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CONVECTION AND CHAOS

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Abstract. We discuss two types of chaotic behaviour exhibited by high Prandtl number convection, that is 'phase chaos' and plumes. In mantle convection, these differing aspects of the motion find their expression in the migration of subduction zones and hot spots, respectively. The analysis of plumes in a single convection cell can be attempted in the framework of Howard's 'bubble' model of convection, using an asymptotic analysis based on a similar method applied to the Lorenz equations. In the partial differential equation case, this leads us in principle to an approximate Poincaré map for the flow. However, we find that Howard's assumption of differing time scales for the processes of growth and flow instability is in error, and (for a single developing plume) the thermal regime is likely to be periodic. For the case of a large aspect ratio, where multiple plume development can take place, the corresponding Poincaré map should lead to a chaotic distribution of plumes in space and time.

For cellular convection, we give a synopsis of the recent scaling theory of the Chicago group. There is a mean flow in the cell, fuelled by thermal plumes which erupt from the boundary layer as they are advected across the cell. The theory is strictly applicable, however, *only to Prandtl numbers of* $O(1)$.

Cellular ('phase') chaos at large Rayleigh number can be modelled using a set of ordinary differential equations for variables which describe the size and location of slowly varying convection cells. The differential equations are parametrised using quasi-stationary boundary layer theory. The same method can in principle be extended to three dimensions, and represents a paradigm for the study of time-dependent motions of the earth's lithospheric plates.

1.TURBULENCE IN CONVECTION

The general picture that we have of the transition to turbulent convection derives from experiments of Krishnamurthi and Busse (Krishnamurthi 1970, Busse and Whitehead 1974, 1976), building on earlier work by Malkus (1954). For fluids with Prandtl number $\sigma \gtrsim O(1)$, this picture is as follows: two-dimensional rolls become unstable to three-dimensional (superimposed) rolls as the Rayleigh number Ra is increased. At higher Ra, time-dependence sets in, the flow becoming periodic and eventually chaotic.

This general picture was augmented by experimental physicists (Ahlers, Libchaber, Gollub and co-workers), beginning in the late seventies, who added such topics of contemporary interest as period-doubling and frequency locking (Libchaber and Maurer 1982), thus relating time-dependent convection to the phenomena exhibited by chaotic solutions of ordinary differential equations. These experiments were done in small aspect ratio boxes, and it is because of this that the fluid behaves like a finite-dimensional system, at least for low enough Rayleigh number. The spectrum of available modes is discrete, and at low Rayleigh number, few are excited. At high Rayleigh number, the length scale of "coherent features" (e.g. plumes) of the flow shrinks, and the box size becomes of decreasing significance.

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(i) Phase chaos

In large aspect ratio boxes, however, finite dimensional notions are more or less irrelevant. In this article we wish to focus on two particular aspects of convection in large boxes (or alternatively, at high Rayleigh number) where the dynamics is fundamentally infinite-dimensional, and in effect occupies a continuous spectrum of wavenumbers. The most obvious example is 'phase chaos', as studied, for instance, by Gollub et al (1982). Here the idea is that at Rayleigh numbers Ra not far above the critical value Ra_c, two-dimensional convection rolls may exhibit a slow large-scale irregular behaviour due to an incoherence of phase between widely separated rolls. That is, the flow is locally regular and two dimensional, but its amplitude and phase, described by a complex amplitude function A, can vary over long space and time scales. This idea is exactly that for which nonlinear stability theory is designed, and expansions of the flow equations lead to equations for A which may be typically represented by the complex Ginzburg-Landau equation

$$A_t = \alpha A + \beta|A|^2 A + \gamma\nabla^2 A \tag{1.1}$$

(which is not actually appropriate to convection (Newell and Whitehead 1969), though it does properly apply to instabilities in some other systems). Depending on the values of α, β, γ, chaotic solutions of (1.1) can appear through the possible mechanisms of spontaneous defect generation (Coullet et al. 1989) or 'homoclinic bursts' (Newell et al 1988), neither of which phenomenon seems easily related to the behaviour of finite-dimensional systems.

This idea, that chaotic solutions can be manifested by large scale evolution of a local, quasi-regular structure, can be extended to high values of the Rayleigh number, at least at high Prandtl number. Busse and Whitehead (1974) showed that there is an instability of bimodal three-dimensional steady convection (which may be thought of as two superimposed rolls) to a 'spoke-pattern', where a quasi-cellular pattern bounded by sheets of descending fluid has a hub of upwelling hot fluid connected to the periphery by sheets of rising hot fluid. This pattern was studied further by Whitehead and Parsons (1978). The spoke pattern evolves irregularly in space and time on a slow time scale, and while the evolution appears random, the local structure is regular. However, low Rayleigh number expansions can not sensibly be applied, and some other analytic technique is necessary.

(ii) Plumes

A different way in which chaotic behavior is manifested in convecting fluids occurs when the Rayleigh number is increased in a box of finite dimension. Thermal boundary layers develop at the base and at the top, and as the cell dimension effectively grows, the thermal boundary layers develop instabilities which erupt as plumes from the boundary layer. At low Rayleigh number, these instabilities may occur as ripples which ride round the cell on the thermal boundary layers, but at higher Rayleigh number, they can break off. This eventually leads to a horizontal spatial incoherence, where plumes break away from the top and bottom at irregular locations. Visualisation of such plumes has been made by Sparrow et al. (1970), for example. A second degree of spatial incoherence can occur when upwelling thermal plumes do not connect to the opposite boundary. In this case they may mix into a mean circulation in the core of the cell, as has been observed by Krishnamurthi and Howard (1981). At high Prandtl number, it may be that such plumes are more isolated events, both in space and time, though this is not certain. Thus if one seeks "coherent structures" in turbulent convection, candidates may be the 'mushroom-cap' heads of thermals, as described by Griffiths (1986), for example.

2. Mantle convection

A good deal of interest exists in the geophysical community in understanding convection, and some observations on the nature of mantle convection will be made here. Convection in the earth's mantle has been reviewed by Hager and Gurnis (1987) and Silver et al. (1988). The literature is enormous, but little concrete understanding of the problem has been gained since the early review by Turcotte and Oxburgh (1972). Constant viscosity convection provides a useful paradigm, but must be (though it is usually not) viewed with a critical perspective, since there are fundamental geophysical variations which must alter the nature of the flow enormously. Chief amongst these are chemical inhomogeneity, phase changes (perhaps) and variable viscosity.

(i) Subduction and plate migration

Despite this, certain features of the flow can be taken as analogues of laboratory experimental features. In particular, subducting lithospheric plates are the descending cold boundary layers of convection 'cells'. The flow is asymmetric because of the enormous temperature dependence of viscosity: thus the cold, subducting plates are thick and stiff, while corresponding ascending plumes are likely to be less positively buoyant. The slabs descend as more or less coherent sheets. Thus, while the details are different, the slabs organise the geometry of the flow in a cellular way. It is known that the cells (bounded by plate margins) are time dependent on the earth, and one ultimate goal of plate tectonics must be to account for and predict this dependence. It is essentially obvious (because of the lateral asymmetry) that as subduction begins, and slabs sink into the mantle, that the subduction zone will migrate backwards towards the ridge. This may of course be compensated by ridge movement, but the point is that secular movement is to be expected, and one ultimate way of modelling this may be by parametrising the cellular structure in the way that is done here. Certainly, the evolution of plate boundaries bears a surficial resemblance to phase-incoherent cellular convection in laboratory fluids.

