# HOMOCLINIC BIFURCATIONS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

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## ABSTRACT

The formal analysis of bifurcations from homoclinic orbits in low-dimensional ordinary differential equations is here extended to deal with ordinary differential equations in n dimensions, and to certain partial differential equations in one space variable on the infinite real axis. For ordinary differential equations, results are equivalent to various cases treated by Shil'nikov: depending on the eigenvalues at the fixed point, an infinite number of periodic orbits can bifurcate at the critical parameter value. By contrast, homoclinic bifurcations for partial differential equations can produce an infinite number of quasi-periodic (modulated travelling wave) solutions.

## 1. INTRODUCTION

One of the few direct methods of analysis which connects chaotic behaviour with trajectories of differential equations is the analysis of homoclinic bifurcations. These occur in the neighbourhood of parameter values for which homoclinic orbits (orbits which are bi-asymptotic to a fixed point as $t \to \pm \infty$) exist. Many examples of ordinary differential equations exist, for which the existence of chaotic behaviour is closely associated with the existence and nature of homoclinic bifurcations.

The most striking example is perhaps that of the Lorenz equations (Lorenz 1963), whose qualitative behaviour can be largely understood (Sparrow 1982) by an analysis of the associated homoclinic structures. The pioneer for these kinds of study was Shil'nikov (1965, 1967, 1970), and his methodology underlies the treatment of later authors (e.g. Arneodo et al. 1982, Tresser 1984, Glendinning and Sparrow 1984, Gaspard et al. 1984). A recent review is by Glendinning (1988).

If homoclinic bifurcations are as important for our understanding of chaotic behaviour in ordinary differential equations as these papers suggest, then it is clear that one should aim to understand their effect in partial differential equations. This is particularly true

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for (flow) systems whose behaviour seems entirely unrelated to finite-dimensional systems, of which the outstanding example is the transition to turbulent flow in parallel wall-bounded shear flows: Poiseuille flow, boundary layer flow. Emmons' turbulent spot is an observed feature of the breakdown of laminar shear flow, which is conceptually at odds with the descriptions afforded by classical linear and nonlinear stability theories (Widnall 1984, Stuart 1960).

Our aim in this brief paper is therefore to examine the possible bifurcations which may arise from the (postulated) existence of homoclinic orbits in partial differential equations, with a view to establishing what may be different to the case of ordinary differential equations. We proceed heuristically; in certain cases, the results may be proved, but there are substantial technical difficulties involved. Furthermore, brevity compels us to be a little inexact.

## 2. ORDINARY DIFFERENTIAL EQUATIONS

We first illustrate the procedure in $\mathbb{R}^n$, in order to draw a connection between the classical Shil'nikov theory, and the more convoluted exercise for partial differential equations.

Suppose the system

$$\dot{x} = f(x,\mu) \ , \tag{2.1}$$

$x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$, with f smooth, has a fixed point $x = 0$ for all $\mu$, and that when $\mu = 0$, there is a homoclinic orbit $\Gamma$:

$$x = x^*(t), \ x^* \to 0 \text{ as } t \to \pm\infty \ . \tag{2.2}$$

It is convenient to choose the phase of $x^*$ so that $x^* = O(1)$ when $t = 0$, whence

$$\begin{aligned} x^* &\sim \exp(tD)\alpha^* \ , \quad t \to -\infty \ , \\ x^* &\sim \exp(tD)\beta^* \ , \quad t \to +\infty \ , \end{aligned} \tag{2.3}$$

where $\alpha^*, \beta^* = O(1)$, and D is the Jacobian of f at 0, which we may assume to be diagonal. Let $W_S$ and $W_U$ be the stable and unstable manifolds of 0, and let $e^S$ and $e^U$ be eigenvectors in $W_S$ and $W_U$, with corresponding eigenvalues whose real part is closest to zero. We define surfaces $\Sigma$ and $\Sigma'$ by

$$\begin{aligned} \Sigma &: |\langle x,e^S\rangle| = \nu \ , \\ \Sigma' &: |\langle x,e^U\rangle| = \nu \ , \end{aligned} \tag{2.4}$$

where $\nu \ll 1$, as shown in Fig. 1. $\Sigma$ is our Poincaré surface and we aim to construct an approximate map from $\Sigma$ to $\Sigma'$, and thence back to $\Sigma$.



