Homoclinic Bifurcations in n Dimensions

By A. C. Fowler

Bifurcations near homoclinic orbits in n dimensions are described. Depending on the eigenvalues of the Jacobian at the fixed point whose real parts are closest to zero, a strange invariant set of periodic and aperiodic orbits can be produced, which can be described by a Bernoulli shift on a finite set of symbols. These results generalize earlier ones of Shil'nikov, Gaspard, Tresser, Glendinning, and Sparrow, amongst others.

1. Introduction

The existence of chaotic motion in solutions of ordinary differential equations in $\mathbb{R}^n$, $n \geq 3$, implies that strange invariant sets can exist, which contain many aperiodic and periodic orbits. Homoclinic bifurcations are a very important mechanism for causing such sets to come into existence, and their identification and analysis is thus crucial for explaining observations of chaos.

Many such analyses have been carried out since the pioneering work by Shil'nikov [10, 14], who analyzed systems of dimension three and four, respectively. In particular, Shil'nikov himself extended his results to systems of arbitrary dimension to ascertain conditions when a single periodic orbit bifurcates from the homoclinic orbit [11, 13] or to find that, at the bifurcation, a strange invariant set exists [14].

These results are complementary, and were notably advanced by Gaspard [5] and Glendinning and Sparrow [7], who were able to extend Shil'nikov's [10] results to a neighborhood of the parameter value $\mu = 0$ at which the homoclinic orbit exists. A similar extension for the four-dimensional case was done by Fowler and Sparrow [4].

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All these results form part of an overall picture which describes the dynamics of n-dimensional systems in a parametric neighborhood of a homoclinic bifurcation. However, there are technical difficulties associated with proving the relevant results, some of which have been addressed by Shil'nikov [14], Gaspard [5], and Tresser [16]. Historically, two approaches have been taken. Shil'nikov rewrites the differential equations in integral form and uses smoothness results for the integral equations to derive (and prove) approximations to a Poincaré map for the system. This technique allows analysis in arbitrary finite dimension *at* the homoclinic bifurcation, but has not been generally applied in a parametric neighborhood of it. A different technique, used by Tresser and coworkers, uses a theorem of Belitskii [1] which extends the Hartman-Grobman theorem [9] to linearize exactly (and smoothly) the equations near the origin (the fixed point). The resultant approximate Poincaré map can then be used to study the bifurcation in a parametric neighborhood of the critical value. However, this linearization (essentially via a normal-form procedure) requires that the eigenvalues of the linearized system at the origin be *nonresonant*, which provides an annoying (and not obviously necessary) restriction on the applicability of the analysis. In particular, it becomes cumbersome for systems of large dimension.

A recent book by Wiggins [17] has shown how the basic analysis of Shil'nikov can be used in n dimensions. In this paper, we follow the methodology of Wiggins closely, but in addition we show how the derived approximate Poincaré map can be further simplified in order to yield explicit results in the general case.

2. An approximate Poincaré map

Consider the system of ordinary differential equations

$$\dot{x} = f(x,\mu), \tag{2.1}$$

where $\mu \in \mathbb{R}$, $x \in \mathbb{R}^n$, and f is a real analytic function. Wiggins [17] has shown that in fact $f \in C^2$ is sufficient. We suppose the origin is a fixed point, namely

$$f(0,\mu) = 0 \tag{2.2}$$

for all μ in a neighborhood of 0, and that when $\mu = 0$, there exists a *homoclinic orbit*

$$\Gamma: x = x^*(t),\ x^* \to 0 \text{ as } t \to \pm\infty. \tag{2.3}$$

It is convenient to specify the phase of x^* in some way, e.g. by defining $t = 0$ such that $|x^*(0)| = \max_t |x^*(t)|$.

We assume that the Jacobian Df of f evaluated at $x = 0$, $\mu = 0$ has distinct eigenvalues of multiplicity one, none of which has real part equal to zero. This situation then remains true in a neighborhood of $\mu = 0$. It follows that, near $\mu = 0$, Df can be diagonalized over $\mathbb{C}$ by a linear transformation $x \to T(\mu)x$.

Furthermore, the stable and unstable manifolds W_S and W_U are analytic functions of x, so that, by a local analytic change of variables, we can write $x = (x_U, x_S)^T$, where $x_S \in W_S$, $x_U \in W_U$, and (locally) $\mathbb{R}^n = W_U \oplus W_S$. In what follows, we assume both transformations have been carried out, and we write the diagonal matrix Df as

$$Df(0,\mu) = D(\mu). \tag{2.4}$$

Let $\{e^j\}$ be the normal basis for $\mathbb{C}^n$. We suppose

$$\begin{aligned} x^* &\sim \sum_j \alpha_j^* e^j \exp(\sigma_j t), \qquad t \to -\infty, \\ x^* &\sim \sum_j \beta_j^* e^j \exp(\sigma_j t), \qquad t \to +\infty, \end{aligned} \tag{2.5}$$

where $D = \operatorname{diag}(\sigma_j)$, and, by our choice of phase for x^*, we may assume $\alpha_j^*, \beta_j^* = O(1)$. (For example, if $x^* = \operatorname{sech} t$, then $\{\sigma_j\} = \{\pm 1\}$, and $\alpha_1^*, \beta_1^* = 2$.) Letting suffixes S and U denote quantities associated with the stable and unstable manifolds of the origin, respectively, we have, with obvious notation,

$$\alpha^* = (\alpha_U^*, 0)^T, \qquad \beta^* = (0, \beta_S^*)^T, \tag{2.6}$$

where $\alpha_U^* \in W_U$, $\beta_S^* \in W_S$. (α^*, β^* are the vectors with components α_j^*, β_j^*.) Equation (2.5) can be written

$$\begin{aligned} x^* &\sim \exp(tD_0)\,\alpha^* \quad \text{as} \quad t \to -\infty, \\ x^* &\sim \exp(tD_0)\,\beta^* \quad \text{as} \quad t \to +\infty, \end{aligned} \tag{2.7}$$

where $D_0 = D(0)\ [= Df(0,0)]$.