(ii) Hot spots

There are numerous regions of anomalous heat flow on the earth, often associated with volcanism; the two obvious examples being Hawaii and Iceland. The plume hypothesis (Morgan 1971) associates these with upwelling thermal plumes in the mantle. Laboratory versions of such plumes in high Prandtl number fluids have been studied by Griffiths (1986) and Olson and Singer (1985); they retain a strong spherical cap-like structure with a thin umbilical tail beneath.

If the whole mantle convects in a more or less coherent fashion (something which is by no means clear), then such plumes could represent instabilities in a basal thermal boundary layer, by analogy to laboratory convection. Mantle plume hot spots seem to be largely independent of plate boundaries, and it may be possible to view the two phenomena as being largely independent. By understanding how thermal plumes are erupted from basal boundary layers, we may therefore be led towards an understanding of the behaviour of hot spots in the mantle.

3. PLUMES

The basic feature about plumes is that they form localized regions in which the temperature is anomalously high (upwelling) or low (downwelling). As such, their basic description involves a boundary layer analysis of convection at high Rayleigh number. Several authors have undertaken such an analysis of steady high Rayleigh number convection; most notable in a geophysical context are the papers by Turcotte and Oxburgh (1967) and Roberts (1979), the latter of which is the best treatment of this subject. We begin by recalling this basic theory. For simplicity in the exposition we will suppose that the Prandtl number is infinite, and will also consider only the case of free slip boundaries.

(i) Boundary layer theory for steady convection

Dimensionless equations governing Boussinesq convection in two dimensions may be written

$$\begin{aligned} \nabla\cdot\mathbf{u} &= 0, \\ \frac{1}{\sigma}\frac{d\mathbf{u}}{dt} &= -\nabla p + \nabla^2\mathbf{u} + Ra\,T\mathbf{k}, \\ \frac{dT}{dt} &= \nabla^2 T, \end{aligned} \tag{3.1}$$

where T is temperature, $\mathbf{u}$ is the velocity, p is pressure, $\mathbf{k}$ is a unit vector pointing vertically upward, and d/dt represents the material derivative $\partial_t + \mathbf{u}\cdot\nabla$. In two dimensions, $\mathbf{u} = (u, v)$, and we define a stream function ψ by $u = -\psi_y$, $v = \psi_x$ (notice the sign convention used here), and the vorticity $\omega = v_x - u_y = \nabla^2\psi$. We take the Prandtl number σ to be infinite, and will suppose that the Rayleigh number Ra is large. Eliminating p, we have

$$\begin{aligned} \omega &= \nabla^2\psi, \\ \nabla^2\omega + \frac{1}{\delta}T_x &= 0, \\ T_t + \psi_x T_y - \psi_y T_x &= \delta^2\nabla^2 T, \end{aligned} \tag{3.2}$$

where we have rescaled ψ and ω with $Ra^{2/3}$, and defined

$$\delta = Ra^{-1/3}; \tag{3.3}$$

this ensures that (for free slip boundaries), both ψ and ω are $O(1)$ in the flow.

Suitable boundary conditions to apply are that

$$\psi = \omega = 0 \text{ on } x = 0, a;\ y = 0, 1, \tag{3.4}$$

and

$$\begin{aligned} T_x &= 0 \text{ on } x = 0, a; \\ T &= 1/2 \text{ on } y = 0, T = -1/2 \text{ on } y = 1. \end{aligned} \tag{3.5}$$

The basic boundary layer theory studied (most comprehensively) by Roberts (1979) is now summarised briefly. The flow consists of a central *core*, two thermal *plumes* at the sidewalls, and two *thermal boundary layers* at the top and bottom. The plumes and thermal boundary layers are connected at the corners, where one can show that the local thermal structure is simply advected round.

A. Core

Here $T = 0$ to all (algebraic) orders of δ, and therefore

$$\nabla^4\psi = 0, \tag{3.6}$$

with appropriate conditions being $\psi = 0$ on all boundaries, $\omega = 0$ on top and bottom, but $\omega = O(1)$ (to be found) at the sides; the non-zero vorticity (which drives the flow) is generated in the thermal plumes at the side walls.

B. Plume

We can suppose the core flow is clockwise, so that $\psi > 0$ and $\omega < 0$, and consider the plume near $x = 0$. Suitable boundary layer scales are

$$x \sim \delta, \quad T \sim 1, \quad \omega \sim 1, \quad \psi \sim \delta, \tag{3.7}$$

whence we find (at leading order) that the rescaled variables satisfy

$$\begin{aligned} \Psi &\sim v_p(y)X, \\ \omega &\sim -\int_0^X T\,dX, \end{aligned} \tag{3.8}$$

whence the core vorticity at the wall is

$$\omega_\infty \sim -\int_0^\infty T\,dX. \tag{3.9}$$

The temperature satisfies

$$\Psi_X T_y - \Psi_y T_X \sim T_{XX}, \tag{3.10}$$

and in Von Mises variables y, Ψ,

$$T_y \sim v_p(y) T_{\Psi\Psi}, \tag{3.11}$$

where v_p is the plume velocity determined by matching to the core flow. Conservation yields $\int_0^\infty T\,d\Psi = \int_0^\infty v_p T\,dX = C$ is a constant, and thus, using (3.9), the sidewall boundary condition for the core flow at $x = 0$ is

$$\psi_x \psi_{xx} = \omega_\infty v_p = -C. \tag{3.12}$$

The core flow can now be solved provided C is determined; this requires solutions of the thermal boundary layer equation.

C. Thermal boundary layer (top)

Putting

(3.13) $$y = 1 - \delta Y, \quad \psi = \delta\Psi, \quad \omega = \delta\Omega,$$

we find

(3.14) $$\Psi \sim u_s(x)Y,$$

where u_s is the surface velocity determined from the core flow, and then in Von Mises variables x, Ψ,

(3.15) $$T_x \sim u_s T_{\Psi\Psi}.$$

Thus

(3.16) $$\frac{\partial T}{\partial s} \sim u\frac{\partial^2 T}{\partial \Psi^2}$$

all the way round the boundary, where u is the core-determined speed, and s is the distance coordinate. Boundary conditions for T are that $T \to 0$ as $\Psi \to \infty$, and alternatively $T = 1/2$, $T_\Psi = 0$, $T = -1/2$, $T_\Psi = 0$ on the four sides of the flow at $\Psi = 0$; T is required to be periodic in s.

D. Solution Strategy

Solve the core flow with $C = 1$, denote the solution with an overhat: $\hat{\psi}$. Thus $\psi = C^{1/2}\hat{\psi}$, $u = C^{1/2}\hat{u}$. The solution of (3.16) is thus

(3.17) $$T = \hat{T}(s, C^{-1/4}\Psi),$$

where $\hat{T}$ is the solution when $C = 1$. Thus $\hat{\psi}$ and $\hat{T}$ are successively solved, and then C is determined from the value of $\int_0^\infty T\,d\Psi$ in the plume, whence

(3.18) $$C = \left\{\int_0^\infty \hat{T}\,d\Psi\right\}^{4/3},$$

which completes the solution.