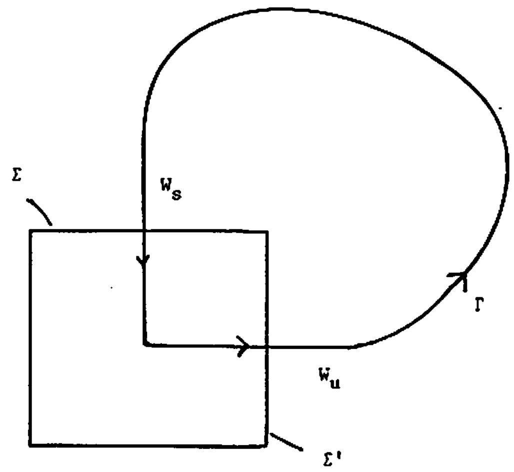


Figure 1

In fact, it is algebraically convenient to use the rescaled coordinates $\alpha$ and $\beta$ on $\Sigma'$ and $\Sigma$, which are defined by analogy with (2.3): if $x^* \in \Sigma'$ at $t = -t_U$, $x^* \in \Sigma$ at $t = t_S$, then we define

$$\begin{aligned} x &= \exp[-t_U D]\alpha \quad \text{on } \Sigma' \ , \\ x &= \exp[t_S D]\beta \quad \text{on } \Sigma \ . \end{aligned} \tag{2.5}$$

The arcane choice of $\alpha$ and $\beta$ ensures that the Poincaré map we shall derive for $\beta$ is properly scaled: in this way, it is transparent to ascertain which terms are in practice small (this is not normally done).

The approximation to the flow consists of two parts. Near 0, from $\Sigma$ to $\Sigma'$, we have

$$\dot{x} \approx Dx \ , \tag{2.6}$$

whence

$$x \approx \exp[(t-t_0)D]x_0 \ , \tag{2.7}$$

and use of (2.5) leads to

$$\alpha \approx e^{PD}\beta \ , \tag{2.8}$$

where $P = t_U + t_S + \bar{t}$, $\bar{t}$ is the transit time from $\Sigma$ to $\Sigma'$ (which depends on $\beta$): P is essentially the interval between passages through $\Sigma$. On the other hand, between $\Sigma'$ and $\Sigma$, we linearise about $x^*$, so that

$$x = x^* + y \ , \quad \dot{y} \approx Df[x^*(t),0]y + \mu \frac{\partial f}{\partial \mu} \ , \tag{2.9}$$

whence

$$y \approx \Psi(t)\Psi^{-1}(t_0)y_0 + \mu\Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\,\frac{\partial f}{\partial \mu}\,ds \,, \tag{2.10}$$

where $\Psi$ is a fundamental matrix for Df. We define

$$\Psi = \exp(tD)H(t) \,, \tag{2.11}$$

and assume $H(-\infty) = I$, $H(\infty) = M$. (Strictly, this requires the eigenvalues of D to be independent of $\mu$.) In this case, we find the image $\beta'$ in $\Sigma$ is given by

$$\beta' - \beta^* \approx M(\alpha - \alpha^*) + \mu c \,, \tag{2.12}$$

where we may take $c = O(1)$ and $M = O(1)$.

The composition of (2.8) and (2.12) now provides an approximate (n-1)-dimensional map (since P is unknown, and $\beta \in \Sigma$, essentially $|\langle\beta, e^S\rangle| = 1$). The further vital observation is that P is large, so that (2.8) and (2.12) are (differentiably) close to a one-dimensional map. To make this explicit, we label elements of $W_S$ and $W_U$ with suffices S and U respectively. With an obvious notation, we write

$$\alpha_U = \alpha_U^* + a_U \,, \quad \beta_S = \beta_S^* + b_S \,, \tag{2.13}$$

so that

$$\begin{aligned} \beta_U &= \exp[-PD_U](\alpha_U^* + a_U) \,, \\ \alpha_S &= \exp[PD_S](\beta_S^* + b_S) \,, \\ b_S' &= M_{SU}a_U + M_{SS}\alpha_S + \mu c_S \,, \\ \beta_U' &= M_{UU}a_U + M_{US}\alpha_S + \mu c_U \,. \end{aligned} \tag{2.14}$$

