temperature fields due to a strongly temperature-dependent viscosity. The analysis is based on a partial description of convection in the earth, in which the boundary-layer nature of the motion is taken into account. We find that realistic parameter values are consistent with thermal runaway occurring on length scales of thousands of kilometers, or time scales of order 10^8 years. If thermal runaway does occur, one would expect partial melting, and probably consequent volcanism.

1. Introduction

For twenty years, it has been accepted that continental drift is caused by the active motion of tectonic plates, some thirteen or so of which make up the earth's surface, and which move relative to each other over time scales of millions of years. This motion is the surface expression of convective motion within the largely solid mantle of the earth (basically the upper 3000 kilometers), which is driven by thermal buoyancy effects. (Compositional effects are also of importance, but in what dynamic way is not known.) The buoyancy sources are surface cooling, basal heating (from the earth's core, which is plausibly cooling), and perhaps also radioactive heating.

Although the mantle is solid, it deforms under applied stresses by various solid-state creep processes, in much the same way as glacier ice. These processes are well understood [53], and many laboratory experiments have been carried out to determine the effective viscosity at various temperatures and stresses [26].

There are many indirect ways to "observe" the convection in the mantle, but no direct ways. Consequently, our knowledge of how convection acts to produce

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the surface observables (such as heat flow, geoid anomalies, seismicity, volcanism, and postglacial rebound, for example) relies heavily on theoretical and experimental studies of convection. It is safe to say that these various studies have engendered many controversies between those with conflicting viewpoints, few of which have yet been satisfactorily resolved. This paper is concerned with one such controversy, but before describing that in detail, it may be useful to give a brief review for the general reader of the various issues at stake. General reviews of mantle convection have been given by Oxburgh and Turcotte [39], Turcotte [55], and Boss [2]. The recent paper by Loper [28] also contains a useful review of the present state of the subject, although from a somewhat different viewpoint.

An early and very successful paper was that of Turcotte and Oxburgh [57], who carried out a boundary-layer analysis of two-dimensional steady-state convection of a constant-viscosity (isoviscous) fluid, with free-slip top and bottom (as is appropriate for the mantle). They showed that when the Rayleigh number Ra is large (and the Prandtl number σ is infinite), the flow consists of an isothermal interior surrounded by thermal boundary layers. They identified the cold near-surface boundary layer with the earth's lithosphere (the upper 100 kilometers or so, over which the temperature jumps from 0°C to about 1300°C), and found that their estimated mean surface velocity compared well with observed lithosphere plate velocities in the range 1–10 cm/year. By taking the horizontal velocity u to be constant, they showed that the upper boundary layer admits a similarity solution, in which the heat flow q decreases with increasing distance x from the upwelling as $q \propto 1/\sqrt{x}$. This is the well-known square-root-of-age law for heat flux, which was shown to agree well with ocean-floor data for ages up to 70 millions years [41]. The convection model thus presents an analogy for mantle convection beneath (active) oceanic lithosphere, which identifies midoceanic ridges with upwelling regions, and oceanic trenches (subduction zones, where the lithosphere plunges into the mantle) with downwellings. There is no obvious application to convection beneath continents.

The apparent relation between heat flux (and, via isostasy, ocean floor topography) and the square root of age provided the starting point for various explanations of why there is a deviation from this relationship for ages greater than 70 million years: the heat flux (and topography) flattens out rather than continuing to decline.

Parsons and McKenzie [40] proposed that "small-scale convection" could occur beneath the lithosphere. They based their analysis on a conceptual model of thermal turbulence due to Howard [22], and some experiments of Curlet in tanks of molten glass (e.g. [8]). This suggestion has generated substantial controversy (e.g. [58]), which still continues (e.g. [11]). However, it is not even clear that the conceptual basis of Parsons and McKenzie's notion is valid.

An alternative explanation for the flattening of old ocean floor was put forward by Schubert and coworkers [49, 61]; this is that viscous heating (i.e. due to mechanical dissipation) could cause a warming at the base of the lithosphere, which would counteract the cooling with age. The analysis was based on solving the boundary-layer equations of convection (analogously to [57]), but with the inclusion of a more realistic rheology law and also including dissipative terms in the energy equation. With realistic earthlike parameters, they found that viscous dissipation could provide a plausible explanation of the observed deviations.

Many other explanations have been offered as well; for example, Jarvis and Peltier [24] suggest that some radioactive heating in the energy equation can bend the heat-flux-versus-age curve in the right direction. In this plethora of activity and debate, some of the more fundamental aspects of the problem seem to have been ignored.

Firstly, Turcotte and Oxburgh's analysis of the boundary layer assumes a constant horizontal velocity; secondly, they suppose a similarity solution is actually consistent with the other parts of the flow and temperature fields. For free boundaries and an isoviscous fluid, *neither* of these assumptions is true. The complete analysis was done by Roberts [48]. For free boundaries, the thermal boundary layer is continuous around the cell periphery, and the steady-state requirement is that the temperature in the boundary layer be periodic (in distance s around the periphery).

The constant-velocity assumption can be ameliorated by an appeal to the "rigidity" of the surface lithospheric plates. This rigidity is due to the well-understood fact that solid-state creep is thermally activated, and the effective viscosity, rather than being constant, actually depends *very* strongly on temperature (and pressure) [16, 26]: Therefore, a cold boundary layer will behave essentially rigidly. However, if one appeals to a temperature-dependent viscosity (thermoviscous fluid) to justify the assumption of constant surface velocity, one must also, for consistency, adopt a thermoviscous rheology in the field equations. This is what Schubert et al. [49, 51] did. However, the boundary-layer equations in that case do not even *have* a similarity solution, so that such similarity as appears must be a local, small-age effect, as Yuen et al. [61] indeed found.

The introduction of a thermoviscous rheology, and of the dissipative heating terms in the energy equation, leads to an interesting feedback between the flow and temperature fields. Viscous heating tends to increase the temperature: in turn this decreases the viscosity. If the flow is driven by a given stress τ, then the viscous heating term τ^2/η (where η is viscosity) increases as the viscosity decreases, so a positive feedback exists. For an exponential dependence of viscosity on temperature, this leads to an explosive local growth of the temperature in a finite time, a phenomenon which is termed thermal runaway. It was studied by Gruntfest [17], and considered in the geophysical context by Rice [45] and Nitsan [36]. Notice that if the strain rate $\dot{e}$ rather than the stress is prescribed, then the feedback is negative, since $\tau^2/\eta = \eta\dot{e}^2$, and one would not expect runaway to occur.

In a sequence of papers Yuen and Schubert [59, 60, 50] and Melosh and Ebel [31, 32] studied the possibility of thermal runaway associated with multiple steady states for the temperature field below the *continental* lithosphere, based on an earlier model of Froidevaux and Schubert [15]. The difference between the oceanic and continental models lies in the neglect of advection of temperature in the latter, on the basis that one can have "steady" (x-independent) temperature and velocity profiles. One imagines the continental lithosphere being rafted over a passive underlying mantle, far from oceanic ridges or trenches, as for example in the Antarctic plate.

Yuen and Schubert found that runaway, associated with multiple steady states, could realistically occur for the earth, but that the stability of the different states depended on whether a surface velocity or surface shear stress was imposed. (In reality, since the surface shear stress ought to be zero, this can be taken to mean a shear stress prescribed at depth, near where deformation begins to occur.)

It seems fair to say that the result of these various studies was rather inconclusive, for two reasons. Firstly, no clear idea of how the various "boundary layer" models exist as subunits of the total process is evident, whereas one knows that, in general, boundary layers and their adjoining interiors are intimately connected via the process of joining the matched asymptotic expansions. Secondly, even if one accepts the equations as representing an appropriate approximation, the physical meaning or implication of multiple steady states is not readily apparent: would such a runaway be manifested as a local phenomenon, or would it lead to catastrophic rearrangement of the whole convection pattern? It may be worthwhile to study these questions a little further, in order to elucidate what may occur, particularly since both local phenomena (such as magmatism) and global phenomena (the breakup of Pangaea) are amongst those which are not fully understood, and which *might* bear some relevance to this topic.

The study of thermal runaway naturally divides into two situations. The first is that studied by Yuen and other workers, of passive flow beneath the *continental* lithosphere. A mathematical study of this situation exists in the thesis by Ryan [47]. The second situation is that beneath the oceanic lithosphere. The topic of thermal runaway was not considered by the previously mentioned authors in this case, for one thing because the presence of advective terms renders the problem two-dimensional, and correspondingly less tractable; however, it naturally presents itself for consideration, in view of the finding that viscous heating can be important beneath the oceanic lithosphere [51]. The present paper will therefore address the issue of possible thermal runaway beneath the oceanic lithosphere, and its potential physical meaning.

2. Geophysical context

In essence, our aim is to solve (approximately) the (boundary layer) equations studied by Schubert et al. [51], whose numerical solutions represent only a locally self-similar approximation near an oceanic ridge. Before attempting this, however, we wish to examine in what way these equations may emerge as a boundary-layer approximation. To do this, it is first necessary to discuss what is currently known about convection at high Rayleigh number of a thermoviscous fluid.

The early boundary-layer analysis, for isoviscous fluids, of Turcotte and Oxburgh [57] was followed by detailed numerical studies by Moore and Weiss [33] and McKenzie et al. [30]. We have already mentioned the temperature dependence of viscosity. All experimental measurements of the thermal activation energy indicate an extremely large dependence of viscosity on temperature, and one should therefore consider the possibility that thermoviscous convection might be fundamentally different in nature to isoviscous convection.

Early attempts to study thermoviscous convection (e.g. [54]) were hampered by the difficulty of obtaining solutions with very large viscosity variations, and also by the fact that the computed flows apparently bore less resemblance to active (oceanic) plate tectonics than the isoviscous computations. This helped prolong

the life span of constant-viscosity calculations (e.g. [24, 20]), but it has now become practical to do serious computations of two-dimensional convection at high Rayleigh number and quite large activation energies, following the efforts of Christensen, Yuen, and coworkers [5, 11, 43, 44]. These have been supplemented by laboratory experiments [35] and by theoretical analyses [34, 14], so that a coherent notion of how thermoviscous convection works has now emerged.