(ii) The Howard 'bubble' model of turbulence

The description of convection by boundary layer theory at high Rayleigh number is idealistic, but not without value. As Ra is increased, convection in real fluids at quite large Prandtl numbers undergoes a transition to steady three-dimensional flow, then to time-dependent, and eventually 'chaotic' or 'turbulent' flow (meaning here simply disordered in space and time). Nevertheless, even when the motion is irregular, the basic structure of a well-mixed, relatively isothermal interior with thermal boundary layers remains appropriate. As explained earlier, the time-dependence arises through the twin features of irregular meandering of cell boundaries, and irregular eruption of plumes from the thermal boundary layers.

An influential paper which seeks to give an approximate quantitative description of this latter phenomenon is that of Howard (1966). This description (sometimes called the 'bubble' model) involves the idea that, if the Prandtl number is large, then convective instability can occur on a time scale which is short compared to that of thermal diffusion. Thus, he visualised turbulent convection at high Rayleigh number *and* high Prandtl number as consisting of a relaxational oscillation, in the slow phase of which, conductive thermal boundary layers grow into a relatively stagnant, isothermal layer of fluid. At some point, the thermal boundary layers become convectively unstable. They then detach rapidly to form a plume, which ascends rapidly through the fluid, and recreates a well-mixed isothermal state.

There is plenty of quantitative experimental evidence which is consistent with this remarkable description (Katsaros et al. 1977, Tamai and Asaida 1984, Tritton et al. 1980), and its conception has become extremely popular in the geophysical literature (e.g. Kenyon and Turcotte 1983). The quantitative theory rests on the notion of a locally-defined 'critical Rayleigh number'. The idea here is that, for a growing boundary layer of dimensionless thickness ϵ, say, the effective local Rayleigh number describing instability is given roughly by $Ra/2\epsilon^3$ (the factor of two coming from the temperature difference between the base and the interior). If one supposes that instability occurs when this is equal to some appropriate critical value Ra_c, then one predicts $\epsilon \sim (Ra/2Ra_c)^{-1/3}$, and thus the Nusselt number (averaged in time) will be $Nu \sim (Ra/Ra_c)^{1/3}$ which is not inconsistent with some experimental results.

This idea of a locally defined Rayleigh number has been widely misappropriated in the geophysical literature, and has foisted such misguided concepts as 'small-scale convection' (Parsons and McKenzie 1978). Howard's idea has never been placed in a consistent mathematical framework. Here we will try to show how this can be done, and in so doing, give a more precise description of the boundary layer instability.

(iii) The Lorenz equations and the Lorenz-Howard connection

The Lorenz equations

(3.19) $$\begin{aligned} \dot{X} &= -\sigma X + \sigma Y, \\ \dot{Y} &= (r - Z)X - Y, \\ \dot{Z} &= XY - bZ. \end{aligned}$$

form both a paradigm for chaos in ordinary differential equations, and have an association with convection, insofar as they represent the amplitudes X, Y, Z of a three-mode Fourier truncation of two-dimensional Boussinesq convection; particularly, X represents the velocity (or vorticity) field, Y represents the horizontal temperature variation, and Z represents the horizontally averaged temperature. Lorenz (1963) found that at $r = 28$, $\sigma = 10$, $b = 8/3$, solutions behaved in an irregular chaotic manner, and the trajectories consisted of alternating excursions of X and Y (and jumps in Z), followed by longer, quiescent phases where X, Y are small and Z decays exponentially. He also showed that the dynamics could be predicted from a one-dimensional contraction of the Poincaré map. The map is one-dimensional because of the large dissipation, which is due to σ being large. Complicated dynamics occurs when r is large, and in later work (Sparrow 1982), it was shown that chaotic motion arose through the occurrence of homoclinic bifurcations associated with homoclinic orbits of the flow.

The approximate dynamics realized by the equations when r and σ are both large suggests that a partial analysis of the equations might be possible. This is, in fact, the case, and Fowler and McGuinness (1982, 1983) showed that when $r \sim \sigma >> 1$, the solutions were characterised by the striking oscillations described above. One can then *derive* an approximate Poincaré map for the system, whose predictions of chaotic behaviour agree both quantitatively and qualitatively with numerical computations.

There is an interesting connection between these relaxational solutions of the Lorenz equations, and the Howard 'bubble' model. Notice first that in the Lorenz equations, σ is the Prandtl number, and r is the 'reduced' Rayleigh number Ra/Ra_c. If one recalls that X represents the velocity (or vorticity) field, while Z represents the horizontally averaged temperature, then the rapid excursions of X and Y correspond to a rapid overturning of the cell, while the slow decay of Z and the small values of X and Y during the quiescent phase correspond to the slow thermal diffusion from the boundaries in the absence of convection. In other words, the Lorenz equations are doing their best to represent Howard's conceptual model at high Ra and σ.

This suggests the following strategy: we seek to give a mathematical description of the bubble model by emulating the analysis which successfully copes with the Lorenz equations. The main complication is that the solutions we seek are those of a relaxation boundary layer flow, and involve singularly perturbed regions in both space and time. We will give a description of one part of this analysis in the following section. First we recall the salient features of the Lorenz analysis (for details, see Fowler and McGuinness (1982)).

In the slow phase, X, Y, are *exponentially small*, but crucially non-zero. Z decays slowly, and X, Y are governed by a *linear* equation with slowly varying coefficients (due to Z), which can be trivially solved using the WKB method. At first, X, Y also decay (rapidly), but eventually they grow. Since their growth rate is rapid, they become of order one suddenly, and there is then a rapid excursion in which thermal diffusion (decay of Z) is irrelevant, but decay of X and Y occurs due to the viscosity (i.e. due to the dissipation term in the X equation). This nonlinear part of the solution only needs to be solved in terms of its initial and final states, which thus gives a map from one side to the other. Solution of the linear slow phase then permits the determination of a *Poincaré map* for the flow. The analysis is, in fact, very similar to that involved in homoclinic bifurcation analysis (Sparrow 1982). Our ultimate aim for the equations of Boussinesq convection is thus to *derive* an approximate Poincaré map. It may be quite possible to *prove* results on strange *attractors* in this way.

4. Homoclinic connections for boundary layer instabilities

(i) Boundary layer instability

We begin by recalling the basic non-dimensional equations:

$$\begin{aligned} \frac{1}{\sigma}\dot{\omega} &= RaT_x + \nabla^2\omega, \\ \omega &= \nabla^2\psi, \\ \dot{T} &= \nabla^2 T, \end{aligned} \tag{4.1}$$

where $\dot{T} = dT/dt$, $\dot{\omega}$ are the advective derivatives. Let us suppose that diffusive thermal boundary layers grow to a distance of order ϵ, and that within this developing boundary layer exponentially small vorticity and buoyancy disturbances eventually grow exponentially on a time scale $t \sim \nu$. In addition, we suppose the natural length of such disturbances is $O(\delta) \lesssim 1$ (we might suppose that $\delta \sim \epsilon$ for very high Ra, but $\delta \sim 1$ at lower Ra).