Our purpose now is to obtain an approximate Poincaré map defined on either of the surfaces Σ or Σ', indicated schematically in Figure 1. Here Σ and Σ' are transverse to W_S and W_U, respectively. Let e^U and e^S denote the eigenvectors which correspond to the least unstable and least stable eigenvalue, respectively, i.e. whose real part is closest to zero (when $\mu = 0$, and hence also for sufficiently small $\mu \neq 0$). (In the case of complex conjugates, either will do.) We choose the surfaces Σ and Σ' by the relations

$$\begin{aligned} |\langle x, e^S \rangle| &= \nu, \\ |\langle x, e^U \rangle| &= k\nu, \qquad k = O(1), \end{aligned} \tag{2.8}$$

where $\nu \ll 1$; the precise choice of Σ' is left slightly flexible at this stage for subsequent convenience. Orbits sufficiently close to Γ on Σ will be mapped

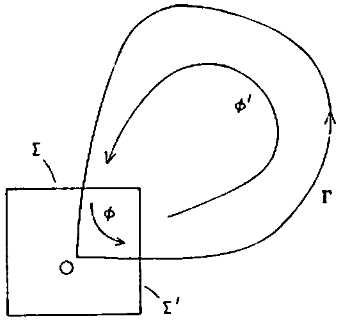

Figure 1. Schematic representation of the flow. The homoclinic orbit Γ passes transversely through the faces Σ and Σ' of a box at the origin. For orbits close to Γ, the map φ takes points on Σ to points on Σ'. For points in φ(Σ) which are still close to Γ, φ' maps them back to Σ. The Poincaré map $\Phi = \varphi' \circ \varphi$ thus maps Σ to itself.

under the flow to points in Σ' close to Γ. Within the box, $x = O(\nu)$ and the flow may be approximately linearized. It is convenient to write the system (2.1) in the form

$$\dot{x} = Dx + g(x), \tag{2.9}$$

where

$$g(x) = f(x,\mu) - Df(0,\mu)x. \tag{2.10}$$

If $x = x_0$ at $t = t_0$, the equation (2.9) can be written as the integral equation

$$x = \exp[(t-t_0)D]\,x_0 + \int_{t_0}^{t} \exp[(t-t_0-\tau)D]\,g[x(\tau)]\,d\tau. \tag{2.11}$$

Since $g = O(x^2)$, standard iterative methods guarantee that x depends analytically on x_0 and smoothly on t for bounded values of t. Further, x is closely approximated by the linear mapping generated by the linearized flow. (Wiggins [17] shows that the error is $O(\nu^2)$.)

We choose to define α^* and β^* in (2.7) precisely by choosing t_U and t_S [both $O(\ln(1/\nu))$] such that

$$\begin{aligned} x^* &= \exp[-t_U D_0]\,\alpha^* \quad \text{on } \Sigma', \quad |\langle \alpha^*, e^U\rangle| = k, \\ x^* &= \exp[t_S D_0]\,\beta^* \quad \text{on } \Sigma, \quad |\langle \beta^*, e^S\rangle| = 1; \end{aligned} \tag{2.12}$$

we are able to do this because x^* and α^* on Σ' are both in the local invariant subspace W_U, and similarly for x^* and β^* on Σ. With these choices of t_U and t_S, we also define α and β by

$$\begin{aligned} x &= \exp[-t_U D_0]\,\alpha \quad \text{on } \Sigma', \quad |\langle \alpha, e^U\rangle| = k, \\ x &= \exp[t_S D_0]\,\beta \quad \text{on } \Sigma, \quad |\langle \beta, e^S\rangle| = 1. \end{aligned} \tag{2.13}$$

If $\tilde{t}$ is the time of transit from Σ to Σ', then (2.11) can be written in the form

$$\alpha = \exp[(t_U + t_S)D_0 + \tilde{t}D]\,\beta + \exp[t_U D_0]\int_0^{\tilde{t}} \exp[(\tilde{t}-\tau)D]\,g[x(\tau)]\,d\tau. \tag{2.14}$$

Note that for x close to Γ in Σ and Σ', we require α and β to be $O(1)$.