The primary feature of the flow is that the cold boundary layer at the surface is so rigid that it will not deform, and sits as a *stagnant* lid on top of a vigorous convection beneath. The fact that it is buoyantly unstable is far outweighed by its extreme reluctance to move. Conversely, large temperature excesses would lead to very small viscosity, and therefore rapid movement. To prevent this the flow only allows a small temperature jump at its base. Thus the temperature profile becomes very asymmetric.

(Incidentally, the fact that a large temperature increase means a very large viscosity decrease has been used by a number of authors as persuasive evidence against the possibility of layered mantle convection [52, 25, 9]: the idea is that a layered convection system would have a pronounced thermal boundary layer at the lower–upper-layer interface; the lower mantle would then have an extremely low viscosity, with accompanying incompatibility with estimates of viscosity based on postglacial rebound, etc., etc. Such an interfacial boundary layer can easily be observed for (essentially) constant-viscosity fluids [46]. The supposition of an interfacial boundary layer in a strongly thermoviscous fluid is an assumption based on constant-viscosity experience. Since no one has yet performed relevant experiments (laboratory or numerical), nor actually done the boundary-layer theory, this supposition remains hypothetical. It is also probably wrong, as the following simple argument suggests.

The Nusselt number N for a strongly thermoviscous fluid is given by [14]

$$N \sim \varepsilon \mathrm{Ra}^{1/5}, \tag{2.1}$$

where ε is the dimensionless inverse activation energy, and Ra is the Rayleigh number *based on the interior viscosity of the convecting layer*. If one supposes that a layered convecting thermoviscous fluid has two "lithospheres" (i.e. each convecting layer resembles a single layer in its own right), then the heat flux through the upper and lower layers will be

$$N \sim \varepsilon_l \mathrm{Ra}_l^{1/5} \sim \varepsilon_u \mathrm{Ra}_u^{1/5}, \tag{2.2}$$

where l and u mean lower and upper, respectively. We can take $\varepsilon_l \approx \varepsilon_u$. The principal difference between Ra_l and Ra_u is in the viscosity. Since (in a steady state) the heat flux through each layer is the same, we require $\mathrm{Ra}_u \sim \mathrm{Ra}_l$, which can only be the case if the two viscosities are of the same order. The implication is that the temperature jump at the interface is small, compatible with a small decrease of viscosity.)

The fact that thermoviscous convection occurs with a stagnant lid means that the earth's mantle is no longer simulated—the plates do not move. However, convection in other planets (e.g. Mars, the Moon, perhaps Venus) can be understood on this basis. Perhaps subcontinental convection on the earth can as well. It then transpires that the subducting oceanic style of convection, which at first seemed so cosily similar to laboratory convection, manifests itself as an extreme problem, at least from the fluid-mechanical point of view. *There is no reason to suppose that the rigid lithosphere would wish to subduct at all.* This statement is unaffected by other considerations, such as the transformation of basalt to denser eclogite in the lithosphere. Now, one can appeal to the fact that below about 700°C, creep is so slow that other effects (elastic, plastic) are of greater importance. While this is true, it is not obvious why a stagnant lid, the top of which is elastic/plastic, would still not exist overlying a vigorous convection zone.

In short, the process of subduction and its initiation is a problem which has still not been resolved. Differing suggestions were offered by Turcotte et al. [56], by McKenzie [29], and more recently by Cloetingh et al. [62]. In the present paper, we are not explicitly concerned with the cause of subduction. What *is* important is that when subduction occurs, it will have a controlling effect on the flow, virtually irrespective of the rest of the dynamics.

To understand this further, let us describe the convective flow of a thermoviscous fluid in the absence of subduction. As mentioned, there is a thick, cold, stagnant lid at the top of the cell. This lid is of variable thickness, and the bulk of the temperature drop across the cell is taken up by it. The flow below is analogous to constant-viscosity convection, having an isothermal (hence isoviscous) interior which is driven by vorticity generated by (weak) buoyancy in the upwelling and downwelling plumes. The temperature difference in these plumes is *small* (so that the viscosity change from the interior is small). Connecting the stagnant lid to the isothermal interior is a shear layer in which the temperature is adjusted. (This is the analogue of Parsons and McKenzie's [40] thermal (as opposed to mechanical) boundary layer beneath the lithosphere.) In this shear layer, the shear stress is larger than in the interior or in the plumes, the temperature jump is small [enough to cause an $O(1)$ change in viscosity], and the velocity is small. Extremely importantly, and unlike constant-viscosity convection with free-slip boundary conditions, this shear layer is *self-determining*, independently of the details of the rest of the flow.

Above the shear layer is the stagnant lid. Because the temperature drop across this lid is so large, and because it is of variable thickness, the buoyancy term in the momentum equation is very large, and so a very large (shear) stress is generated within the lid. An even larger longitudinal stress occurs in a thin skin at the top of the lid, whose function is to enable shear stress to decrease to zero at the top surface.

Thus the dominant features of the system are the thick lid, with its large stresses, and the adjoining shear-thermal layer. These parts of the flow are essentially self-controlled. Now suppose that some unspecified mechanism weakens the lithosphere at its margin (above the downwelling) so that subduction occurs. Evidently, the effect on the interior will be drastic. The negative buoyancy of the downgoing slab is much larger than that produced by any upward plume, so that it is plausible that the flow will be locally controlled by the slab (and so also its attached plate).

Since the theoretical explanation of subduction is not resolved, an asymptotic theory for thermoviscous convection with active plates does not exist, and so we cannot deductively establish the appropriate scales for the flow. In what follows, we will examine two possible alternatives. The first of these is a "whole-mantle" type of flow, in which we suppose that the descending slab penetrates to near the core-mantle boundary [7], *and* that the interior viscosity is relatively constant (more on this below).

The second possibility is a "shallow-mantle" type of flow, in which, *even if the slab penetrates to great depth*, the bulk of the return flow occurs at shallow levels. (This is rather different to the usual notion of strictly layered convection.) Such a shallow return flow implies a sublithospheric lubrication flow, and as we shall see, a shear layer (the asthenosphere) just beneath the lithosphere. Such a shear layer was found numerically by Schubert et al. [51].

The reason this second possibility may occur is that viscosity increases with pressure (i.e. depth). For typical values of activation energy and activation volume, one finds that (assuming an adiabatic temperature) viscosity will increase with depth. Consequently, the interior flow may be more sluggish. Depending on the degree of increase, the flow may preferentially concentrate in regions of lower viscosity. Simply for this reason, one may obtain a relatively shallow return flow. In this case, the large slab stress may not be felt beneath the lithosphere, and the flow *there* may just be driven by the shear stress generated across the lithosphere [18, 19]. Any excess stress in the slab might then determine the rate of trench migration, which in general will be nonzero.

3. Mathematical model

We wish to study a model of two-dimensional convection in a rectangular cell, of a fluid whose viscosity depends on temperature and pressure (and also stress). Appropriate equations for this situation are

$$
\begin{aligned}
u_x + v_y &= 0,\\
p_x &= \tau_{1x} + \tau_{2y},\\
p_y &= \tau_{2x} - \tau_{1y} - \rho g,\\
\rho &= \rho_0[1-\alpha T],\\
\tau_1 &= 2\eta u_x,\\
\tau_2 &= \eta(u_y + v_x),\\
\rho_0 c_p \frac{dT}{dt} - \alpha T \frac{dp}{dt} &= k\nabla^2 T + \frac{\tau^2}{\eta},\\
\tau^2 &= \tau_1^2 + \tau_2^2,\\
\eta &= \frac{A}{\tau^{n-1}} \exp\left[\frac{E^* + pV^*}{RT}\right].
\end{aligned}
\tag{3.1}
$$

In these equations, cartesian axes are (x, y) (y pointing vertically upwards), velocity is (u, v), p is the pressure, τ_1 and τ_2 are the stress deviator components ($\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$, $\tau_{11} = -\tau_{22} = \tau_1$, $\tau_{12} = \tau_{21} = \tau_2$), ρ is the density, T is the temperature, g is gravity, α is the thermal expansion coefficient, η is the viscosity, c_p is the specific heat, k is the thermal conductivity, E^* and V^* are activation energy and volume, respectively, and R is the gas constant. In (3.1), we have already adopted the Boussinesq approximation, meaning here exactly that the dimensionless parameter $\alpha\Delta T$ is small, and is therefore neglected (here ΔT is an appropriate temperature scale). Various other assumptions are made, notably of zero isothermal compressibility ($\partial\rho/\partial p = 0$), zero radiogenic heating, infinite Prandtl number, and also constant g, α, c_p, k. The functional form chosen for η is of a common type, appropriate for thermally activated dislocation creep [16].

Boundary conditions are as follows. Suppose $y = 0$ is a reference level, e.g. the ocean surface. The lithosphere surface is $y = -w(x)$, where w is the ocean depth. If w is small enough, we linearize about $y = 0$, to find appropriate conditions

$$T = T_0, \quad v = 0, \quad \tau_2 = 0, \quad p + \tau_1 = \rho_w g w(x) \tag{3.2}$$

at $y = 0$, where ρ_w is the density of seawater. These conditions represent prescribed temperature, continuity of stress, and the kinematic condition. Other boundary conditions will be discussed later.