Notice $\alpha_U^*, \beta_S^* \sim O(1)$, $a_U, b_S, b_S' \ll 1$; if we restrict attention to only those points in $\Sigma$ and $\Sigma'$ which will be in the invariant set of trajectories, i.e. which remain close to $\Gamma$ for all past and future iterations, then clearly we must have

$$\beta_U \approx \exp[-PD_U]\alpha_U^* \,,$$

$$\alpha_S \approx \exp[PD_S]\beta_S^* \,, \tag{2.15}$$

(note Re $\sigma(D_U) > 0$, Re $\sigma(D_S) < 0$), and so $\beta_U'$ must be approximately given by

$$\beta_U' \approx \exp[-P'D_U]\alpha_U^* \tag{2.16}$$

for some P' (otherwise subsequent iterates would not be close to $\Gamma$). Hence $(2.14)_4$ is an approximate equation for $a_U$; however, one can show that rank $(M_{UU}) = k-1$ (if dim $W_U = k$, so that $M_{UU}$ is $k \times k$) (this is because of the autonomy, i.e. time-translation invariance, of (2.1)). If $\eta$ is the unique eigenvector of the Hermitian adjoint of $M_{UU}$, then P' is explicitly determined by the orthogonality criterion

$$\langle\eta, \exp(-P'D_U)\alpha_U^*\rangle = \langle\eta, M_{US}\exp(PD_S)\beta_S^*\rangle + \mu\langle\eta, c_U\rangle \,. \tag{2.17}$$

The exact map is differentiably close to this one.



It is straightforward to read from this map the usual Shil'nikov (1965,1967) results for fixed points: the proof uses the implicit function theorem in a straightforward manner. In particular, the results extend to n dimensions in a seemingly straightforward manner. In order to make statements about aperiodic orbits (that there can be uncountably many which are homeomorphic to a shift on N symbols, where $N \to \infty$ as $\mu \to 0$), one has to be more careful. One cannot use the one-dimensional map on its own. In fact, the proof involves the full (n-1)-dimensional map, but uses the fact that the action of the map is naturally partitioned into the effect on P, together with a hyperbolic structure on the remainder of $W_U$ and $W_S$.

## 3. PARTIAL DIFFERENTIAL EQUATIONS

Here we cursorily sketch the analogous ideas to those of the preceding section. We consider the partial differential equation

$$A_t = N[\partial_x](A;\mu) \,, \quad -\infty < x < \infty \,, \tag{3.1}$$

where N is autonomous in both space and time. We assume $N(0;\mu) = 0$, and that when $\mu = 0$, there is a homoclinic orbit $\Gamma$:

$$A = A^*(x,t) \,, \quad A^* \to 0 \ \ \underline{\text{uniformly}} \text{ as } \ t \to \pm\infty \tag{3.2}$$

(i.e. not a soliton). Moreover, we assume $A^*$ is localised in both space and time. If the dispersion relation at the origin for modes $\exp[ikx + \sigma(k)t]$ is $\sigma = \sigma(k)$, then we choose the phase of $A^*$ so that

$$A^* - \int_{-\infty}^{\infty} \alpha^*(k)\exp[ikx + \sigma(k)t]dk \quad \text{as} \quad t \to -\infty \,,$$

$$A^* - \int_{-\infty}^{\infty} \beta^*(k)\exp[ikx + \sigma(k)t]dk \quad \text{as} \quad t \to +\infty \,; \tag{3.3}$$

the definitions of $\Sigma$ and $\Sigma'$ are entirely analogous to the finite-dimensional case, as are those of $\alpha$ and $\beta$, and we quickly derive