In order to compute the flow from Σ' back to Σ, we suppose that x is sufficiently close to Γ that a linearization about x^* can be made. We write

$$x = x^* + y, \tag{2.15}$$

so that y satisfies

$$\dot{y} = A_\Gamma(t)y + \mu\frac{\partial f}{\partial \mu}(x^*,0) + G(t;y), \tag{2.16}$$

where

$$\begin{aligned} A_\Gamma &= Df(x^*,0), \\ G &= f(x^*+y,0) - f(x^*,0) - Df(x^*,0)y - \mu\frac{\partial f}{\partial \mu}(x^*,0). \end{aligned} \tag{2.17}$$

Let $\Phi(t)$ be a fundamental matrix for the linear equation

$$\dot{y} = A_\Gamma y, \tag{2.18}$$

and define the heteroclinic matrix H by

$$\Phi = \exp(tD_0)\,H(t). \tag{2.19}$$

By extending Floquet's theorem to orbits of infinite period, we have (see [3], Theorem 8.1) that H tends to a constant matrix as $t \to \pm\infty$. We suppose

$$H(\infty)H^{-1}(-\infty) = M_0. \tag{2.20}$$

The solution of (2.16) satisfying $y = y_0$ on $t = t_0$ can be written

$$y = \Phi(t)\Phi^{-1}(t_0)y_0 + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\left[G(s;y(s)) + \mu\frac{\partial f}{\partial\mu}(x^*,0)\right]ds. \quad (2.21)$$

With α and β defined as in (2.13), we now *choose* k such that the time of transit from Σ' to Σ is $t_U + t_S$. Putting $t = t_S$ and $t_0 = -t_U$, and using (2.19), we obtain

$$\beta' - \beta^* = H(t_S)H^{-1}(-t_U)(\alpha - \alpha^*)$$
$$+ H(t_S)\int_{-t_U}^{t_S} H^{-1}(s)e^{-sD_0}\left[G + \mu\frac{\partial f}{\partial\mu}(x^*,0)\right]ds. \quad (2.22)$$

The integral term contains two constituents. That containing G is (crudely) $O(y^2)$, and was shown by Wiggins [17] to be $O(\nu^2)$ (when scaled as here). [This assertion, Proposition (3.2.7) on p. 198, follows from a Taylor expansion of (2.22) about α^* when $\mu = 0$; see equation (3.2.59). There may be some doubt about this, insofar as Wiggins's $O(u'^2)$—our $O(y^2)$—will be multiplied by a coefficient which grows as $\nu \to 0$ (due to the exponential growth near 0). However, it is nevertheless feasible that the term is indeed $O(\nu^2)$, since the values of x in Σ' are *restricted* so that the image in Σ satisfies $|x| = O(\nu)$. Nevertheless, the justification for this step is not as straightforward as it may appear.] The second term in μ can be estimated for large t_S and t_U, since as $s \to \pm\infty$,

$$\frac{\partial f}{\partial\mu}(x^*,0) = D_0'x^* + O(x^{*2}), \qquad \text{where} \quad D_0' = \left.\frac{dD}{d\mu}\right|_{\mu=0}.$$

Since $x^* \sim e^{sD_0}$, it follows that this term is $O[\mu\{t_S + t_U + O(1)\}]$, where the $O(1)$ term comes from the nondivergent part of the integral. Lastly, $H(t_S)H^{-1}(-t_U) = M_0[1 + o(1)]$, where the error terms are $\exp[-O(t_s,t_u)]$. Since the divergent part of the μ term arises solely because of the dependence of D on μ, it is clearer to include this in the matrix M_0, so that we write the map (2.22) in the form

$$\beta' - \beta^* = M(\alpha - \alpha^*) + \mu c + O(\nu), \quad (2.23)$$

where

$$M = M(\mu) = M_0[1 + O(\mu P)], \quad (2.24)$$

and where the recurrence time between intersections with Σ is

$$P = t_U + t_S + \bar{t}. \quad (2.25)$$

In a similar way we can write (2.14) in the form

$$\alpha = \exp[PD_0][1 + O(\mu P)]\beta + O(\nu). \quad (2.26)$$

The composition of (2.26) with (2.14) defines the Poincaré map.

In what follows, we shall establish results concerning periodic and aperiodic orbits of this map. Wiggins [17] has shown that it suffices to consider C^1 approximations of this map in doing the analysis. Therefore we henceforth consider the smooth perturbations to the map determined by neglecting the terms of $O(\mu P, \nu)$ (assuming $\nu \gtrsim \mu P$), and drop the suffix zero on M_0, D_0; the resulting approximation consists of the linear flow

$$\alpha \approx e^{PD}\beta, \quad (2.27)$$

composed with the affine map

$$\beta' - \beta^* = M(\alpha - \alpha^*) + \mu c; \quad (2.28)$$

this defines a map $\beta \to \beta'$ on subsets of Σ, with the recurrence time P being determined via the subsidiary condition which defines Σ, which is

$$|\beta^S| = 1. \quad (2.29)$$

The general form of this approximate map is not particularly new, although the style of the derivation is rather novel. In particular, by keeping track of the sizes of various terms, we will be able to study the way in which (2.27) and (2.28) approximate the full map, and also see how an approximate one-dimensional map may be further derived from (2.27) and (2.28). The use of further C^1 approximations to this map may be justified in the same way as before.