Rescale the variables as

$$t \sim \nu, \quad y \sim \epsilon, \quad x \sim \delta, \quad \psi \sim \epsilon\delta/\nu, \quad \omega \sim \delta/\epsilon\nu, \tag{4.2}$$

where the choice of ψ scaling is in order for the Jacobian advective term to remain comparable to the time derivative. We define

$$Ra = \frac{\delta^2}{\epsilon^3\nu}, \quad \beta = \epsilon/\delta; \tag{4.3}$$

the equations become (in the rescaled variables)

$$\begin{aligned} \frac{1}{\sigma b}\dot{\omega} &= T_x + \nabla_\beta^2\omega, \\ \omega &= \nabla_\beta^2\psi, \\ \dot{T} &= b\nabla_\beta^2 T, \end{aligned} \tag{4.4}$$

where

$$b = \nu/\epsilon^2, \quad \nabla_\beta^2 = \frac{\partial^2}{\partial y^2} + \beta^2\frac{\partial^2}{\partial x^2}. \tag{4.5}$$

Before choosing b, let us set out a strategy based on Howard's model, and analogous to the treatment of the Lorenz equations. If $\psi << 1$, then the horizontally average temperature $\overline{T}$ satisfies (approximately)

$$\overline{T}_t = b\overline{T}_{yy}, \tag{4.6}$$

with $\overline{T} = 1/2$ on $y = 0$, $\overline{T} \to 0$ as $y \to \infty$. In fact we take (4.6) as the definition of $\overline{T}$, put

$$T = \overline{T} + \tilde{T}, \tag{4.7}$$

so that

$$\tilde{T}_t + \psi_x \overline{T}_y + (\psi_x \tilde{T}_y - \psi_y \tilde{T}_x) = b\nabla_\beta^2 \tilde{T}, \tag{4.8}$$

and the vorticity equation in full is

$$\frac{1}{\sigma b}[\omega_t + (\psi_x \omega_y - \psi_y \omega_x)] = \tilde{T}_x + \nabla_\beta^2 \omega. \tag{4.9}$$

Thus our ansatz will be that $\psi, \omega, \tilde{T}$ are all exponentially small, so that the following linear approximation holds:

$$\begin{aligned} \frac{1}{\sigma b}\omega_t &= \tilde{T}_x + \nabla_\beta^2 \omega, \\ \omega &= \nabla_\beta^2 \psi, \\ \tilde{T}_t + \psi_x \overline{T}_y &= b\nabla_\beta^2 \tilde{T}. \end{aligned} \tag{4.10}$$

If we now pursue a strict analogy to the Lorenz equations (where the equations for X, Y, Z correspond to those for ω, $\tilde{T}$, $\overline{T}$ respectively), we would choose $b = 1/\sigma << 1$, so that $\overline{T}$ would be a function of the slow time t/σ, while ω and $\tilde{T}$ would evolve on the fast time. Let us postpone this assumption for the time being. For simplicity, suppose $\delta >> \epsilon$, so that $\beta << 1$. We have

$$\begin{aligned} \frac{1}{\sigma b}\omega_t &= \tilde{T}_x + \omega_{yy}, \\ \omega &= \psi_{yy}, \\ \tilde{T}_t + \psi_x \overline{T}_y &= b\tilde{T}_{yy}. \end{aligned} \tag{4.11}$$

Elimination of $\tilde{T}$ and ω yields the equation

$$\left(\frac{\partial}{\partial t} - b\frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{\sigma b}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2}\right)\psi_{yy} + \psi_{xx}\overline{T}_y = 0, \tag{4.12}$$

or (neglecting a term of $0(1/\sigma)$)

$$\frac{1}{\sigma b}\psi_{yytt} - \psi_t^{iv} + b\psi^{vi} + \psi_{xx}\overline{T}_y = 0, \tag{4.13}$$

where superscript Roman numerals indicate partial derivatives with respect to y. Relevant boundary conditions for conditions of free slip at the base are

$$\begin{aligned} \psi = \psi^{ii} = \psi^{iv} = 0 \quad &\text{at } y = 0, \\ \tilde{T}, \psi_y \to 0 \quad &\text{as } y \to \infty. \end{aligned} \tag{4.14}$$

Now we find that if $b = 1/\sigma$ (and we neglect the third term in (4.13)) then the solution grows unstably for all t. That is, there is no period of decay, and the strategy adopted for the Lorenz system fails. Instead, suppose we choose

$$b = 1, \tag{4.15}$$

so that

$$\nu = \epsilon^2, \quad \delta^2/\epsilon^3 = Ra. \tag{4.16}$$

Then the boundary layer equation governing the growth of ψ is , to leading order,

$$\psi_t^{iv} = \psi^{vi} + \psi_{xx}\overline{T}_y. \tag{4.17}$$

Solutions are exponential in x but not in t (since $\overline{T}_y$ is time-dependent), thus

$$\psi = \phi e^{ikx}, \tag{4.18}$$

and

$$\phi_t^{iv} = \phi^{vi} + k^2(-\overline{T}_y)\phi. \tag{4.19}$$

An initially developing conductive thermal profile has

$$\overline{T} = \frac{1}{2}\mathrm{erfc}(y/2t^{1/2}), \quad -\overline{T}_y = \frac{1}{2\sqrt{\pi t}}e^{-y^2/4t}. \tag{4.20}$$

Some idea of the nature of solutions of (4.19) can be obtained by using the quasi-static approximation, in which $\overline{T}$ varies slowly with t (Robinson 1976). Then

$$\phi \sim \exp\left[\int^t \mu(t)\,dt\right] f(y) \tag{4.21}$$

gives the relevant WKB approximation, and

$$\mu f^{iv} = f^{vi} + k^2(-\overline{T}_y)f. \tag{4.22}$$

The eigenvalue μ can be determined by a variational principle, which is obtained from

$$\mu = \frac{\rho k^2 \int_0^\infty (-\overline{T}_y)|f|^2\,dy - \int_0^y |f'''|^2\,dy}{\int_0^\infty |f''|^2\,dy}. \tag{4.23}$$

Since $-\overline{T}_y \to 0$ as $t \to 0$ for $y > 0$, it is clear from (4.23) that $\mu < 0$ for small values of t, but as t increases, μ will become positive. There will thus be an initial decay of ψ towards zero, followed by a resurgence, and the eruption of the next plume.

In fact, a more complete analysis shows that both terms in b in (4.13) need to be included; the solutions for ψ have multiple scale behaviour in y, and a thorough analysis is necessary.

(ii) Plume eruption dynamics

Suppose an erupting thermal plume is (relative to the boundary layer scales in (4.14), with $b = 1$) of thickness δ_p. Then $x \sim \delta_p$, $\psi \sim 1$, whence the plume velocity $v_p \sim 1/\delta_p >> 1$; if the plume rises in stagnant fluid, then (if $\delta_p << \beta$) $\omega \sim \beta^2/\delta_p^2$, and also the vorticity-buoyancy balance implies that $\beta^2\omega/\delta_p^2 \sim 1/\delta_p$; thus

$$\delta_p \sim \beta^{4/3}, \tag{4.24}$$

and the plume dynamics are described by the approximate equations (where we take $y \sim \beta^{-2/3}$, $t \sim \beta^{2/3}$, in order to obtain the vertical scale over which heat conduction acts):

$$\begin{aligned} 0 &= T_x + \omega_{zz} \\ \omega &= \psi_{xx}, \\ \dot{T} &= T_{xx}. \end{aligned} \tag{4.25}$$