The equations are now made dimensionless in the following way:

$$\begin{aligned} T &= T_a\theta, \\ x &= lx^*, \quad y = -dy^*, \\ u &= Uu^*, \quad v = -\frac{Ud}{l}v^*, \quad t = \frac{l}{U}t^*, \\ \tau_2 &= [\tau]\tau_2^*, \quad \tau_1 = \frac{d}{l}[\tau]\tau_1^*, \\ p &= \rho_0 g d(1 - y^*) + \frac{l}{d}[\tau]p^*, \\ \eta &= \eta_a\eta^*, \end{aligned} \tag{3.3}$$

where the various scales will be specified in due course. Notice that y^* and v^* now point downwards. The Boussinesq parameter is defined as

$$\Delta = \alpha T_a, \tag{3.4}$$

and the assumption in writing (3.1) was that $\Delta \ll 1$. Substituting (3.3) into (3.1),

we find

$$\begin{aligned} u^*_{x^*} + v^*_{y^*} &= 0, \\ p^*_{x^*} &= \nu^2\tau^*_{1x^*} - \tau^*_{2y^*}, \\ -p^*_{y^*} &= \nu^2[\tau^*_{2x^*} + \tau^*_{1y^*}] + \mu_1(\theta - \theta_{ad}), \\ \mu_3\tau_1^* &= 2\eta^* u_{x^*} \\ \mu_3\tau_2^* &= -\eta^*(u^*_{y^*} + \nu^2 v^*_{x^*}), \\ \eta^* &= \frac{1}{\tau^{*n-1}}\exp\left[\frac{1 - \theta + \mu y^*}{\varepsilon\theta}\right], \\ \tau^* &= \{\tau_2^{*2} + \nu^2\tau_1^{*2}\}^{1/2}, \\ \frac{d\theta}{dt^*} - Dv^*\theta &= \mu_2[\theta_{y^*y^*} + \nu^2\theta_{x^*x^*}] + \frac{\beta\tau^{*2}}{\eta^*}. \end{aligned} \tag{3.5}$$

The material derivative is $d/dt^* = \partial/\partial t^* + \mathbf{u}^*\cdot\nabla^*$, and the parameters in (3.5) are the following:

$$\begin{aligned} \nu &= d/l, \\ \varepsilon &= RT_a/E^*, \\ \mu &= \rho_0 g d V^*/E^*, \\ D &= \alpha g d/c_p, \\ \beta &= \frac{[\tau]^2 l}{\eta_a\rho_0 c_p U T_a}, \\ \mu_1 &= \frac{\alpha T_a \rho_0 g d^2}{l[\tau]}, \\ \mu_2 &= kl/Ud^2, \\ \mu_3 &= \frac{d[\tau]}{\eta_a U}. \end{aligned} \tag{3.6}$$

(Here $\kappa = k/\rho_0 c_p$ is the thermal diffusivity.) We have chosen

$$\eta_a = \frac{A}{[\tau]^{n-1}}\exp\left(\frac{1}{\varepsilon}\right). \tag{3.7}$$

The adiabatic temperature θ_{ad} is

$$\theta_{ad} = e^{Dy^*}; \tag{3.8}$$

strictly, its introduction into $(3.5)_3$ requires the lithostatic pressure in (3.3) to be $\rho_0 gd[1-y^*+O(\Delta)]$. The $O(\Delta)$ term can, however, be ignored, as we have done.

It remains to choose the scales T_a, U, d, l, and $[\tau]$. The temperature scale must be determined by connection to the rest of the flow. We expect that the temperature should be adiabatic for large y^*, and by choice of T_a, we can ensure that the particular adiabat is θ_{ad}. This means that T_a is essentially the asthenospheric temperature, $T_a \approx 1500$ K. The length scale l is given, and can be taken as a representative plate length (1000–10,000 km) or comparable to the depth, that is, $l \sim 3000$ km. There remain U, d, and $[\tau]$, or equivalently μ_1, μ_2, μ_3.

We choose these scales in order to focus on the lithosphere boundary layer. Anticipating that $\nu \ll 1$ $(d \ll l)$, balance of τ_2^*, p^*, and θ in $(3.5)_2$ suggests

$$\mu_1 \sim 1; \tag{3.9}$$

balance of advection and conduction in $(3.5)_8$ suggests

$$\mu_2 \sim 1; \tag{3.10}$$

finally, the choice of μ_3 depends on the assumption of whether there is or is not an asthenospheric shear layer. If so, then $\mu_3 \sim 1$, but if not, then $\mu_3 \ll 1$. Precise specification of μ_3, and also of μ_1 and μ_2, is postponed to the following sections. It may, however, be of some use to compute representative values of the parameters, based on observed (or inferred) values:

$$\begin{aligned} U &\approx 3\ \text{cm}_y^{-1}, \\ [\tau] &\approx 10\ \text{bars}, \\ l &\approx 3000\ \text{km}, \\ d &\approx 100\ \text{km}; \end{aligned} \tag{3.11}$$

with other standard values $R \approx 8.3$ J mole^{-1}K^{-1}, $T_a \approx 1500$ K, $E^* \approx 122$ kcal mole^{-1} (1 cal = 4.2 J), $\alpha \approx 3\times 10^{-5}$ K^{-1}, $g \approx 10$ m s^{-2}, $c_p \approx 0.27$ cal g^{-1} K^{-1}, $\rho_0 \approx 3.5$ g cm^{-3}, $\kappa \approx 10^{-2}$ cm^2 s^{-1}, $V^* \approx 11$ cm^3 mole^{-1}, $\eta_a \approx 10^{21}$ poise, we find

$$\begin{aligned} \nu &\approx \tfrac{1}{30}, \\ \varepsilon &\approx 0.024, \\ \mu &\approx 0.075, \\ D &\approx 0.026, \\ \beta &\approx 0.005, \\ \mu_2 &\approx 0.3, \\ \mu_3 &\approx 1, \\ \mu_1 &\approx 5.25. \end{aligned} \tag{3.12}$$

We see that μ_1, μ_2, μ_3 are reasonably ~ 1, and that $\nu, \varepsilon, \mu, D, \beta$ are all small.

We now simplify the equations (3.5) a little. First, we drop the asterisks. Define a stream function Ψ by

$$u = \Psi_y, \qquad v = -\Psi_x. \tag{3.13}$$

Define a potential temperature Φ by

$$\theta = \theta_{ad}\Phi = e^{Dy}\Phi. \tag{3.14}$$

Neglecting terms of $O(D^2/\varepsilon)$ and $O(\mu D/\varepsilon)$, the viscosity is then given by

$$\eta = \frac{1}{\tau^{n-1}} \exp\left[\frac{1-\Phi+\varepsilon G y}{\varepsilon\Phi}\right], \tag{3.15}$$

where

$$G = \frac{\mu - D}{\varepsilon} \sim 2, \tag{3.16}$$

on the basis of (3.6) and (3.11). We now neglect terms of $O(\nu^2)$ and of $O(D)$ (except in the viscous heating term) in the equations (3.5). Using (3.15) and (3.13), we then find the simplified set

$$\begin{aligned} \tau_{yy} &= \mu_1 \Phi_x, \\ \Psi_{yy} &= -\mu_3 |\tau|^{n-1}\tau \exp\left[-\left\{\frac{1-\Phi+\varepsilon G y}{\varepsilon\Phi}\right\}\right], \\ \Psi_y\Phi_x - \Psi_x\Phi_y &= \mu_2 \Phi_{yy} + \varepsilon\mu_4 |\tau|^{n+1} \exp\left[-\left\{\frac{1-\Phi+\varepsilon G y}{\varepsilon\Phi}\right\}\right], \end{aligned} \tag{3.17}$$

where we have defined

$$\mu_4 = \frac{\beta}{\varepsilon} = \frac{\mu_3 D}{\mu_1 \varepsilon} \tag{3.18}$$

for reasons which will be made clear in Section 4. We have written $\tau_2 = \tau$. From (3.12), a typical value of μ_4 is 0.2.

The above set (3.17) is to be solved subject to μ_1, μ_2 "near" to 1, $\mu_3 \sim 1$ or $\ll 1$, and with the boundary conditions

$$\Phi = \Phi_0, \quad \Psi = 0, \quad \tau = 0 \quad \text{on} \quad y = 0. \tag{3.19}$$

As $y \to \infty$, the solutions must match to the interior solutions. These depend on what is assumed about G. If G is neglected (isoviscous interior), then we expect "whole-mantle" convection and consequently

$$\Phi \to 1, \qquad \Psi \sim u^* y, \tag{3.20}$$

where u^* is (in principle) determined from the interior flow. However, if $G \sim 1$, $\Phi \to 1$ is not obviously consistent with (3.17), and the possible limiting forms of the variables will have to be considered further in Section 5.

4. Analysis (i): $G \ll 1$, "whole mantle"

Following the discussion of Section 3, the assumption that $G \ll 1$, and concomitant vigorous whole-mantle convection, will require $\mu_3 \ll 1$ in (3.17), in order that Ψ and τ will balance in the interior. Let us suppose that $y \sim \Lambda$ is the depth scale of the mantle: $\Lambda \gg 1$ (e.g. $\Lambda \sim 30$). Matching of Ψ to the interior will require $\Psi \sim \Lambda$ when $y \sim \Lambda$. Now if the interior sees a free-slip upper boundary (i.e. if $\tau \sim \Lambda$ in the interior), then the surface velocity u^* will be determined from the interior as a function of x, which will in turn create enormous longitudinal stresses. This seems unlikely, and observations do not suggest much variation in u^*. Therefore, this suggests that the interior will see an $O(1)$ stress and a prescribed velocity: that is, the third boundary condition for (3.17) to supplement (3.20) would be

$$\tau \to \tau^*(x). \tag{4.1}$$

At any rate, this requires that we choose

$$\mu_3 = 1/\Lambda. \tag{4.2}$$

Neither this, nor the supposition that G is small, is compatible with actual numerical estimates of the parameters in the earth. Nevertheless, we now turn to a solution of (3.17), with $\mu_3 \ll 1$, but μ_1 and μ_2 (yet to be specified) of "approximate" order 1. The solution is sought under the assumption $\varepsilon \ll 1$.