3. Geometry of the invariant set

We now wish to approximate (2.27) and (2.28) by using the fact that P is large for small μ and ν. The idea involved is as follows: a neighborhood of α^* in Σ' of size δ will be mapped under (2.28) to a neighborhood of β^* in Σ of size δ, provided $\delta \gtrsim \mu$. The part of this $N_\delta(\beta^*)$ in Σ which can map through to $N_\delta(\alpha^*)$ again is close to

$$\beta_U = e^{-PD_U}\alpha_U^*, \quad (3.1)$$

where the suffix U signifies a restriction to W_U. Furthermore, under the map φ from Σ to Σ', Σ is mapped into $\varphi(\Sigma)$, which is close to

$$\alpha_S = e^{PD_S}\beta_S^*. \quad (3.2)$$

(3.1) denotes a one-dimensional set in W_U, while (3.2) denotes a one-dimensional set in W_S. If $\dim W_U = k$, $\dim W_S = n-k$, then $\dim(W_S \cap \Sigma) = n-k-1$, $\dim(W_U \cap \Sigma') = k-1$, and we denote the $(n-k)$-dimensional set given by (3.1) in $(W_S \oplus W_U) \cap \Sigma$ as Λ_0, and the k-dimensional set given by (3.2) in $(W_S \oplus W_U) \cap \Sigma'$ as Λ'_0. The corresponding neighborhoods of Λ_0 and Λ'_0 are denoted Σ_0 and Σ'_0, and we note that our restriction to δ-neighborhoods of α^* and β^* ensures that $|\Sigma_0 \cap W_S| \sim \delta$, $|\Sigma_0 \cap W_U| \sim \delta \exp[-O(P)]$, $|\Sigma'_0 \cap W_U| \sim \delta$, $|\Sigma'_0 \cap W_S| \sim \delta \exp[-O(P)]$, where the modulus signs denote the size of the indicated set: i.e., Σ_0 is quasi-$(n-k)$-dimensional, and Σ'_0 is quasi-k-dimensional.

Under the affine transformation (2.28) the quasi-k-dimensional set Σ'_0 is rotated and translated. For $\mu = 0$, the point $\alpha = \alpha^*$ in Σ' is mapped to $\beta = \beta^*$ in Σ. In general, the intersection of the k-dimensional Λ'_0 with the $(n-k)$-dimensional Λ_0 would have $n-(n-k)-k$ dimensions, i.e. be a point, but we shall see that when $\mu = 0$, the restriction of $M = M_0$ to W_U has rank $k-1$, so that the image $\varphi'_0(\Lambda'_0)$ will be $(k-1)$-dimensional. Hence $\varphi'_0(\Lambda'_0) \cap \Lambda_0 = \Lambda_1$ is one-dimensional, where $\varphi'_0 = \varphi'$ at $\mu = 0$, and it follows that $\varphi'(\Sigma'_0) \cap \Sigma_0 \equiv \Sigma_1$ is a neighborhood (of dimension $n-1$) of a one-dimensional set. The invariant set for the flow lies inside this set, and it is on this basis that a one-dimensional approximation is of some real use.

Beyond our assumption of distinct eigenvalues (so that Df could be diagonalized), we have not assumed any further restriction on the eigenvalues. However, it is evident that if the set of eigenvalues whose real part is closest to zero consists of either two reals, a real and a complex conjugate pair, or two complex conjugate pairs, then a further reduction is possible (and these are the generic possibilities). The latter two cases were considered by Shil'nikov [14], and more recently by Glendinning and Sparrow [7], Gaspard [5], and Fowler and Sparrow [4]. The previous statements remain valid, but the intersection of Σ_0 and Σ'_0 can be visualized.

We now turn to a justification of the approximations involved in the above discussion. Using suffixes U and S to denote elements in W_U and W_S, we write (2.27) as

$$\begin{aligned} \beta_U &= e^{-PD_U}\alpha_U, \\ \alpha_S &= e^{PD_S}\beta_S. \end{aligned} \tag{3.3}$$

Since $\alpha \approx \alpha^*$, $\beta \approx \beta^*$, it is convenient to define

$$\alpha_U = \alpha^*_U + a_U, \quad \beta_S = \beta^*_S + b_S, \qquad a_U, b_S \ll 1, \tag{3.4}$$

and thus

$$\begin{aligned} \beta_U &= e^{-PD_U}(\alpha^*_U + a_U), \\ \alpha_S &= e^{PD_S}(\beta^*_S + b_S). \end{aligned} \tag{3.5}$$

Notice that, since $\beta_U \in W_U$, we can choose a_U in a $(k-1)$-dimensional subspace, e.g. by choosing $a^U_U = 0$ (this simply defines P). The Poincaré map $\Phi = \varphi' \circ \varphi$ from Σ to Σ is defined on the $(n-1)$-dimensional hypersurface Σ. Equivalently, we map b_S ($n-k-1$ dimensions), a_U ($k-1$ dimensions), P (1 dimension) to subsequent values b'_S, a'_U, P'. Thus we define

$$\beta'_U = e^{-P'D_U}(\alpha^*_U + a'_U). \tag{3.6}$$

We write

$$M = \begin{pmatrix} M_{UU} & M_{US} \\ M_{SU} & M_{SS} \end{pmatrix}, \tag{3.7}$$

where M_{UU} is $k \times k$, M_{US} is $k \times (n-k)$, etc. Thus (2.28) is

$$b'_S = M_{SU}a_U + M_{SS}\alpha_S + \mu c_S, \tag{3.8}$$

and, using (3.6),

$$e^{-P'D_U}(\alpha^*_U + a'_U) = M_{UU}a_U + M_{US}\alpha_S + \mu c_U. \tag{3.9}$$