A further integration yields

$$\begin{aligned} \psi_{xxx} &= -T, \\ T_t + \psi_x T_y - \psi_y T_x &= T_{xx} \end{aligned} \tag{4.26}$$

and one must seek numerical solutions; the important point from the point of view of the thermal boundary layer is that the plume ejection is *rapid* if $\beta << 1$. This leads us to a schematic description of the dynamics of the boundary layer, which is consistent with Howard's bubble model. This is the following: we solve the linear equation (4.17) for the evolution of the stream function field in the boundary layer, starting from a (small) initial value $\psi_0(x)$. There are two distinct cases. Firstly, suppose the domain is $(0, 2\pi)$, and assume that ψ develops a maximum within the cell (i.e. use (4.18) with $k = 1$); when $\psi = 0(1)$, nonlinear terms become important and lead (we suppose) to the rapid ejection of a plume. Howard's idea would then be that after the plume ejection, the average temperature profile reverts to $\overline{T} = 0$ in $y > 0$, and ψ evolves as before, till ejection of the next plume. This leads naturally to periodicity in plume emission with a periodicity of $O(1)$, which in dimensional terms is of order $(d^2/\kappa)\epsilon^2 = (d^2/\kappa)(Ra/\delta^2)^{2/5}$. Chaos is essentially irrelevant here. The second, more relevant case is where $\delta << 1$ (as we suppose), so that the boundary layer length scale is large; then multiple plumes can form. We divide the x axis into 'catchment areas', divided by stagnation points (where $\psi_y = 0$). Then initial evolution of a stream function $\psi_0(x)$ will lead to ψ first becoming $O(1)$ in one of the catchment areas. The rapid plume eruption then takes place, and we could suppose that in this catchment area, ψ rapidly drops to zero, and gives us a new initial $\tilde{\psi}_0(x)$. In this way, we can in principle derive a Poincaré map taking $\psi_0 \to \tilde{\psi}_0$. Such a map is conceptually very similar to the way one would construct a Poincaré map for , e.g., the complex Ginzburg-Landau equation, cf. Newell et al. (1988). However, because the time scale of evolution of the instability is the same as the interval between plume ejection, then really $\psi = O(1)$ all the time, and the

formalisation of the idea is not asymptotically valid. It is likely that homoclinic trajectories connecting $\psi = 0$ to itself exist, but in practice they would not be obtained. Nevertheless, the discussion here raises the interesting idea that chaos in extended systems may arise through the phase indeterminacy of the homoclinic structures (here, plumes), and this can be compared to the homoclinic bifurcation analysis for partial differential equations on an infinite domain (Fowler 1990).

(iii) The KZZ model

The above effort to analyse Howard's scaling analysis may in fact be rendered obsolete by the scaling analysis done by the Chicago group (Kadanoff, Zaleski and Zanetti being those responsible) (Castaing et al. 1989). Here we paraphrase this model, and comment on its applicability. In particular, it requires the Prandtl number σ to be $O(1)$, and is hence *inapplicable* to geophysical flows. We begin with the dimensionless equations (4.1) in stream function/vorticity variables:

$$\begin{aligned} \frac{1}{\sigma}\dot{\omega} &= RaT_x + \nabla^2\omega, \\ \omega &= \nabla^2\psi, \\ \dot{T} &= \nabla^2 T; \end{aligned} \tag{4.27}$$

equivalently,

$$\begin{aligned} \frac{1}{\sigma}\dot{\mathbf{u}} &= -\nabla p + RaT\mathbf{k} + \nabla^2\mathbf{u}, \\ \nabla \cdot \mathbf{u} &= 0, \\ \dot{T} &= \nabla^2 T. \end{aligned} \tag{4.28}$$

The KZZ argument is the following. There is a basal thermal boundary layer of thickness $\epsilon \sim 1/\mathrm{Nu}$, where Nu is the Nusselt number, a 'mixing region' of height δ, in which plumes erupted from the boundary layer mix with the core flow. Suppose that eddies retaining a temperature excess Δ_c (over the average, zero) move with a typical velocity u_c in the core (note that the quantities δ_c, u_c used here are *dimensionless*, scaled as in (4.28)). Suppose that plumes ascend from the basal boundary layer with velocity u_p and have temperature excess δ_p. Then we have

$$u_c \sim (\sigma Ra\Delta_c)^{1/2}; \tag{4.29}$$

this represents a balance of acceleration and buoyancy in the core of the cell; it assumes that the Reynolds number of the flow is large, specifically that

$$u_c \gtrsim \sigma. \tag{4.30}$$

If we integrate $(4.28)_3$ over a closed volume V, and assume that T is statistically stationary, then the divergence theorem implies

$$\int_S (Tu_n - \frac{\partial T}{\partial n})\, ds = \mathrm{Nu}, \tag{4.31}$$

where S is any horizontal line in the cell. If hot parcels rise and cold parcels descend, this gives

$$\text{Nu} \sim \Delta_c u_c \sim \Delta_p u_p. \tag{4.32}$$

A third balance arises by balancing vorticity and buoyancy in the upwelling plumes. *Assuming* these have representative thickness ϵ, *but have temperature excess* $O(1)$(!), then

$$u_p \sim \epsilon^2 Ra. \tag{4.33}$$

Note the seeming contradiction in this statement (plumes had temperature excess Δ_p, which as we shall see is *not* $O(1)$). Closure of the model is effected by assuming either $u_c \sim u_p$ (the first KZZ prescription) or $\Delta_p \sim \Delta_c$ (the second); evidently these are equivalent.

The KZZ theory is not, as it stands, an entirely happy one; its best feature is that it gives the exponents

$$\Delta_c \sim Ra^{-1/7}, \quad u_c \sim Ra^{3/7}, \quad \text{Nu} \sim Ra^{2/7}, \tag{4.34}$$

observed in the experiment. We shall therefore assume it is correct, and try to give a prescription for the flow geometry which can be consistent with the assumptions. A further drawback of the theory is that the 'mixing length' seems irrelevant to the determination of the scales (4.34).

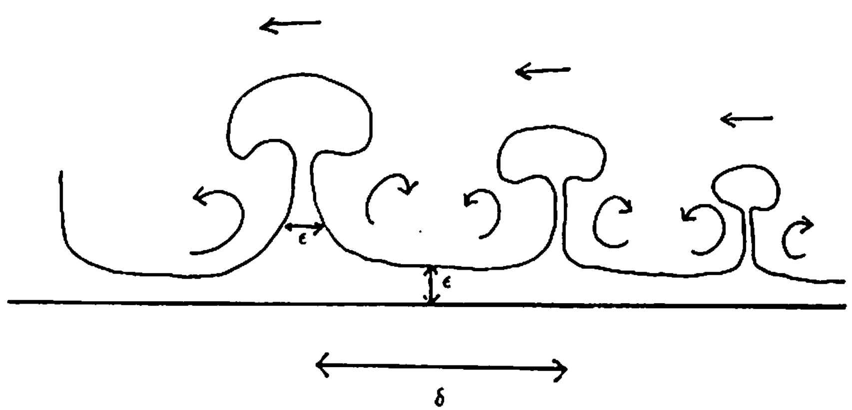

FIGURE 4.1

We now offer a cartoon of the flow which may be consistent with the KZZ model (see Figure 4.1). There is a core circulation with velocity $\sim u_c$. As this

sweeps past the base, a thermal boundary layer of thickness ϵ is formed. If the Prandtl number σ is $O(1)$, then there is also a shear layer of thickness $0(\epsilon)$. As the boundary layer is advected along the base, instabilities grow and erupt upwards as plumes. We suppose these plumes are separated by a horizontal distance $\sim \delta$ (this will be the mixing length). Observations of thermal plumes (e.g. Sparrow et al. 1970) suggest that they typically evolve into mushroom-shaped objects by entraining surrounding fluid. Thus the thin tails of the plumes will maintain a temperature excess $T \sim 1$ (just as in boundary layer theory), while we suppose the cap temperature is Δ. There is no real distinction between Δ_p and Δ_c: the mixing length here is a horizontal measure of plume spacing. Also, because of the viscous shear layer at the base, the core velocity u_c is comparable to the plume ascent velocity.