(a) Lithosphere

Let us first assume $\partial/\partial t = 0$. The lithosphere is where the exponential terms in (3.17) are negligible, $\Phi < 1$. We put

$$\mu_1 = 1/\mu_2; \tag{4.3}$$

neglecting the exponentials, we have (to leading order)

$$\begin{aligned} \Psi &\sim u^* y, \\ u^*\Phi_x &= \mu_2 \Phi_{yy}, \\ \tau_{yy} &= \Phi_x/\mu_2; \end{aligned} \tag{4.4}$$

it follows that

$$\tau = \frac{\Phi - \Phi_0}{u^*} + By. \tag{4.5}$$

Evidently, $(4.4)_2$ admits a similarity solution, and presumably this is what is appropriate [42]. The similarity variable is

$$\eta = \tfrac{1}{2} y \left(\frac{u^*}{\mu_2 x} \right)^{1/2}, \tag{4.6}$$

and, generally, we seek a solution in the form

$$\Phi \sim m_0(\eta) + x m_1(\eta) + \cdots. \tag{4.7}$$

The functions m_r satisfy

$$m_r'' + 2\eta m_r' = 4 r m_r, \tag{4.8}$$

with solutions satisfying $\Phi = \Phi_0$ on $\eta = 0$,

$$\begin{aligned} m_0 &= \Phi_0 + A_0 \operatorname{erf}(\eta), \\ m_r &= A_r\{ i^{2r}\operatorname{erfc}(\eta) - i^{2r}\operatorname{erfc}(-\eta)\}, \qquad r = 1,2,\ldots, \end{aligned} \tag{4.9}$$

where A_j are constants; the repeated integrals $i^{2r}\operatorname{erfc}(\eta)$ are defined by Abramowitz and Stegun [1, p. 299].

This solution, of similarity type, becomes invalid when $1 - \Phi \sim \varepsilon$. For sufficiently small x, we concentrate on m_0. To match to the region below the lithosphere, we expect $A_0 \sim 1 - \Phi_0$, so that $1 - \Phi \sim \varepsilon$ when $\eta \gg 1$, i.e. $y \gg \mu_2$. It is because we wish the lithosphere thickness to be ~ 1 that we *formally* define μ_2 so that $1 - m_0 \sim \varepsilon$ when $y \sim 1$. Specifically,

$$m_0 \sim \Phi_0 + A_0\left[1 - \frac{2}{\sqrt{\pi}} \frac{e^{-\eta^2}}{(2\eta)} \left\{1 + O\left(\frac{1}{\eta^2}\right)\right\}\right] \quad \text{as} \quad \eta \to \infty, \tag{4.10}$$

and putting

$$A_0 = 1 - \Phi_0, \tag{4.11}$$

we have

$$m_0 \sim 1 - \frac{(1-\Phi_0)}{\sqrt{\pi}} e^{-\eta^2}\left[\frac{1}{\eta} + \cdots\right], \tag{4.12}$$

i.e.

$$1 - m_0 \sim \varepsilon \quad \text{when} \quad e^{-\eta^2}/\eta \sim \varepsilon. \tag{4.13}$$

Define

$$e^{-\eta^{*2}}/\eta^* = \varepsilon. \tag{4.14}$$

Thus

$$\eta^* \sim [\ln(1/\varepsilon)]^{1/2}. \tag{4.15}$$

Consequently define

$$\mu_2^{1/2} = \frac{1}{\eta^*} \sim \left[\frac{1}{\ln(1/\varepsilon)}\right]^{1/2}. \tag{4.16}$$

Then $\Phi = 1 + O(\varepsilon)$ for $y \approx y^* = 2(x/u^*)^{1/2}$, and near this lithosphere base, there is a (formally) thin transition region, which we shall call the asthenosphere. Notice that for $\varepsilon \approx \frac{1}{40}$, $[\ln(1/\varepsilon)]^{1/2} \approx 1.8$. Obviously, this is not very large; nevertheless, we may expect the analysis to be qualitatively valid.

(b) Asthenosphere

At the base of the lithosphere, we put

$$y = y^* + \nu Y,$$
$$\Phi = 1 + \varepsilon\phi. \tag{4.17}$$

The small parameter ν (to be chosen) defined here is not to be confused with use of the same symbol in other sections for the reciprocal of the aspect ratio. In the transition region, we choose

$$\nu = \mu_2 \sim \frac{1}{\ln(1/\varepsilon)}, \tag{4.18}$$

and find, neglecting $O(\varepsilon)$,

$$\nu u^* \phi_x = \phi_{YY} + u^* y^{*\prime} \phi_Y + \nu\mu_4 |\tau|^{n+1} e^{\phi}, \tag{4.19}$$
$$\tau_{YY} = \varepsilon\nu\phi_x.$$

It follows from the second of these that, in order to match through the asthenosphere to an interior in which $\tau \sim 1$, we require

$$B = 0 \tag{4.20}$$

in (4.5), and consequently

$$\tau^* = \frac{1 - \Phi_0}{u^*}, \tag{4.21}$$

which determines asthenospheric stress level.

Now

$$\eta = \eta^* + \frac{1}{2\eta^*} Y\left(\frac{u^*}{x}\right)^{1/2}: \tag{4.22}$$

therefore the matching condition on ϕ as $Y \to -\infty$ can be determined from (4.7). For large η, this gives

$$\Phi \sim 1 - \frac{1 - \Phi_0}{\sqrt{\pi}} e^{-\eta^2}\left(\frac{1}{\eta} + \cdots\right) - A_1 x(\eta^2 + \cdots) - A_2 x^2\left(\frac{\eta^4}{12} + \cdots\right)\cdots, \tag{4.23}$$

which suggests

$$A_1 = \varepsilon\nu a_1,$$
$$A_2 = \varepsilon\nu^2 a_2, \tag{4.24}$$

etc. Then using (4.22) and $(4.17)_2$, we can write (4.23) as

$$\phi \sim -\frac{1 - \Phi_0}{\sqrt{\pi}} \exp\left[-\left(\frac{u^*}{x}\right)^{1/2} Y\right] - \left\{a_1 x + \frac{a_2 x^2}{12} + \cdots\right\} \tag{4.25}$$

plus higher-order terms. This is the matching condition for the solution of (4.19) to $O(\nu)$.

A simplification ensues by using the asthenospheric similarity variable

$$\zeta^* = \left(\frac{u^*}{x}\right)^{1/2} Y. \tag{4.26}$$

Then (4.19) takes the form (neglecting a further term of order ν)

$$\nu x \phi_x = \phi_{\zeta^*\zeta^*} + \phi_{\zeta^*} + s(x) e^{\phi}, \tag{4.27}$$

where

$$s = \frac{\nu\mu_4 x |\tau^*|^{n+1}}{u^*}, \tag{4.28}$$

and ϕ satisfies

$$\phi \sim b(x) - \frac{1-\Phi_0}{\sqrt{\pi}} e^{-\zeta^*} \quad \text{as} \quad \zeta^* \to -\infty. \tag{4.29}$$

The purpose of retaining the otherwise negligible $\nu x \phi_x$ term in (6.32) will appear below. The Taylor coefficients of $b(x)$ (assumed regular at $x = 0$) will determine the a_i, and hence A_1 in (4.10). Evidently the preexponential constant in (4.29) can be absorbed in the location of the origin of ζ^*, and we therefore ignore it henceforth: that is, put

$$\zeta = \zeta^* - \ln\left(\frac{1-\Phi_0}{\sqrt{\pi}}\right), \tag{4.30}$$

so that we can drop the asterisks in (4.27), and

$$\phi \sim b(x) - e^{-\zeta} \quad \text{as} \quad \zeta \to -\infty. \tag{4.31}$$

At leading order in ν, we drop the x-derivative term in (4.27); ϕ then satisfies the quasistationary equation

$$\phi_{\zeta\zeta} + \phi_\zeta + s e^{\zeta} = 0, \tag{4.32}$$

with boundary condition at $-\infty$. For given b and s, the solutions are easily studied in the phase plane. We put

$$\phi = -\ln s + \psi; \tag{4.33}$$

then a typical phase plot of ψ versus ψ_ζ is shown in Figure 1. Typical graphs of ψ and e^{ψ} versus ζ are shown in Figure 2. In particular,

$$\phi = -\ln \zeta - \ln s + o(1) \tag{4.34}$$

as $\zeta \to \infty$ for each b and s.

(c) Mesosphere

If we assume $\mu_4 \sim 1/\nu$, then $s \sim 1$, and the solution in Section 4(b) becomes large and negative when $\zeta \to \infty$. When $\zeta \sim 1/\nu$ $(Y \sim 1/\nu)$, then $\phi \sim -\ln(1/\nu)$, so $e^{\phi} \sim \nu$, and the balance of terms in (4.27) breaks down. There is thus a further part of the thermal boundary layer in which $y - y^* = O(1)$, and ϕ adjusts to its deep mantle (interior) value, at $y \to \infty$.

With

$$\phi = -\ln(1/\nu) + \tilde{\phi}, \qquad \mu_4 = \bar{\mu}/\nu, \tag{4.35}$$

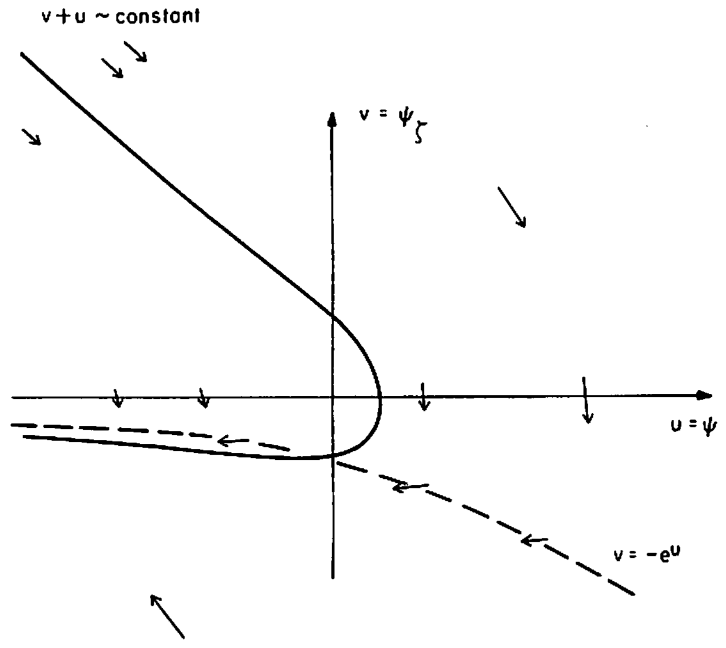

Figure 1. Solution behavior of ψ in the phase plane $\psi = u$, $\psi_\zeta = v$.