The idea is that, given a_U, P, and b_S, (3.5) gives α_S, and then (3.8) gives b'_S, whereas (3.9) determines P' and a'_U. This is effected by the observation that the rank of M_{UU} is $k-1$, for the following reason. Since $\dot{x}^*$ is an exact solution of $\dot{y} = Df(x^*, 0)y$, it follows (since $\dot{x}^* \sim e^{tD_0}D_0\alpha^*$ as $t \to -\infty$, $\dot{x}^* \sim e^{tD_0}D_0\beta^*$ as $t \to +\infty$), that, with relative error $o(1)$, using (2.21) with $G = \mu = 0$,

$$\beta^* = D_0^{-1}M_0 D_0 \alpha^*. \tag{3.10}$$

Since D_0 is diagonal and $\alpha^*_S = 0 = \beta^*_U$, it follows that

$$M_{UU}(D_U\alpha^*_U) = 0, \tag{3.11}$$

so that $\det M_{UU} = 0$. We can assume rank $M_{UU} = k-1$, for otherwise, there exists another α^{**}_U such that $M_{UU}D_U\alpha^{**}_U = 0$, and hence (at $\mu = 0$) another principal homoclinic orbit. This is nongeneric (in the absence of symmetry), and we ignore the possibility.

Suppose η is the (unique) eigenvector of the Hermitian adjoint of M_{UU}; then it follows that we require P' to satisfy, from (3.9) and (3.5),

$$\begin{aligned} \langle \eta, e^{-P'D_U}\alpha^*_U \rangle &= \langle \eta, M_{US}e^{PD_S}\beta^*_S \rangle + \mu \langle \eta, c_U \rangle \\ &\quad + \langle \eta, -e^{P'D_U}a'_U + M_{US}e^{PD_S}b_S \rangle. \end{aligned} \tag{3.12}$$

Since the last term is small, then (3.12) is approximately a one-dimensional map for P to P'. The Poincaré map $\Phi:\Sigma \to \Sigma$ is thus defined by (3.12) (to determine P'), (3.9) (to determine a'_U), and (3.8) (to determine b'_S). All of the various error terms in these equations are analytic functions of a_U, b_S, and μ, and depend smoothly on P.

4. Periodic and aperiodic orbits

It is convenient to write the Poincaré map given by (3.12), (3.9), and (3.8) in the form

$$\begin{aligned} \langle \eta, e^{-P'D_U}\alpha_U^* \rangle &= \langle M_{US}^*\eta, e^{PD_S}\beta_S^* \rangle + \mu\langle \eta, c_U \rangle + \psi_1(P, P', a'_U, b_S), \\ a_U &= \psi_2(P, P', a'_U, b_S; \mu), \\ b'_S &= \psi_3(a_U, P, b_S; \mu), \end{aligned} \tag{4.1}$$

where the functions ψ_i are analytic in each of their arguments. The equations are defined in a neighborhood of $a_U = 0 = b_S$, $P = \infty$, and if this neighborhood is $|a_U| \le \delta$, $|b_S| \le \delta$, $|e^{-\sigma_m P}| \le \delta$, where σ_m is the magnitude of the real part of the eigenvalue of D closest to zero, then $\psi_1 = O(\delta^2)$, $\psi_2 = O(\delta)$ with $\partial\psi_2/\partial b_S, \partial\psi_2/\partial a'_U = O(\delta)$, and $\psi_3 = O(\delta)$ with $\partial\psi_3/\partial b_S = O(\delta)$, $\partial\psi_3/\partial a_U = O(1)$ (we assume $\mu \lesssim \delta$). Equation $(4.1)_2$ defines a'_U implicitly, and it is evident that in mapping a ball of size δ in Σ_1, the components of a_U are multiplied by $O(1/\delta)$, while those of b_S are multiplied by $O(\delta)$.

It is clear that the choice of coordinates (P, a_U, b_S) represents a natural decomposition of Σ_1, which we write as $\Sigma_1 = \Lambda_1 \oplus \Sigma_U \oplus \Sigma_S$, insofar as P parametrizes the one-dimensional part of the set, whereas $\Sigma_U \oplus \Sigma_S$ provides a hyperbolic structure for that part of Σ_1 orthogonal to Λ_1, as is illustrated in Figure 2, which represents a transverse section of the set Σ_1 (i.e., it is spanned

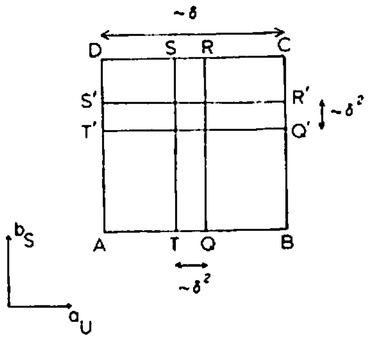

Figure 2. The set Σ_1 is a δ-neighborhood of the origin in $(e^{-\sigma_m P}, a_U, b_S)$ space $(\Lambda_1 \oplus \Sigma_U \oplus \Sigma_S)$ which maps at least partly into itself under Φ. This figure illustrates schematically the strong expansion and contraction which the map generates in $\Sigma_U \oplus \Sigma_S$. In a cross section at constant P, only strips such as $TQRS$ (near a_U = constant) are mapped back to $ABCD$. The image is $T'Q'R'S'$, but at a different value of $P \in \Lambda_1$.