If we write the equations in a moving coordinate system with $\xi = x + u_c t$ (thus the core flow is clockwise) and put $\mathbf{u} = -u_c\mathbf{i} + \mathbf{u}'$, where $\mathbf{i}$ is the horizontal unit vector, we regain

$$\frac{dT}{dt'} = \nabla^2 T, \tag{4.35}$$

here d/dt' is the corresponding material derivative. In effect, the plumes are fixed in this moving frame, and it is then plausible to *choose* the mixing length δ as the length over which the thermal boundary layer grows to $O(\epsilon)$. This gives

$$u_c/\delta \sim 1/\epsilon^2, \quad \text{i.e. } \delta \sim \epsilon^2 u_c. \tag{4.36}$$

Next, it is because the boundary layer is a viscous shear layer that (compare boundary layer theory with free slip conditions) the plume velocity is comparable, and hence the plume tail thickness is also $\sim \epsilon$. If σ is large, then the boundary layer velocity will be small, and then the plume tail thickness will be thinner. Hence the KZZ scaling requires $\sigma \lesssim O(1)$. The upwelling plume tail generates a vorticity which drives the eddy flow between plumes of $O(u_c)$. The plumes extend to the core, and the notion of a depth of the mixing layer may be redundant.

Thus we have

$$u_c \sim \epsilon^2 Ra, \tag{4.37}$$

and

$$\text{Nu} \sim \Delta u_c \tag{4.38}$$

as before. This gives the KZZ scalings, and in addition that

$$\delta \sim \epsilon^2 u_c \sim Ra^{-1/7}, \tag{4.39}$$

which is also determined by KZZ.

The last scale successfully determined by KZZ is the periodicity of thermal oscillations in the cell. The data indicates that these consist of more severe fluctuations in temperature which are superimposed on top of a higher frequency oscillation which would correspond to the passage of developing plumes. At a spacing δ, with a velocity u_c, these would have a characteristic frequency

$$\omega_{pl} \sim u_c/\delta \sim Ra^{4/7}. \tag{4.40}$$

KZZ associate the longer periodicity with a Brunt-Väisälä frequency, which they associate with oscillations of stably stratified cooler eddies below warmer ones. The frequency is defined as

$$\omega_{BV} \sim (\sigma Ra\Delta/\delta)^{1/2} \sim Ra^{1/2}, \tag{4.41}$$

since $\delta \sim \Delta$, and corresponds to a time scale

$$t_{BV} \sim \delta^{1/2}/u_c. \tag{4.42}$$

The data is consistent with this estimate, but it is not straightforward to incorporate it in the model. For one thing, the idea of a gravity wave oscillation in a horizontally stratified medium is not obviously consistent with the picture presented here. For another, the data might suggest that the oscillations (of $O(1)$) occur in the boundary layer; nor do they necessarily look like a slow modulation of the faster oscillation – they seem to be associated with particularly sharp spikes. Our natural explanation would then be that they represent oscillations of the boundary layer itself within the quasi-steady evolving picture we have described. But a time scale t_{BV} in (4.22) is difficult to isolate in (4.28).

On the other hand, there may be problems in associating t_{BV} with a Brunt-Väisälä frequency. This frequency describes oscillations in which density variations are important in the continuity equation. If we include compressibility in (4.28), we have

$$\begin{aligned} \frac{1}{\sigma}\dot{\mathbf{u}} &= -\nabla p + RaT\mathbf{k} + \nabla^2\mathbf{u} \\ \nabla\cdot\mathbf{u} &= -B\dot{T} \\ \dot{T} &= \nabla^2 T, \end{aligned} \tag{4.43}$$

where

$$B = \alpha\Delta T \tag{4.44}$$

might be called the Boussinesq number. The Boussinesq approximation involves the limit $B \to 0$. If, for the moment we consider T to be a diffusionless passive scalar, then $\dot{T} = 0$, and small oscillations of (4.43) involve the basic equations

$$\begin{aligned} v_t &\sim \sigma RaT, \\ T_t &\sim -vT_y, \end{aligned} \tag{4.45}$$

whence

$$v_{tt} \sim -\sigma RaT_y v, \tag{4.46}$$

which gives the Brunt-Väisälä frequency (4.42) (with $T_y \sim \Delta/\delta$). It is not immediately clear that the same idea will apply here, since T is not passive. Actually the diffusion time for T over a length δ is $\delta^2 \sim Ra^{-2/7} >> Ra^{-1/2} \sim t_{BV}$. Therefore, it is in fact consistent to consider oscillations on a time $O(t_{BV})$, and we suppose the mechanism is just (4.45), with oscillations taking place in the eddies above the boundary layer.

Thus the Brunt-Väisälä oscillation is consistent with the model. The eddies between plumes are described by the approximate equations

$$\begin{aligned} \frac{1}{\sigma}\dot{\mathbf{u}} &= -\nabla p + RaT\mathbf{k}, \\ \nabla\cdot\mathbf{u} &= 0, \\ \dot{T} &= 0, \end{aligned} \tag{4.47}$$

and as illustrated, the recirculating eddies have a temperature variation Δ. This is as opposed to the isothermal temperature in steady state boundary layer theory, and may be thought of as being due to re-incorporation of the ascending plume cap, which will be eroded by shear from the downwelling eddy. Thus the core velocity will oscillate, and this will lead to a modulation of the plume emission temperature, as we suppose is observed.

This completes our discussion of the remarkable KZZ model. In summary of this section, the Howard model is not really viable as a model of convection in a circulating flow. The KZZ model represents a significant improvement, but is restricted to values of $\sigma = O(1)$. The obvious next step is to extend the KZZ model to values of $\sigma >> 1$. We leave this undertaking for a future publication.

5. Cellular chaos at high Rayleigh number

In this section we indicate a way of modelling time-dependent convection, when this takes the form of quasi-stationary convection cells whose boundaries can migrate. The type of convection we are thinking of is represented in the laboratory by spoke-pattern convection (Whitehead and Parsons 1978), and is akin to that which occurs in the earth's mantle (although the precise physics is then rather different). The idea is to parametrise the convective motion within any cell in terms of the cell dimension and Rayleigh number, and then to determine via boundary layer theory the relative velocities of the thermal plumes which divide different cells. The philosophy is thus one of combined asymptotic and numerical methods, since we hope to be able to model efficiently high Rayleigh number convection in large containers by first analysing the boundary layers. The computational problem then becomes much simpler to perform.

The strategy we adopt in this section is the following. The boundary layer theory for steady, infinite Prandtl number convection with free-slip boundaries has been summarised in section 3. We now show how this theory is modified for the case of slowly-varying cell boundaries (the fact that the boundary plumes move slowly is a consequence of the analysis). We begin by describing the theory for the simplest case of two convective cells in $0 < x < a$, $0 < y < 1$, with a simple moving plume at $x = \xi(y,t)$ which divides the cells. Later we will discuss possible generalisations to multiple cells, and to real, three-dimensional convection.