$(3.17)_3$ can be written

$$u^* \tilde{\phi}_x = \nu \tilde{\phi}_{YY} + \bar{\mu} |\tau|^{n+1} e^{\tilde{\phi}}. \tag{4.36}$$

At leading order conduction is negligible, and $\tilde{\phi}$ satisfies

$$\tilde{\phi} \sim \ln\left[\frac{u^*}{\bar{\mu}\tau^{*n+1}\{p(y) - x\}}\right] \tag{4.37}$$

for some $p(y)$.

Rewriting (4.34) in terms of y, the matching condition for $\tilde{\phi}$ as $y \to y^*$ is

$$\tilde{\phi} \sim \ln\left[\frac{1}{s(y-y^*)}\right]\left(\frac{x}{u^*}\right)^{1/2}\Bigg]. \tag{4.38}$$

Since $y^* = 2(x/u^*)^{1/2}$, i.e. $x = u^* y^{*2}/4$, we choose

$$p(y) = \tfrac{1}{4} u^* y^2 \tag{4.39}$$

in (4.37), and then (4.37) and (4.38) are automatically identical. The mesosphere

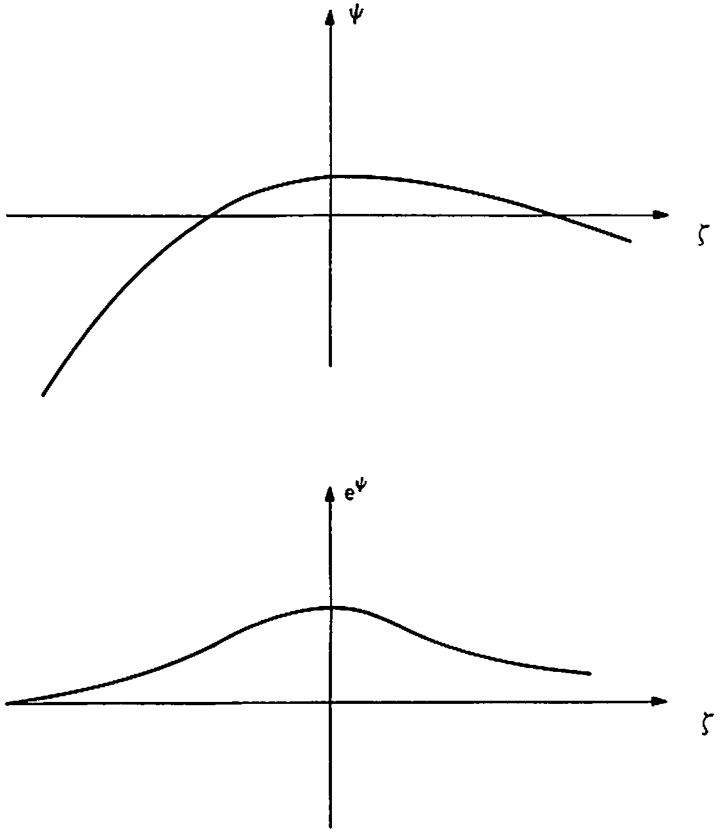

Figure 2. Typical profiles of ψ and e^ψ versus ζ.

solution (4.37) should suffice to match automatically to the deep interior, since no (obvious) further balance of terms arises.

(d) Instability in the asthenosphere

At this point, the basic structure of the solution is complete. It is very similar to the solutions given by Ockendon [38] and Pearson [42]—see Figure 3 in the latter paper in particular. In both these papers, it is conjectured that a thermal runaway will take place due to the exponential source term in the equation, and we make a similar conjecture here: that is, we expect ϕ to reach infinity at a finite value of x. Ockendon [38] exhibits the equations which are assumed to "blow up"; Pearson [42] does not (since he does not examine the region analogous to the asthenosphere), but both authors construct a local similarity solution as the maximum value of ϕ becomes large.

The region where a blowup may occur is the asthenosphere, where ϕ satisfies [to $O(\nu)$]

$$\nu x \phi_x = \phi_{\zeta\zeta} + \phi_\zeta + s e^\phi,$$

$$\phi \sim b - e^{-\zeta}, \quad \zeta \to -\infty,$$

$$\phi \sim -\ln \zeta + C, \quad \zeta \to +\infty. \tag{4.40}$$

If we continue to ignore $\nu x \phi_x$, then the quasistationary solution of (4.40) develops a sharp maximum when s becomes large. In fact, one finds (with an analysis similar to Pearson's [42] one) $\phi \sim -\ln s$ at $\zeta \sim -\ln \ln s$. At this point $\nu x \phi_x$ becomes nonnegligible when $\ln s \sim 1/\nu$, and a runaway might be expected to occur. However, there is a much more dramatic runaway mechanism present: the quasistationary solution of (4.40) is potentially linearly unstable (as is easily seen, since the stability problem can be written as a standard Schrödinger eigenvalue problem). Therefore, any slight x dependence of ϕ may be rapidly amplified and produce an active runaway.

To elucidate this, we put

$$x = x^* e^{\nu\tau}, \tag{4.41}$$

so ϕ satisfies (to leading order)

$$\phi_\tau = \phi_{\zeta\zeta} + \phi_\zeta + s e^\phi, \tag{4.42}$$

where s is a slowly varying function of τ. This equation can be transformed to Gruntfest's [17] runaway problem $\phi_\tau = \phi_{ZZ} + s e^\phi$, with $Z = \zeta + \tau$, and since (4.42) is to be solved on an infinite ζ domain, it is feasible that ϕ will blow up at a finite value of τ. The explosive runaway would occur rapidly, the temperature rising significantly within a distance $\sim \nu$ of the runaway point. One can also consider the equation (4.42) as representing the time-dependent evolution of a disturbance at fixed x. This is because the missing time-derivative term is $\nu \phi_t$, which shows that the asthenosphere responds on a fast time $t \sim \nu$ to an initial (local) disturbance, whereas the other regions are essentially static on this time scale. Runaway would occur in a time of (dimensional) order $\nu[x]/U$, where $[x]$ is distance from a midocean ridge. Obvious candidates for such disturbances would be deep-mantle plumes.

The hypothesis is thus that $\phi \to \infty$ at some ζ for $\tau = \tau_0 \sim O(1)$. In order to verify this, I solved (4.42) numerically. The boundary conditions are those in (4.40), and I took $b = 0$ as an example. The procedure is similar to that described by Hocking et al. [21]. We denote the stationary solution of (4.42) as ϕ_0, and put $\theta = \phi - \phi_0$, so that

$$\theta_\tau = \theta_{\zeta\zeta} + \theta_\zeta + \sigma(\zeta)[e^\theta - 1], \tag{4.43}$$

where

$$\sigma(\zeta) = se^{\phi_0}; \tag{4.44}$$

boundary conditions for θ are

$$\theta \to 0 \quad \text{as} \quad \zeta \to +\infty,$$
$$\theta_\zeta \to 0 \quad \text{as} \quad \zeta \to -\infty. \tag{4.45}$$

I used a Crank-Nicholson method and linearized the nonlinear term by writing

$$e^\theta \approx \tfrac{1}{2}[e^{\theta_{i,j}} + e^{\theta_{i,j+1}}]$$
$$\approx e^{\theta_{i,j}}\left[1 + \tfrac{1}{2}(\theta_{i,j+1} - \theta_{i,j})\right], \tag{4.46}$$

which is consistent with the numerical approximation used. I solved for θ_0 using a Runge-Kutta fourth-order method (with $b = 0$), and then solved the partial differential equation on a finite grid (usually starting with $-5 < \zeta < 5$), with $\theta = 0$ at the end points. Whenever the values of θ next to the boundaries exceeded some small value (usually 10^{-4}), the ζ domain of integration was increased by one in the appropriate direction. The time step k was divided by two every time the maximum value of ϕ (not θ) exceeded some chosen value, generally if $k\exp[\phi_{max}] < 0.05$. In this way one could follow the solution up to maximum values of 10^{12} and beyond, although well before this stage the limited spatial resolution makes the solution not worth pursuing. I generally used a spatial step size of 0.05. Lower values produced quantitatively slightly different, but qualitatively similar results. For initial values, I put $\phi = \bar{\phi}$ in $0 < \zeta < 1$ at $\tau = -3$; sometimes I used $0 < \zeta < 2$. Too low values of ϕ led to initial diffusion and an eventual (slow) rise of ϕ_{max}. Generally, maximum values of ϕ greater than zero heralded the beginning of blowup, lesser values growing very slowly. A typical plot of ϕ_{max} versus τ is given in Figure 3. We tried s values of 0.5, 1, 2; $\bar{\phi}$ values of 0.5, 1, 1.5. All the plots are similar to Figure 3, and we conclude that an ~ 1 initial value of θ in (4.43) leads to an infinite value of ϕ at some finite "time" $\tau = \tau_0$; this singularity occurs at a specific value of ζ, say $\zeta = \zeta_0$.