by a_U and b_S). From $(4.1)_2$, we see that if $ABCD$ is a δ-neighborhood of the origin, then only a_U-values close to those satisfying $a_U = \psi_2(P, P'; 0, 0; \mu)$ are mapped back into $ABCD$ [where $a'_U = O(\delta)$]. Since Σ_U is expanding and Σ_S is contracting, it follows that the image of $QRST$ in Figure 2 is $Q'R'S'T'$ as shown (not necessarily with the same orientation), where $|QT| \sim \delta^2$, $|Q'R'| \sim \delta^2$. Repeating this process in $\Sigma_U \oplus \Sigma_S$ leads to the usual invariant set consisting of a Cantor set (embedded in $n-2$ dimensions), if there is more than one value $a_U = \psi_2(P, P'; \mu)$ with $|a_U| \le \delta$. A much more thorough discussion of the geometry of the Poincaré map is given by Wiggins [17]. As we shall see, this may be the case if either σ^U or σ^S is complex. It is not difficult to see that the set of values $a_U = \psi_2(P, P'; \mu)$ is exactly given [we compare (3.12) with (3.9)] by

$$a_U = M_{UU}^{\perp\,-1}\left(e^{-P'D_U}\alpha_U^* - M_{US}e^{PD_S}\beta_S^* - \mu c_U\right), \tag{4.2}$$

where $M_{UU}^{\perp\,-1}$ is the inverse of M_{UU} on the orthogonal complement of η in W_U and P' is related to P by the approximate one-dimensional map

$$\langle \eta, e^{-P'D_U}\alpha_U^* \rangle = \langle M_{US}^*\eta, e^{PD_S}\beta_S^* \rangle + \mu\langle \eta, c_U \rangle. \tag{4.3}$$

From the above discussion, it is straightforward to deduce several features of the bifurcations. Periodic orbits for the flow correspond to fixed points of the map, and are (at least) C^1-close to (P, a_U, b_S) satisfying $P' = P$ in (4.3), $a_U = \psi_2(P, P; \mu)$, $b_S = \psi_3(a_U, P; \mu)$. By the implicit-function theorem, the existence of (nondegenerate) fixed points of (4.3) therefore guarantees the existence of corresponding fixed points for the full Poincaré map. Similarly, period-two orbits, period-four orbits, etc. of (4.3) have corresponding orbits in the flow, and suggest the existence of period-doubling windows. As a consequence, this suggests the possible existence of aperiodic orbits. However, their existence cannot be straightforwardly deduced from the existence of corresponding orbits for the one-dimensional map (4.3), since the use of Bowen's shadowing lemma (see [8]), by which one might prove their existence, relies on some approximated aperiodic sequence $(P, a_U, b_S)_n$ existing on a hyperbolic invariant set, and while $\Sigma_U \oplus \Sigma_S$ has the appropriate structure, Λ_1 in general will not [because of turning points in the map $P \to P'$ given by (4.3)].

However, we can construct a horseshoe for (4.1) if certain conditions are met. Suppose (4.3) is multivalued, so that there are (at least) two values $a_U = \psi_2(P, P'; \mu) = \psi^1(P; \mu)$, $\psi^2(P, \mu)$, say, neighborhoods V_1 and V_2 of which map back to $|a_U| \le \delta$. (This occurs if σ^U is complex.) Suppose further that $|\mathrm{Re}\,\sigma^S| > \mathrm{Re}\,\sigma^U$, so that (4.3) is strongly contractive for "most" values of P. If we can find fixed points P_1, P_2 of (4.3) on the two components V_1 and V_2, respectively, such that (4.3) contracts the interval (P_1, P_2) in both components, then the Poincaré map induces a hyperbolic structure on $o(\delta)$ neighborhoods of the fixed points corresponding to P_1 and P_2.

The situation is illustrated in Figure 3. A vertical strip V_1 through P_1 of dimensions [in $(a_U, b_S, e^{-\sigma_m P})$] of $O(\delta^2, \delta, \delta)$ (respectively) is mapped to a set

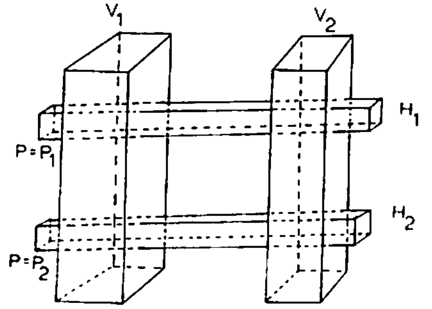

Figure 3. The vertical sets V_1 and V_2 containing the fixed points with $P = P_1$ and P_2, respectively, are mapped to the horizontal bars H_1 and H_2. This construction is analogous to that which one obtains in a horseshoe map.

H_1 enclosing P_1 of dimensions $(\delta, \delta^2, o(\delta))$. A similar set V_2 through P_2 is mapped to a similar image H_2 through P_2. The intersections of these four sets, $H_1 \cap V_1 = W_{11}$, $H_1 \cap V_2 = W_{12}$, $H_2 \cap V_1 = W_{21}$, $H_2 \cap V_2 = W_{22}$, when iterated under the Poincaré map Φ, define eight subsets of W_{ij} as

$$\begin{aligned} W_{111} &= \Phi(W_{11}) \cap V_1, \ldots, \\ W_{ijk} &= \Phi(W_{ij}) \cap V_k, \qquad i,j,k = 1,2, \end{aligned} \tag{4.4}$$

which by construction are nonempty. Iterating Φ forwards and backwards, we construct (part of) the invariant set for Φ, $I[P_1, P_2] = W_{\ldots ijk \ldots}$, which is homeomorphic to a symbolic sequence on two symbols. It follows that in this case, the flow contains a countably infinite number of periodic orbits, and uncountably many aperiodic ones. Furthermore, if there are N branches $a_U = \psi^i(P;\mu)$, with N distinct fixed points, a similar construction leads to a symbolic sequence defined on N symbols. Since the same statements are true for the time-reversed flow, the condition $|\mathrm{Re}\,\sigma^S| > \mathrm{Re}\,\sigma^U$, $\sigma^U \in \mathbb{C}$, may be replaced by $|\mathrm{Re}\,\sigma^S| < \mathrm{Re}\,\sigma^U$, $\sigma^S \in \mathbb{C}$ (Shil'nikov's condition).