$$\alpha(k) \approx \beta(k)e^{\sigma(k)P} \,, \tag{3.4}$$

in analogy to (2.8). Going round the outside is a little more cumbersome, since for example if (3.1) is a semi-flow (e.g. if parabolic) then the analogy to $\Psi^{-1}$ in (2.10) does not exist (in the same sense). However, with appropriate care, one can indeed derive the analogue of (2.12). Specifically, put

$$A = A^* + v \,, \quad v_t \approx N'(A^*)v + \mu\,\frac{\partial N}{\partial \mu} \,, \tag{3.5}$$

where N' is the derivative of N. Let $T(t,\tau)$ be the (semi) flow corresponding to (3.5), with $\mu = 0$, and let $\psi(x,k;t,\tau)$ be a family of orthonormal eigenfunctions of $T^*T$ satisfying

$$\int_{-\infty}^{\infty} \psi(x,k)\,\bar{\psi}(x,\ell)dx = \delta(k-\ell) \,; \tag{3.6}$$

then we assume

$$\psi(s,k;t,0) \to \psi_\pm(s,k) \quad \text{as} \quad t \to \pm\infty \tag{3.7}$$

(the definition of $\psi$ for $t < 0$ follows from the formal inverse). In this case, we find

$$\beta'(k) - \beta^*(k) = \int_{-\infty}^{\infty} M(k,\ell)(\alpha(\ell) - \alpha^*(\ell))d\ell + \mu c(k) \qquad (3.8)$$

as an analogue to (2.12).

One continues by observing that $P \gg 1$. The procedure is essentially the same as before except that the subsequent homoclinic pulse will in general be phase shifted by a length L, say; thus (with $\beta_U = \beta(k)|_{k\epsilon U}$, $U = \{k \text{ s.t. } Re\sigma > 0\}$)

$$\beta_U \approx \exp[-\sigma_U(k)P]\alpha_U^* \quad \text{in} \quad \Sigma , \qquad (3.9)$$

and

$$\beta_U' \approx \exp[-\sigma_U(k)P - ikL]\alpha_U^* \quad \text{in} \quad \Sigma , \qquad (3.10)$$

for some P' and L. The analogue to the matrix $M_{UU}$ is the operator $\int_U M_U(k,\ell)a_U(\ell)d\ell$ ($M_U = M(k,\ell)|_{k\epsilon U}$), which has two null vectors, corresponding to time translation invariance and spatial translation invariance, and hence we get two orthogonality conditions which serve to define P' and L. The general form is

$$\int_U \exp[-\sigma_U(k)P' - ikL]w_j(k)dk$$
$$= \int_S \exp[\sigma_S(k)P]y_j(k)dk + \mu \qquad (3.11)$$

with j = 1,2 and $w_j, y_j$ known functions.

Yet further simplification uses the fact that $P' \gg 1$ ($= O(1/\mu)$) to approximate the integrals in (3.11) using the method of steepest descents. The result now depends on the dispersion relation. A typical case is $\sigma = \alpha - \gamma k^2$, corresponding to

$$A_t = \alpha A + \gamma A_{xx} + f(A) , \qquad (3.12)$$

where f is a nonlinear operator. Then one gets

$$\sum_m c_{jm}\exp[-i\omega_m P' - ik_m L]/(P' + i\lambda_m L)$$
$$= \sum_m d_{jm}\exp[i\omega_m P]/P + \mu , \qquad (3.13)$$

where $\omega_m, k_m, \lambda_m$ depend on $\sigma(k)$. A simple example is where N is real and symmetric, and $k_m, \lambda_m, \omega_m$ occur in complex conjugate pairs. Then (3.13) can be simplified to

$$\frac{e^{-ikL}}{P' + i\lambda L} = \frac{B}{P} + \mu C , \qquad (3.14)$$

where we assume one positive zero of Re $\sigma$ at k, and B and C are generally complex.

Periodic orbits in the flow would correspond to P = P' with L = 0, and in general do not exist. Multiple fixed points with P = P', L ≠ 0, do exist, and have $P \sim 1/\mu$ as $\mu \to 0$. (This contrasts to the logarithmic dependence, $P \sim \ell n(1/\mu)$, in $\mathbb{R}^n$.) For example, if B and C are real, then

$$L \sim n\pi/k , \quad \mu \sim P^{-1} \qquad (3.15)$$



as $P \to \infty$, $n \in Z^+$. There are countably many of these solutions which have the form of travelling modulated 'solitary' waves. The period of modulation is $P \sim \mu^{-1}$ and the wave speed is $L/P \sim n\pi/\mu k$.