As $\tau \to \tau_0$, the solution has a local similarity structure

$$\phi \sim -\ln[\tau_0 - \tau] + g(\eta), \tag{4.47}$$

$$\eta = \frac{(\zeta - \zeta_0) + (\tau - \tau_0)}{(\tau_0 - \tau)^{1/2}}, \tag{4.48}$$

where g satisfies

$$g'' - \tfrac{1}{2}\eta g' + se^g - 1 = 0. \tag{4.49}$$

This solution is given by Lacey [27]. For the singularity to occur at $\zeta = \zeta_0$ only, we

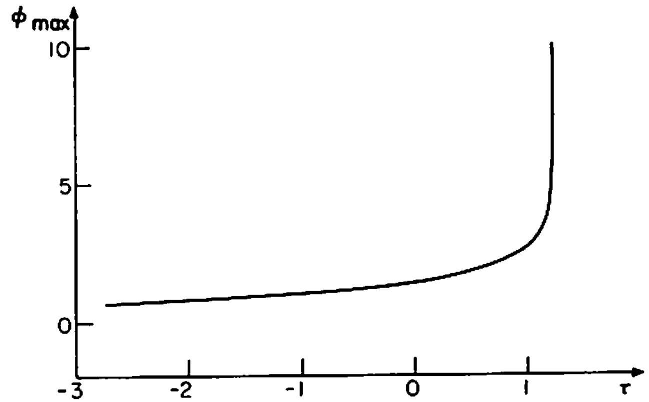

Figure 3. A typical numerical solution of (4.43) with boundary conditions as in (4.45), $b = 0$. We plot the maximum value of ϕ, ϕ_{max}, versus τ. For this particular solution, we took $s = 0.5$, and an "initial" perturbation (at $\tau = -3$) from the steady state, $\phi - \phi_0 = \theta = 1$, $0 < \zeta \le 4$, $= 0$ otherwise. In this instance blowup occurs at $\tau = 1.23$. This figure is typical. Other typical results are for $s = 1$, same initial conditions (blowup at $\tau = -1.50$), and $s = 2$, $\theta = 0.5$ in $0 < \zeta \le 2$, $= 0$ otherwise, at $\tau = -3$, which gives blowup at $\tau = 1.06$. Step size was 0.05 for these results. Calculations were done on an HP-3000 at MIT.

require

$$g \sim -2\ln|\eta|, \qquad \eta \to \pm\infty, \tag{4.50}$$

as is consistent with (4.49); in fact one would need

$$g \sim -2\ln|\eta| - \ln s, \qquad \eta \to \pm\infty. \tag{4.51}$$

Because of the logarithmic dependence on $\tau_0 - \tau$, the runaway is *extremely* precipitous.

(e) Summary

There is an apparent problem with the condition (4.21), which acts as a boundary condition for the interior flow, and that is, that we have *additionally* assumed u^* constant. The apparent resolution is that the interior flow must be solved in the region $y > y^*$, where y^* is unknown. Thus, for given $y^*(x)$, the solution of the interior flow uses the two conditions of no flow (approximately) though y^*, and (4.21), which yields a specific surface velocity of the form

$$u^* = U(x, y^*(x)), \tag{4.52}$$

where U is some determinable (in principle) function. Thus y^* is, in reality, determined by the condition of constant u^*, which yields

$$y^{*\prime}(x) = -\frac{\partial U}{\partial x} \Big/ \frac{\partial U}{\partial y^*}. \tag{4.53}$$

which is to be solved subject to $y^*(0) = 0$. In seeking a quasisimilarity solution $y^* \propto \sqrt{x}$, we are simply assuming that this is consistent (for small x) with (4.53), motivated more by observation than by whether this will actually be the case. However, it should be pointed out that even if η^* in (4.14) is *not* constant, the similarity solution in the lithosphere is still valid, since $\eta^* \gg 1$, and in fact the gist of the analysis is unaltered.

The results of this section are based on the assumption $G \approx 0$ (i.e., $G \ll 1$), and require

$$\mu_3 \sim 1/\Lambda \ll 1,$$
$$\mu_1 \sim 1/\nu,$$
$$\mu_2 \sim \nu \sim \frac{1}{\ln[1/\varepsilon]},$$
$$\mu_4 \sim 1/\nu \tag{4.54}$$

for their validity. The prescription of μ_1, μ_2, and μ_3 is required; that of μ_4 is required if a thermal runaway is to occur before the hot region is washed away.

It may help to rewrite (3.11) as follows: if d, U, τ, l, η are measured in units of 100 km, 3 cm y^{-1}, 10 bars, 3000 km, 10^{21} P, then

$$d \sim 0.8\mu_1\mu_4/\mu_3,$$
$$U \sim 1.2(\mu_2\mu_4/\eta)^{1/2},$$
$$\tau \sim 1.5\frac{\mu_3(\mu_2\eta/\mu_4)^{1/2}}{\mu_1},$$
$$l \sim 2.2\frac{\mu_1^2\mu_4^2(\mu_4\eta/\mu_2)^{1/2}}{\mu_3^3}. \tag{4.55}$$

If $\mu_3 \ll 1$ (e.g. $\Lambda \sim 30$), it is clear from $(4.55)_4$ that μ_4 must be very small indeed, and the prospects for thermal runaway are very remote. Forgetting this aspect, we need $\mu_4 \approx \mu_3$ for $(4.55)_{1,4}$ to make sense. This means U is going to be larger than observed, unless large $O(1)$ factors come into play, or unless η is less than 10^{21} P. If $\mu_2 \sim \frac{1}{4}$, $\mu_4 \sim 10^{-3}$, then $U \sim 2\times10^{-2}/\eta$, which is feasible if $\eta \sim 10^{20}$ P and a factor of ten is inserted. Then $l \approx 4.4\times10^3$, which requires a factor of a thousand less; and so on: the point is that (a) $G \approx 0$ is not suggested by prevalent values of the parameters; (b) the thermal runaway hypothesis is difficult to uphold if $G \approx 0$, (c) it is difficult to fit the boundary-layer theory into a $G = 0$, whole-mantle convection scheme *anyway*, leaving aside the problem of thermal runaway.

Rather than press forward with a less likely parameter set, we will now turn to what we hope to be a more realistic system.

5. Analysis (ii): $G \sim 1$, "layered mantle"

In order to analyse the equations (3.17) in the case $G \sim 1$, $\varepsilon \ll 1$, we have to reconsider the boundary conditions: this is because the style of flow is likely to be very different. In particular, the viscosity increases exponentially outside the boundary layer, and one has to wonder whether there will be *any* mantle flow beyond the boundary layer. Obviously, one needs some global idea of the overall convective style, before being able to apply appropriate boundary conditions.

We have already discussed some of the physical consequences of a temperature-dependent viscosity in Section 2. Here we will adopt the scenario suggested by Fowler [12], in which the temperature below the lithosphere increases adiabatically until the viscosity reaches a certain value, at which point the velocity has decreased to such an extent that the temperature can no longer be adiabatic. It should be mentioned that the suggestion of that paper has recently been criticised on various grounds [4, 6, 10, 28]. Like the original paper, these criticisms are, to an extent, speculative, and it remains to be seen whether its conclusions are plausible or not.

If they are, then there is a natural depth scale $y = y_m$, which distinguishes two regions of convection: a fast, vigorous, adiabatic flow ($y < y_m$), and a more sluggish, isoviscous flow ($y > y_m$). The depth y_m depends on the viscosity exponent G and on the ratio of mantle depth to boundary layer thickness, Λ. In order to obtain a balance between conduction and advection in the (lower) mantle and also in the constitutive relation, one needs, from (3.17) (and supposing $\Lambda - y_m \sim \Lambda$, $\mu_2, \mu_3 \sim 1$),

$$\psi \sim \Lambda^2\exp(-Gy_m),$$
$$\Lambda\exp[-Gy_m] \sim 1/\Lambda^2,$$

i.e.,

$$\Psi \sim 1/\Lambda,$$
$$y_m \sim \frac{3\ln\Lambda}{G}. \tag{5.1}$$

Formally, this is large, though for $\Lambda \approx 30$, $G \approx 2$, one has $y_m \approx 5$, which is not *too* large.

It should be mentioned that the above scenario is conjectural, but it does satisfactorily tie in with the notions of a constant-viscosity mantle, layered mantle convection, and core-mantle temperatures $\gtrsim 3000$ K. All of these "observables" are to some extent also conjectural, and there may be other problems with the layered style suggested here (e.g. stability, boundary-layer structure, etc.), but one could argue that there is no materially better model available at the moment.

(a) Boundary conditions

The equations are still given by (3.17). Again, on the surface, we have

$$\tau = 0, \quad \Psi = 0, \quad \Phi = \Phi_0 \quad \text{on} \quad y = 0. \tag{5.2}$$

If we suppose y_m segregates the mantle into two distinct convecting regions, then (5.1) suggests that Ψ should (approximately) satisfy

$$\Psi, \Psi_y = 0 \quad \text{on} \quad y = y_m; \tag{5.3a}$$

additionally

$$\Phi = 1 + O(\varepsilon) \quad \text{on} \quad y = y_m; \tag{5.3b}$$

we would surmise that the $O(\varepsilon)$ contribution to (5.3b) would be determined, in principle, from the dynamics in the lower mantle. But in particular, we assume that no interior boundary layers (for $\Phi \sim 1$) can exist. (Such layers would lead to a very low-viscosity lower mantle; see [25], for example.) If we suppose $y_m \gg 1$, then (5.3) would be

$$\Psi \to 0 \quad \text{as} \quad y \to \infty,$$
$$\Phi = 1 + O(\varepsilon) \quad \text{as} \quad y \to \infty, \tag{5.4}$$

the first of these being tantamount to two conditions on Ψ (cf. [37]). The second condition in (5.4) we shall treat rather circumspectly, since analytical [47] and numerical [51] considerations suggest that this condition may not be applicable at infinity, whereas (5.3b) may be possible (if y_m is not too large). There is some reason to suppose that the limits $y_m \to \infty$ and $\varepsilon \to 0$ do not commute [47].

(b) Lithosphere

Rather than introduce the definitions of μ_i as we proceed, we present them now:

$$\mu_1 = 1/\nu,$$
$$\mu_2 = \nu,$$
$$\mu_3 = 1/\nu,$$
$$\mu_4 = \bar{\mu}/\nu, \qquad \bar{\mu} \sim 1. \tag{5.5}$$

These are as before, except that $\mu_3 \sim 1/\nu$: this choice is motivated by the need to

have nontrivial shearing in the "mesosphere"—so that the boundary conditions at y_m can be applied.

The analysis proceeds much as in Section 4. There is a quasirigid region $\Phi < 1$, in which the equations are (again in steady-state form)

$$u^*\Phi_x = \nu\Phi_{yy},$$
$$\tau_{yy} = \Phi_x/\nu,$$
$$\Psi_{yy} = 0, \qquad \Psi = u^*y. \tag{5.6}$$

We suppose u^* is constant in (5.6), to conform with the rigidity of the lithosphere. We have quasisimilarity solutions as before:

$$\tau = \frac{\Phi - \Phi_0}{u^*} - \beta(x)y,$$
$$\Phi \sim \Phi_0 + (1 - \Phi_0)\operatorname{erf}\eta + \varepsilon\sum_1^\infty \nu^r a_r x^r\{i^{2r}\operatorname{erfc}(\eta) - i^{2r}\operatorname{erfc}(-\eta)\},$$
$$\eta = \tfrac{1}{2}y\left(\frac{u^*}{\nu x}\right)^{1/2}, \tag{5.7}$$

and $\Psi = u^*y$.