Applications

There are three distinct generic cases:

(i) $\sigma^U = \lambda^U$, $\sigma^S = -\lambda^S$ (Lorenz case),
(ii) $\sigma^U = \lambda^U$, $\sigma^S = -\lambda^S \pm i\omega^S$ (Shil'nikov case),
(iii) $\sigma^U = \lambda^U \pm i\omega^U$, $\sigma^S = -\lambda^S \pm i\omega^S$ (bifocal case),

in which $\lambda^{U,S}, \omega^{U,S}$ are real, and no other eigenvalues have the same real parts. We consider these in turn.

(i) *Lorenz Case.* Two single real eigenvalues are closest to zero: $\lambda^U > 0$, $-\lambda^S < 0$. If we define

$$\xi = e^{-\lambda^U P}, \tag{4.5}$$

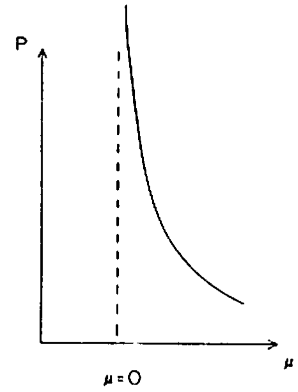

Figure 4. Exponential approach of P to ∞ as $\mu \to 0$ for the Lorenz case, or the Shil'nikov case with $\sigma^U \in \mathbb{C}$, $\lambda^S < \lambda^U$.

then, after suitable rescaling, (4.3) can be written approximately as

$$\xi' = a\xi^\gamma + \mu, \tag{4.6}$$

where

$$\gamma = \lambda^S/\lambda^U. \tag{4.7}$$

This has the form of the Lorenz map for $\xi > 0$ [15], which for the Lorenz equations themselves can be extended antisymmetrically to $\xi < 0$, because of the symmetry of the equations. There is a single fixed point of (4.6) for small ξ and μ, satisfying $\xi \approx \mu$ for $\gamma > 1$, $\xi \approx (-\mu/a)^{1/\gamma}$ for $\gamma < 1$, and thus in either event

$$\mu \sim e^{-\lambda_m P}, \tag{4.8}$$

where $\lambda_m = \min(\lambda^S, \lambda^U)$, as shown in Figure 4. This was shown by Shil'nikov [11].

(ii) *Shil'nikov Case.* One single real and one single complex pair of eigenvalues have real parts closest to zero. By appealing to time reversal if necessary, we can assume either that λ^U is real and $-\lambda^S \pm i\omega^S$ are complex, or vice versa. In view of our previous discussion, we take $-\lambda^S$ real, $\lambda^U \pm i\omega^U$ complex. Then (4.3) can be uniformly approximated, possibly after rescaling, by

$$e^{-\lambda^U P'} \cos \omega^U P' = a e^{-\lambda^S P} + \mu, \tag{4.9}$$

or [using (4.5)]

$$\xi' \cos\left(\Omega \ln \frac{1}{\xi'}\right) = a\xi^\gamma + \mu, \tag{4.10}$$

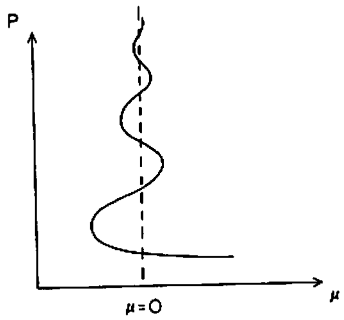

Figure 5. Oscillatory approach of P to ∞ as $\mu \to 0$ for the Shil'nikov case ($\sigma^U \in \mathbb{C}$, $\lambda^S > \lambda^U$) or the bifocal case.

where

$$\Omega = \omega^U/\lambda^U. \tag{4.11}$$

Evidently, when $\gamma < 1$ ($\lambda^S < \lambda^U$), there is a unique fixed point of (4.10), and the bifurcation diagram is as shown in Figure 4. On the other hand, when $\gamma > 1$ ($\lambda^S > \lambda^U$), there will be many fixed points satisfying

$$\xi \cos\left(\Omega \ln \frac{1}{\xi}\right) \approx \mu, \tag{4.12}$$

and it is graphically obvious that each fixed point lies on a different branch $a_U = \psi(P, \mu)$. In this case, we have orbits corresponding to symbolic sequences on N symbols, where N is the number of nondegenerate roots of (4.9) (i.e. where the intersection is not tangent). As μ tends to zero, the period P of the principal periodic orbit is given by

$$\mu \sim e^{-\lambda^U P} \cos \omega^U P \tag{4.13}$$

as shown in Figure 5 (more generally, there would be a phase addition to P). The number N of distinct components (which would be sets $V_1, V_2, \ldots, V_N$ in Figure 3) is asymptotically given by the number of roots of (4.13), and is thus

$$N \sim \frac{\Omega}{\pi} \ln \frac{1}{\mu} \tag{4.14}$$

as $\mu \to 0$. When $\mu = 0$, there are an infinite number of components, which are essentially labeled by the value of P and which image P' we choose [more specifically, the components can be labeled by the index r, where for given P the image P' lies in $((k-r-1)\pi, (k-r)\pi)$ for some integer k]. It then follows

that when $\mu = 0$, the component I_r can only map into I_s for those s satisfying $s/r \le \lambda^S/\lambda^U$ (since we require $ae^{\lambda^U P' - \lambda^S P} \le 1$), at least for large enough P. This theorem was proved by Shil'nikov [14].