(i) Boundary layer theory for slowly varying cells

We follow the recipe given in section 3. The flow configuration is as shown in Figure 5.1, and the equations are given by (3.2).

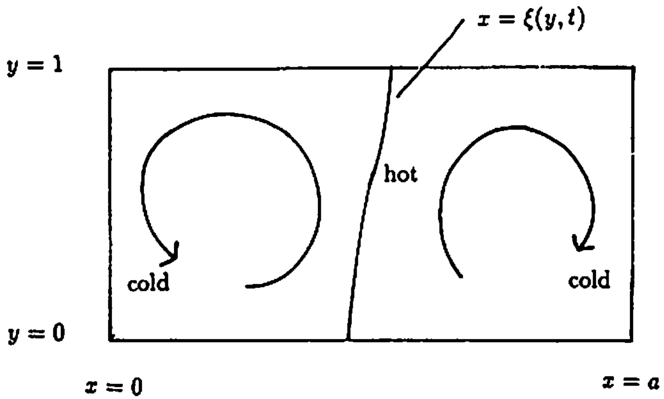

FIGURE 5.1

A. Cores

This is as before; $T = 0$ to all orders of $\delta, \omega \sim O(1)$ at the sides, and at the central plume. (We here anticipate the result that $\partial\xi/\partial t << 1$.) The stream function satisfies the biharmonic equation of Stokes flow $\nabla^4\psi = 0$ in both cores.

B. Plumes

For the stationary plumes at $x = 0$ and $x = a$, the analysis is as before, and we have the boundary conditions for the core flow:

(5.1)
$$\psi_x\omega = -C_L \text{ at } x = 0, \qquad \psi_x\omega = -C_R \text{ at } x = a;$$

for the central plume, we put

(5.2)
$$x = \xi + \delta X, \quad \tau = \delta t, \quad \psi = \delta\Psi, \quad \omega \sim 1, \quad T \sim 1,$$

and anticipate that $\xi = \xi(y,\tau)$ (as is necessary to obtain a sensible balance below); then we have, at leading order,

(5.3)
$$\Psi \sim X v_p(y),$$

(5.4)
$$(1 + \xi_y^2)\omega_X + T \sim 0,$$

whence

(5.5)
$$(1 + \xi_y^2)[\omega]_{\xi-}^{\xi+} \sim -\int_{-\infty}^{\infty} T\,dX$$

gives a boundary jump condition for the two core flows.

The temperature equation is quasi-static on the time scale $\tau = O(1)$, and T satisfies

(5.6)
$$-\xi_\tau T_X + \Psi_X T_y - \Psi_y T_X \sim (1 + \xi_y^2)T_{XX}.$$

In von Mises coordinates y, Ψ, we get

(5.7)
$$T_y - \xi_\tau T_\Psi \sim v_p(1 + \xi_y^2)T_{\Psi\Psi},$$

whence we have

(5.8)
$$\int_{-\infty}^{+\infty} T\,d\Psi = v_p \int_{-\infty}^{\infty} T\,dX = C(\tau)$$

(note also C_L, C_R are functions of τ), so that (5.5) is

(5.9)
$$(1 + \xi_y^2)v_p[\omega]_{\xi-}^{\xi+} = -C.$$

C. Thermal boundary layers

As before, we find

(5.10)
$$T_s \sim u T_{\Psi\Psi},$$

where s is arc length, and u is the tangential velocity.

D. Solution strategy

We have to solve $\nabla^4\psi = 0$ in each core, with $\psi = 0$ on the boundaries of each cell, $\omega = 0$ on top and bottom, $\psi_x\psi_{xx} = -C_{L,R}$ at the left and right boundaries, and

(5.11)
$$\psi = 0, \quad [\psi_x]_-^+ = 0, \qquad (1 + \xi_y^2)[\psi_x\psi_{xx}]_-^+ = -C \qquad \text{at } x = \xi.$$

Given C_L, C_R, C *and* the free boundary ξ, these are sufficient to determine both flows. The free boundary has therefore to be determined, along with C_L, C_R, C, from the solution to the temperature equation.

This is not a very attractive proposition in general, but the whole procedure becomes much simpler if one assumes that the cells are reasonably narrow, at least formally. Roberts (1979) has given approximations for this case. Basically, $\partial/\partial x >> \partial/\partial y$, so the flow is essentially vertical. Moreover, the transit time in the plumes is much longer than in the horizontal boundary layers, and therefore the temperature T at the beginning of the boundary layer can be taken as approximately zero: hence a similarity solution can be written down. This enables us to reduce the entire problem to a single (ordinary) differential equation for the plume position ξ.

E. Practical implementation

We now restrict ourselves to the case where, at least formally, a is small, so that the cells are tall and thin. In practice, we hope such an approximation will give qualitatively accurate results even for $a \sim O(1)$. The stream function in each cell now satisfies, approximately,

$$\psi_{xxxx} \sim 0. \tag{5.12}$$

Let $-v_L$, ω_L denote the velocity and vorticity in the plume at $x = 0$, and v_R, $-\omega_R$ denote those at $x = a$. Then

$$\begin{aligned} \psi &= -v_L x + \frac{1}{2}\omega_L x^2 + A_L x^3, \qquad x < \xi, \\ \psi &= v_R(a - x) - \frac{1}{2}\omega_R(a - x)^2 + A_R(a - x)^3, \qquad x > \xi, \end{aligned} \tag{5.13}$$

and it follows that we require, from the boundary conditions, that

$$\begin{aligned} v_L\omega_L &= C_L, \\ v_R\omega_R &= C_R, \\ -v_L\xi + \frac{1}{2}\omega_L\xi^2 + A_L\xi^3 &= 0, \\ v_R(a-\xi) - \frac{1}{2}\omega_R(a-\xi)^2 + A_R(a-\xi)^3 &= 0, \\ -v_L + \omega_L\xi + 3A_L\xi^2 &= v_p, \\ -v_R + \omega_R(a-\xi) - 3A_R(a-\xi)^2 &= v_p, \\ v_p[\omega_L + 6A_L\xi + \omega_R - 6A_R(a-\xi)] &= C, \end{aligned} \tag{5.14}$$

where we have taken $\xi_y << 1$. At this stage, we have twelve unknowns: v_L, ω_L, A_L, v_R, ω_R, A_R, v_p, C_L, C_R, C, ξ and ξ_τ; but only seven equations. To determine the rest, we must solve the temperature equations in each cell.