(c) Asthenosphere

As before, define

$$e^{-\eta^{*2}}/\eta^* = \varepsilon, \qquad \eta^* \sim [\ln(1/\varepsilon)]^{1/2},$$
$$\nu = 1/\eta^{*2}; \tag{5.8}$$

when $\Phi \to 1$, we put

$$\Phi = 1 + \varepsilon\phi,$$
$$y = y^* + \nu Y,$$
$$\Psi = u^*y^* + \nu\psi. \tag{5.9}$$

Then, to leading order, the equations (3.17) give

$$\tau \approx \tau^* = \frac{1 - \Phi_0}{u^*} - \beta(x)y^*,$$
$$\psi_{YY} = -|\tau^*|^{n-1}\tau^*\exp[-Gy^*]e^{\phi},$$
$$\nu[\psi_Y\phi_x - \psi_x\phi_Y] = \phi_{YY} + u^*y^{*\prime}\phi_y + \bar{\mu}|\tau^*|^{n+1}\exp(-Gy^*)e^{\phi}, \tag{5.10}$$

where we retain the left-hand side of $(5.10)_3$ analogously to Section 4. Notice that the choice of $\mu_3 = 1/\nu$ ensures a balance in $(5.10)_2$, though this choice of μ_3 is *not* made for this reason [see Section 5(d) below]; however, the balance indicates that the asthenosphere *is* a shear layer, in which a large change of velocity can take place.

Boundary conditions on ϕ and ψ are

$$\psi \sim u^* Y,$$
$$\phi \sim b(x) - \frac{1-\Phi_0}{\sqrt{\pi}} \exp\left[-\left(\frac{u^*}{x}\right)^{1/2} Y\right], \quad \text{as} \quad Y \to -\infty, \tag{5.11}$$

the latter stemming from (4.29). Again, putting

$$\zeta = \left(\frac{u^*}{x}\right)^{1/2} Y - \ln\left[\frac{1-\Phi_0}{\sqrt{\pi}}\right], \tag{5.12}$$

the quasistationary solution of (5.10) [neglecting $O(\nu)$] satisfies [with $y^* = 2(x/u^*)^{1/2}$]

$$\phi_{\zeta\zeta} + \phi_\zeta + s e^{\phi} = 0,$$
$$\phi \sim b(x) - e^{-\zeta}, \qquad \zeta \to -\infty, \tag{5.13}$$

where

$$s = \frac{\bar{\mu} x \tau^{*n+1} \exp(-Gy^*)}{u^*}, \tag{5.14}$$

and we can assume $\tau^* > 0$ from physical considerations. Once ϕ is determined from (5.13), ψ is found by quadrature from $(5.10)_2$: we have

$$\psi_{YY}(\bar{\mu} x \tau^*/u^*) = -s e^{\phi};$$

thus

$$\bar{\mu}\tau^* \psi_{\zeta\zeta} = \phi_{\zeta\zeta} + \phi_\zeta,$$

so

$$\bar{\mu}\tau^*\left[\psi_\zeta - (xu^*)^{1/2}\right] = \phi_\zeta + \phi - b(x), \tag{5.15}$$

to satisfy the condition $\psi \sim u^* Y$ as $Y \to -\infty$.

(d) Mesosphere

When $Y \sim 1/\nu$, then $\phi \sim -\ln(1/\nu)$; we put

$$\phi = -\ln(1/\nu) + \tilde{\phi}, \qquad y - y^* = O(1); \tag{5.16}$$

then we have

$$\tau \approx \frac{1-\Phi_0}{u^*} - \beta(x) y, \tag{5.17}$$

$$\Psi_{yy} = -|\tau|^{n-1}\tau \exp[\tilde{\phi} - Gy]. \tag{5.18}$$

$$\Psi_y \tilde{\phi}_x - \Psi_x \tilde{\phi}_y = \bar{\mu}|\tau|^{n+1} \exp[\tilde{\phi} - Gy], \tag{5.19}$$

where we have neglected the term $\nu\tilde{\phi}_{yy}$ in (5.19). The matching condition for $\tilde{\phi}$ is (4.38):

$$\tilde{\phi} \sim \ln\left[\frac{1}{s(y-y^*)}\left(\frac{x}{u^*}\right)^{1/2}\right], \qquad y \to y^*. \tag{5.20}$$

From (5.15), it also follows that

$$\Psi = u^* y + \frac{1}{\bar{\mu}\tau^*}\left(\frac{u^*}{x}\right)^{1/2}(y-y^*)$$
$$\times\left[-\ln\left(\frac{1}{\nu}\right) + 1 - b - \ln\left\{s\left(\frac{u^*}{x}\right)^{1/2}(y-y^*)\right\}\right]$$
$$+ O(\nu \ln(1/\nu)),$$
$$\Psi_y \sim -\frac{1}{\bar{\mu}\tau^*}\left(\frac{u^*}{x}\right)^{1/2}\left[\ln(y-y^*) + \left\{\ln\left(\frac{1}{\nu}\right) + b + \ln s + \tfrac{1}{2}\ln\left(\frac{u^*}{x}\right)\right\}\right] + u^*, \tag{5.21}$$

as $y \to y^*$. We thus attempt to solve (5.17)–(5.19), with the boundary condition $\Psi = u^* y^*$ on $y = y^*$, although neither $\tilde{\phi}$ nor Ψ will be analytic there. In fact, it seems that a Frobenius type expansion for $\tilde{\phi}$ and Ψ is appropriate. The other boundary condition for Ψ is that $\Psi = 0$ when $y = y_m$, and $\beta(x)$ is chosen so that also $\Psi_y = 0$ there. There seems to be no closure in this problem to determine u^*; however, it is plausible that this must be determined by the dynamics of the subducting slab, and its interaction with the pressure gradient β. Again, we might then expect that the requirement of constant u^* will determine y^*, but the lithosphere similarity solution remains valid, and the analysis is essentially unaltered.

It may be noted that the asymptotic form for Ψ as $y \to y^*$ contains a term $\ln(1/\nu)$. Since the complete equations do not contain such terms, it seems that one can simply telescope them in the expansion, i.e. include them as $O(1)$. In any case, if $\nu \approx \frac{1}{3}$, then $\ln(1/\nu) \approx 1$, and it is needlessly pedantic to consider such terms as other than $O(1)$. [This is not true for $\ln(1/\nu)$ in (5.16), because of the exponentiation.]

The other question of interest concerning (5.17)–(5.19) is the behavior of $\bar{\phi}$ for large y (or large y_m). We inquire whether $\bar{\phi}$ can approach zero (i.e. an adiabat). The answer is evidently no, since if $\bar{\phi} \to 0$, then as $y \to \infty$ (and $y_m \to \infty$)

$$\tau \sim -\beta y,$$

$$\Psi_y = O(y^n e^{-Gy}),$$

so that

$$\bar{\phi}_x = O(y),$$

contradicting the assumption. On the other hand, $\bar{\phi} < Gy$, so that $\Psi \to 0$ as $y \to \infty$. If we assume

$$\bar{\phi} \sim yf(x), \qquad y \to \infty, \tag{5.22}$$

then we find that f must satisfy

$$f = \frac{G\bar{\mu}\int_x^{x_0} \beta\, dx}{1 + \bar{\mu}\int_x^{x_0} \beta\, dx}, \tag{5.23}$$

where x_0 is some constant. Therefore (5.22) is a consistent limiting form for $\bar{\phi}$, and we can see that the limits $y_m \to \infty$ and $\varepsilon \to 0$ need to be treated carefully, at least in a numerical solution. This sheds some light on the numerical results of Schubert et al. [51]. If we choose x_0 as the subduction zone, as seems appropriate for the specification of initial conditions for (5.19), then (5.23) yields a super-adiabatic, but subisoviscous temperature; an adiabat is restored if $G = 0$ (no activation volume) or $\bar{\mu} = 0$ (no viscous heating). However, (5.22) can be expected to break down for very large y, due to the neglect of other terms in (3.17).

(e) Instability in the asthenosphere

We now revisit the equations (5.10). We expect the quasistationary solution to be unstable, and therefore we put

$$x = x^* e^{\nu\xi}; \tag{5.24}$$

we also define

$$\psi = (u^* x)^{1/2} \tilde{\psi}, \tag{5.25}$$

to remove the dominant x dependence from ψ: with ζ defined by (5.12), $(5.10)_{1,2}$ are

$$\tilde{\psi}_\zeta \phi_\xi - \tilde{\psi}_\xi \phi_\zeta = \phi_{\zeta\zeta} + \phi_\zeta + s^* e^{\phi}, \tag{5.26}$$

$$\tilde{\psi}_{\zeta\zeta} = -p^* e^{\phi} \tag{5.27}$$

to leading order, where

$$s^* = s(x^*), \tag{5.28}$$

$$p^* = x^{*1/2} \frac{\tau^{*n}}{u^{*3/2}} \exp[-Gy^*]. \tag{5.29}$$

The boundary conditions on these equations are that

$$\tilde{\psi} \sim \zeta, \qquad \zeta \to -\infty$$

$$\phi \sim \phi_0(\zeta), \qquad \zeta \to \pm\infty, \tag{5.30}$$

where ϕ_0 is the quasistationary solution of (5.26) (with no left-hand side.)

I have not numerically integrated these equations. The contention is that the blowup result in Section 4 is sufficient to suggest strongly that a similar blowup will occur here. The equations themselves are not particularly more difficult to formulate numerically, but they describe a circulatory motion, and hence involve backwards diffusion when $\tilde{\psi}_\zeta < 0$ as ξ increases. Some care is necessary in formulating the "initial" conditions, therefore. A problem of this type has been encountered and solved recently by Howard [23] in the context of salt fingers in double diffusion. Notice, however, that the time evolution of an initial disturbance still obeys (4.42) and hence thermal runaway can be predicted in that case.