(iii) *Bifocal Case.* This case, where two single pairs $\lambda^U \pm i\omega^U$, $-\lambda^S \pm i\omega^S$ of eigenvalues have real parts closest to zero (one with positive real part, one with negative real part) was also considered by Shil'nikov [14], and the same theorem quoted above applies. Simplification of (4.3) yields the approximate

$$e^{-\lambda^U P'} \cos \omega^U P' = ae^{-\lambda^S P} \cos(\omega^S P + \varphi) + \mu. \tag{4.15}$$

Supposing, now without loss of generality, that $\lambda^U < \lambda^S$, then fixed points are given by

$$e^{-\lambda^U P} \cos \omega^U P \approx \mu, \tag{4.16}$$

and the same considerations as for the Shil'nikov case apply, with the same conclusions. The secondary oscillations due to $\omega^S \ne 0$ are essentially irrelevant, and (4.13) and (4.14) still apply. When $\lambda^S \approx \lambda^U$ in the bifocal case (and in the Shil'nikov case), the situation is more complicated, and has been implicitly considered by Fowler and Sparrow [4], who consider in more detail the geometry of the component ψ^i when $\lambda^S \approx \lambda^U$, and by Gaspard et al. [6] and Bernoff [2], who consider the Shil'nikov case when $\lambda^S \approx \lambda^U$. The bifocal results also apply if $\lambda^U > \lambda^S$, and the necessary counterpart of (4.16) is

$$\mu \sim e^{-\lambda^S P} \cos \omega^S P, \tag{4.17}$$

and similarly for (4.14).

Stability

Glendinning and Sparrow [7] were able to demonstrate for the three-dimensional Shil'nikov system that tangencies of the one-dimensional map (4.3) [i.e., if we write (4.3) in the form $L(P') = R(P)$, then $dL/dP = dR/dP$ *and* $P = P'$] correspond to saddle-node bifurcations of the principal periodic orbit in the flow, and that such orbits may be stable. This is due to the fact that in their case W_U is one-dimensional, so that $(4.1)_2$ is degenerate, a_U does not exist independently of P, and $M_{UU} = (0)$ in (3.9), which thus defines P' directly. In this case the slope of the map (4.3) [$= R'(P)/L'(P)$] helps decide the stability of the orbits. However, if $\dim W_U \ge 2$, then all periodic orbits will generally be unstable (typically saddles). In general, we cannot say much about how the periodic orbits disappear near a tangency of (4.3). In cases (ii) and (iii), passage through tangency is associated (when there is a strange invariant set) with disappearance of one of the components V_i. Thus many of the orbits are annihilated. It is tempting to suppose that the principal periodic orbit undergoes a saddle-node bifurcation, and moreover that passage of the slope R'/L' of (4.3) through -1

is associated with period doubling. However, while the *existence* of "period-doubled" orbits can be guaranteed by the implicit-function theorem, the bifurcation sequence cannot be so easily ascertained, since although the Jacobian (and its spectrum) depends continuously on μ, we cannot be sure that the passage of an eigenvalue through 1 occurs for any particular orbit. Passage through tangency removes many orbits on the invariant set in a complicated way.

5. Conclusions

Homoclinic bifurcations in n-dimensional flows can cause "explosions," in which strange invariant sets are created. These invariant sets contain countably many periodic orbits and uncountably many aperiodic orbits, all of which are unstable for most values of the bifurcation parameter μ (excluding tangencies). If the unstable manifold W_U of the origin has dimension k, then (most of) the orbits in the invariant set will have $k-1$ positive Lyapounov exponents if $\lambda^S > \lambda^U$, and k if $\lambda^S < \lambda^U$, where λ^S, λ^U are the absolute values of the real parts of the eigenvalues at the origin which are closest to zero. In general, only three types of bifurcation occur, which are counterparts of their low-dimensional equivalents (Lorenz, Shil'nikov, bifocal examples). The extension to systems with symmetry is obviously an important one here, since such systems will naturally arise in practice (for example, in Fourier truncations of partial differential equations).

Most of our discussion has been descriptive rather than rigorous; a thorough analytic description of the derivation of the Poincaré map, and a description of its geometry, is given by Wiggins [17]. However, Wiggins stops short of analyzing the n-dimensional case in any detail, referring only to two papers of Shil'nikov [13, 14]. What is novel in this paper, therefore, is the further reduction of the approximate Poincaré map to a perturbation of a one-dimensional map. This further reduction explicitly uses the fact that as $\mu \to 0$, the recurrence time P becomes large. The use of this approximation dramatically simplifies the analysis. In a subsequent paper, we will show how the same formalism can be applied to partial differential equations.

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