If N(A) is real but not symmetric, then $\omega_m \neq 0$ in (3.13); if $k_m = \pm k$ as before, then $\lambda_m = \pm \lambda$, and the equivalent of (3.14) is

$$\exp[-i\omega P' - ikL]/(P' + i\lambda L) = (B/P)\cos(\omega P + \theta) + \mu C . \qquad (3.16)$$

Fixed points P = P', L = constant exist, with

$$\mu \sim \frac{1}{P}\cos(\omega P + \tilde{\theta}) ,$$
$$\omega P + kL = n\pi , \quad n \in Z^+ , \qquad (3.17)$$

corresponding to Shil'nikov's results (Glendinning and Sparrow 1984). There are countably many such modulated travelling waves, with periods P and $2\pi/\omega$, and the wave speed is $L/P \sim -\omega/k$. Figures 2 and 3 illustrate the general dependence of P on $\mu$ and L.

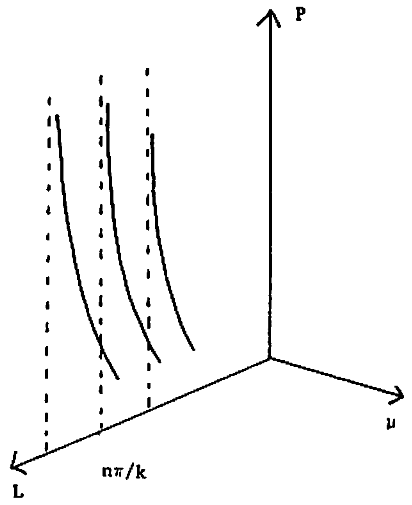

Figure 2. (3.15)

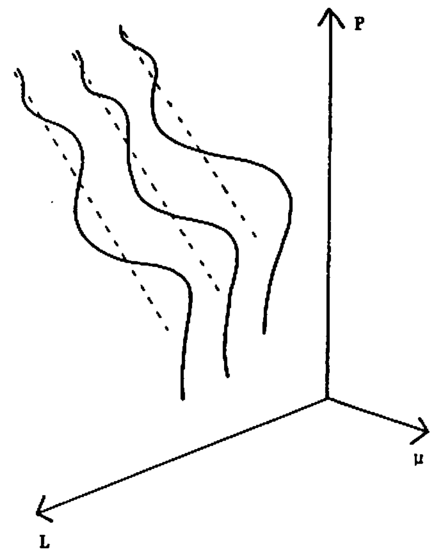

Figure 3. (3.17)

## 4. DISCUSSION

The resumé here is obviously very truncated, and will be developed elsewhere. However, it is clearly formally possible to develop results for partial differential equations which are analogous to those which can be obtained for ordinary differential equations. The principal conclusion is that periodic solutions are not generated, but rather modulated travelling waves with one very long period. There is plenty of suggestive evidence of such types of solutions in results reported at this conference. However, the details obviously depend closely on the form of the dispersion relation $\sigma(k)$, amongst other things.

As a final comment, let us note that if, in (3.12), $\alpha$, $\gamma$ or f(A) is complex, then in general, we obtain two complex equations for the two real unknowns P' and L. (When $\alpha,\gamma,f(A)$ are real, these are complex conjugates of each other.) This is inconsistent, and we pose as a preliminary conjecture the idea that in this case, pair production occurs: two homoclinic pulses are produced by one parent, with values $P_i', L_i$ determined by the two complex equations. This conjecture is preliminary, since it would be destroyed by a further two orthogonality relations, which one might suppose could arise from taking the complex conjugte of the equation. At the moment, this question is unresolved. However, if homoclinic orbits induce pair production, then the consequent trajectories have no counterpart in finite systems, but might provide a possible framework for the interpretation of turbulent spot generation in shear flows.

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