(f) Summary

We claim that if

$$\mu_1 = \frac{1}{\nu}, \quad \mu_2 = \nu, \quad \mu_3 = \frac{1}{\nu}, \quad \mu_4 \sim \frac{1}{\nu} \sim \ln\left[\frac{1}{\varepsilon}\right], \tag{5.31}$$

then active mantle convection is consistent with the possible occurrence of thermal runaway in the asthenosphere. The crucial test is whether the parameter values (5.31) are at all consistent with the various "observed" values. To examine this, we again give the values of d, U, τ, l, measured in units of 100 km, 3 cm y^{-1}, 10 bars, 3000 km, and the (asthenospheric) viscosity η measured in units of 10^{21} poise, using (5.31) (and writing $\mu_4 = \bar{\mu}/\nu$),

$$d \sim 0.8\bar{\mu}/\nu,$$

$$U \sim 1.2(\bar{\mu}/\eta)^{1/2},$$

$$\tau \sim 1.5(\eta/\bar{\mu})^{1/2}\nu,$$

$$l \sim 2.2\bar{\mu}^{5/2}\eta^{1/2}/\nu^2. \tag{5.32}$$

It is evident, since $\nu \approx \frac{1}{3}$ [for $\varepsilon = \frac{1}{40}$, from (4.19)], that values of $\bar{\mu}$ and η of order unity give values of d, U, τ, and l which are in fair agreement with observed values. For example, taking $\bar{\mu} = \frac{1}{3}$ gives a lithosphere thickness of ≈ 80 km, and taking $\eta = \frac{1}{3}$ (corresponding to an asthenospheric minimum viscosity of 3.3×10^{20} P; cf. [3]) gives a velocity $\approx 4\ \mathrm{cm\,yr^{-1}}$, stress ≈ 5 bars, and (importantly) a length scale of about 2600 km. We could hardly ask for a better vindication of the assumptions (5.31). On this basis we hypothesize that thermal runaway may be an important process in the earth's asthenosphere. It would occur as a local phenomenon, leading to partial melting and (perhaps) midplate or subduction volcanism. Fowler [13] in fact suggested that thermal runaway of this type could generate the partial melt which was required by Turcotte et al.'s [56] mechanism for the initiation of subduction. It is not clear at the moment, though, whether viscous heating can be of major significance in stagnant plate convection. If not, then the initiation of subduction must find its explanation elsewhere.

6. Discussion

In this paper we have attempted to do a partial boundary-layer analysis of the equations governing the upper part of the earth's oceanic lithosphere. This attempt is, in part, motivated by a wish to update Turcotte and Oxburgh's [57] isoviscous analysis to cope with a more pertinent rheology, and, in part, by earlier work by Schubert and others concerned with the numerical solution of this problem, and possible effects of thermal runaway and nonuniqueness (although the latter topics were considered in the context of subcontinental lithosphere). Particular issues involved are whether and why one should really expect quasisimilarity at young ages, and whether viscous heating by mechanical dissipation might realistically (1) explain the departure from the square-root-of-age behavior of heat flow at large ages, and (2) cause thermal runaway to occur, with subsequent attendant volcanism.

The appropriate boundary-layer equations involve three important dimensionless parameters: $\varepsilon, G, \bar{\mu}$. These measure, respectively, the temperature sensitivity of viscosity (small ε means very sensitive), the pressure dependence of viscosity (more specifically, the sensitivity of viscosity to the difference between isoviscous and adiabatic temperatures), and viscous heating ($\bar{\mu} \sim 1$ means viscous heating is important). Based on typical earth-inferred values, we estimate typical values $\varepsilon \sim \frac{1}{40}$, $G \sim 2$, $\bar{\mu} \sim 0.2$. We then attempt to solve the boundary-layer equations asymptotically in the limit $\varepsilon \to 0$. The cases $G \ll 1$, $G \sim 1$ are both considered; also $\bar{\mu} \lesssim 1$.

The case of $G \ll 1$ corresponds to a fortuitously isoviscous adiabat. It does not correspond to earth-type parameters, but does coincide with the notion of a constant-viscosity mantle, based on postglacial rebound, and also with many people's concept of whole-mantle convection. Our analysis of this case suggests that viscous heating is unlikely to be of any significance, and that, even if G is small, one requires a further parameter μ_3 to be small ($\sim \frac{1}{30}$), whereas inferred values are ~ 1 [see (3.12)].

The analysis of the case $G \sim 1$ is similar to the previous case, except that we hypothesize that the increase of viscosity with depth will preferentially force a return flow to occur at relatively shallow depth. This is circumstantially compatible with some sort of "layering," at least away from subducting slabs. Other than that, the analysis is similar, but now $G \sim 1$ is comparable with earth-type values, and moreover, $\bar{\mu} \sim 1$ is sufficiently large that viscous heating is important, primarily in the "asthenosphere."

In either case, the quasi-similarity solution at young ages is asymptotically predicted (even if the lithosphere thickness is not precisely proportional to the square root of age), although the full boundary-layer equations do not admit such a solution. Concentrating now on the shallow-mantle, $G \sim 1$ case, we find that there is a steady solution which divides naturally into lithosphere, asthenosphere (with a local temperature maximum with depth), and mesosphere, in which the return flow takes place. This mesosphere is of thickness of the order of the upper mantle.

To the extent that they are comparable, our results are consistent with those of Schubert et al. [51], who also found that viscous heating could be important. In particular, our analysis explains some curious features of their numerical results. When $\bar{\mu} \sim 1$ (viscous heating is important), the matching condition $\tilde{\phi}_y \to 0$ (far-field adiabat) is inconsistent with the equations. On the other hand, prescription of $\tilde{\phi} = 0$ at some large (but fixed) depth is consistent with the notion of a vigorous upper-mantle return flow. Schubert et al. experienced some difficulty with their far-field thermal boundary condition, which can be ascribed to this fact. In addition, prescription of a constant u^* and (e.g.) constant $\tilde{\phi}$ at the "base" will not yield zero surface stress, and constant "basal" $\tilde{\phi}$ will not yield constant u^*. A possible resolution of this situation is discussed below.

Of course, Schubert et al. [51] did not find evidence of thermal runaway in their study. This is because they sought local nonsimilarity solutions, which is much the same as seeking solutions to $\phi_x = e^{\phi}$, $\phi(0) = 0$, in the form of Taylor series. One finds $\phi = x + \frac{1}{2}x^2 + \cdots$, which shows no sign of the sudden explosion at $x = 1$. Our analysis predicts that, for typical earth-type parameters, thermal runaway can and will occur as a local phenomenon. This could (for active plates) occur in two principal ways. A small excess temperature in the asthenosphere (above the steady state) warms up gradually as the flow proceeds, and undergoes thermal runaway as x increases, leading to partial melting at some critical x^*. It is actually possible that this could occur in a steady state, if x^* is identified with an oceanic trench, and if subduction (more precisely, behind-arc volcanism) is therefore associated with thermal runaway *underneath* the slab. It seems consistent that this could be the case, but it is by no means certain.

Alternatively, an initial disturbance to the steady flow will evolve (following the material element) on a "rapid" time scale $\sim 10^8$ yr, leading to a rapid runaway towards the end. Since the blowup is logarithmic, the "noticeable" part of this runaway is much shorter. For $\phi = -\ln(1-\tau)$, we have $\phi(0) = 0$, $\phi(0.8) = 1.6$, $\phi(0.9) = 2.3$, $\phi(0.99) = 4.6$. If a tenfold increase may reasonably be considered "noticeable" (remember, with $\varepsilon = \frac{1}{40}$, this represents an actual rise of some $0.25T_a$, i.e. 400 K, perhaps heralding the onset of partial melting), then one has to wait until $1 - \tau \sim 10^{-4}$ or 10^{-5}. In terms of a time scale of 10^8 yr, this is "only" 10^3–10^4 years. Obvious candidates for such disturbances are deep-mantle plumes, and an analysis suggests that such a plume (or blob), arriving in the asthenosphere with an excess temperature ~ 50 K, will lead to runaway at a time of $\sim 10^8$ yr later—at a different location, since the disturbance travels with the flow as it evolves. It is consistent with our analysis that hot spots (of plume-type origin) generate magma through the occurrence of thermal runaway.

We have sought to analyse convection in the earth as it occurs. The simplifications we have made are not all rational ones, but we feel that the essential physics is described. Unfortunately, it seems unlikely that any laboratory experiment will be able to be done to verify or disprove the predicted structure. The attainment of $\bar{\mu}$ and $G \sim 1$ seems prohibitive. A better chance exists for numerical experiments, but at the moment the problem is beyond one's grasp. Here, one can (in principle) have $G, \bar{\mu} \sim 1$, $\varepsilon \ll 1$, at least to a certain extent, but the simulation of the return flow in a channel considered here poses some problems. For one thing, subduction can not yet be simulated on the basis of a given, material-only-dependent rheology. In fact, subduction is not understood in the earth, either. To get around that, one can simply prescribe a surface velocity u^*. Then the problem is overprescribed ($\Psi = 0$, $\tau = 0$) at the surface. Following Schubert et al. [51], one might simply ignore the τ boundary condition, and indeed, that might be reasonable. Otherwise, one possibility is to prescribe the temperature $\phi = \phi_b(x)$ at the "base" and the velocity u^* at the surface. Then one computes the surface stress as $\tau = \tau(x, \phi_b(x))$, and (in view of the discussion about ϕ in Section 5) solves $\tau(x, \phi_b(x)) = 0$ for ϕ_b.

A more serious problem is that the equation (3.17) for Φ is of backwards-diffusion type when $\Psi_y < 0$. One therefore has problem of mixed-diffusion type, requiring initial conditions at different ends. One way out of this is to include the small horizontal diffusion term, and although this raises further problems for the numerical analyst, it may be the easiest way to achieve a result.

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