Now when the cells are narrow, a similarity solution becomes appropriate; if u is flow speed, then the relevant solutions are

$$T = \pm\frac{1}{2}\,\mathrm{erfc}\left[\frac{\Psi}{2\left\{\int_0^s u\,ds\right\}^{1/2}}\right], \tag{5.15}$$

where we have + for the lower boundary layers, and − for the upper ones. In addition, we change the sign of Ψ in these boundary layers to be positive in the left cell. The heat added through the lower boundary layers is then

$$h_L = \int_0^\xi -u\frac{\partial T}{\partial \Psi}\,ds \tag{5.16}$$

for the left cell, and similarly for the right cell h_R, and also for the heat losses h_L', h_R' through the top boundary layers. Using the relation

$$\frac{d}{ds}\int_0^\infty T\,d\Psi = -u\frac{\partial T}{\partial \Psi}\Big|_{\Psi=0}, \tag{5.17}$$

we then have that

$$C = -C_L + h_L - C_R + h_R, \tag{5.18}$$

where also

$$\begin{aligned} C_L &= -\int_0^\infty T\,d\Psi, \\ C_R &= -\int_0^\infty T\,d\Psi, \end{aligned} \tag{5.19}$$

on the left and right walls, respectively. Moreover, it can be shown that for isothermal interior temperatures equal to zero, the net buoyancy flux is zero, that is,

$$C = C_L + C_R. \tag{5.20}$$

Now consider the central plume. Here $-\infty < \Psi < \infty$, and $\Psi < 0$ denotes the 'left hand' part of the plume. If

$$\begin{aligned} C_L' &= \int_{-\infty}^0 T\,d\Psi, \\ C_R' &= \int_0^\infty T\,d\Psi \qquad \text{at } s = 1 \text{ (top of plume)}, \end{aligned} \tag{5.21}$$

then

$$\begin{aligned} -C_L &= C_L' - h_L', \\ -C_R &= C_R' - h_R'. \end{aligned} \tag{5.22}$$

Furthermore, it follows from (5.4) and (5.13) that, if we assume $\omega = 0$ on $\Psi = 0$,

$$\begin{aligned} v_p(\omega_L + 6A_L\xi) &= C_L', \\ v_p(-\omega_R + 6A_R(a-\xi)) &= -C_R'. \end{aligned} \tag{5.23}$$

We have now introduced a further six unknowns: h_L, h_R, h_L', h_R', C_L' and C_R'; so we are missing eleven equations. Five are given by (5.18), (5.22) and (5.23). The remainder follow from (5.19), (5.20) and (5.21), and the prescriptions such as (5.16) for the h's. Since determination of C_L' and C_R' involves solution of the plume equation (5.7) involving $\dot{\xi} = \xi_\tau$, this is how the plume integration speed is incorporated.

Adopting the similarity solutions for the thermal boundary layer temperatures, we find

$$h_L = (t_L/\pi)^{1/2}, \quad h_R = (t_R/\pi)^{1/2}, \tag{5.24}$$

where $t_L = \int_0^\xi u\,ds$ (similarly t_R) is the transit time across the left lower layer. In the case where the flow in the cell is laterally symmetric (and $\psi_x\psi_{xx} = \pm C$ at each side), Roberts finds numerically that $t_L \approx .266C^{1/2}a^{3/2}$; in our case the flow will really be asymmetric, but we nevertheless suppose that a simple weighted average is appropriate, and (hence) moreover that the transit time in upper and lower layers in the same.

Then

$$\text{(5.25)}\qquad \begin{aligned} h_L' = h_L &= 0{\cdot}29\left[\frac{C_L^{1/4}+(C/2)^{1/4}}{2}\right]\xi^{3/4},\\ h_R' = h_R &= 0{\cdot}29\left[\frac{C_R^{1/4}+(C/2)^{1/4}}{2}\right](a-\xi)^{3/4}, \end{aligned}$$

which determines the h's.

Lastly, we find C_L' and C_R' by solving the plume equation

$$\text{(5.26)}\qquad T_s - UT_\Psi = v_p T_{\Psi\Psi},$$

where $\dot{\xi} = U$, with

$$\text{(5.27)}\qquad \begin{aligned} T = T_0(\Psi) &= \frac{1}{2}\text{erfc}\left[\frac{-\Psi}{2t_L^{1/2}}\right], \quad \Psi < 0,\\ &= \frac{1}{2}\text{erfc}\left[\frac{\Psi}{2t_R^{1/2}}\right], \quad \Psi > 0, \end{aligned}$$

at $s = 0$, and

$$\text{(5.28)}\qquad T \to 0 \text{ as } \Psi \to \infty.$$

When $a << 1$, we only need the large time solution of this, which is (at $s = 1$)

$$\text{(5.29)}\qquad T \sim \frac{1}{2\sqrt{\pi v_p}}\left\{\int_{-\infty}^{\infty} T_0(\theta)\,d\theta\right\}\exp\left[-\frac{(\Psi+U)^2}{4v_p}\right]$$

(since v_p is constant). After some algebra, we find

$$\text{(5.30)}\qquad C_R' = \frac{1}{2}(h_R + h_L)\text{erfc}[U/2v_p^{1/2}] - C_R.$$

The equations (5.14), (5.20), (5.22), (5.23), (5.25) and (5.30) give seventeen algebraic equations for the eighteen unknowns v_L, ω_L, A_L, v_R, ω_R, A_R, v_p, C_L, C_R, C, ξ, $U = \dot{\xi}$, h_L, h_L', h_R, h_R', C_L' and C_R'. The equation (5.18) is redundant, being equivalent to $(5.14)_7$, (5.22), (5.23) and (5.25). Elimination of the surrogate variables thus gives, in principle, a first order differential equation of the form $\dot{\xi} = f(\xi)$. To date, solutions have not been obtained. However, since the system is first order, the flow can do very little of interest: either tend to a state with one cell, or tend to one with two cells.

However, the important point is that this procedure can be expanded easily (in principle) to flows with multiple cells. It is obvious that such flows will be determined by systems of first order ordinary differential equations. Therefore, as soon as four or more cells exist, chaotic behaviour is a possibility. We will pursue this investigation in future work, but here we wish to examine the possibility of extension of these ideas to three dimensions. Firstly, let us summarise the mechanics by which the plume is required to move.

If the cells are of different size, than in general $u_L \neq u_R$, and so the initial profile T_0 for T in the plume is skewed. Thus if $U = 0$, the diffusion of T in the plume will lead to a buoyancy transfer: for example if $u_L > u_R$, the left profile of T is steeper, and so buoyancy diffuses to the right. This causes an increase in the buoyancy on the right (thus increasing u_R), but also an increased delivery of heat at the right. Both effects will cause a change in cell size, because of the dependence of u_R on cell size (or vice versa).

(ii) Extension to three dimensions

The extension of the method outlined above to multiple plumes is obvious in principle. In order to extend it to three-dimensional flows, which is our ultimate aim, we must first do three-dimensional boundary layer theory. This has not been done previously, and here we sketch the procedure.

The equations can be written in the form

$$\text{(5.31)}\qquad \begin{aligned} \nabla\cdot\mathbf{u} &= 0,\\ p_x &= \nabla^2 u,\\ p_y &= \nabla^2 v,\\ p_z &= \nabla^2 \omega + \frac{1}{\delta}T,\\ \mathbf{u}\cdot\nabla T &= \delta^2\nabla^2 T, \end{aligned}$$

in analogy to (3.2). In the core, $T = 0$ to all orders of δ, and we have Stokes flow

$$\text{(5.32)}\qquad \begin{aligned} \nabla p &= \nabla^2\mathbf{u},\\ \nabla\cdot\mathbf{u} &= 0, \end{aligned}$$

with free slip boundary conditions at top and bottom, $\mathbf{u}\cdot\mathbf{n} = 0$ at the sides, but the shear stress ($\sim \partial w/\partial n$) will be non-zero due to the plume bouyancy.

It is convenient to write

$$\text{(5.33)}\qquad \mathbf{u} = (\mathbf{U}, w), \quad \nabla = \left(\nabla_H, \frac{\partial}{\partial z}\right),$$

where $\mathbf{U}$ is the horizontal component of velocity, and ∇_H the horizontal gradient derivative. In the plume, $\mathbf{u}$ takes its value determined by the core flow. If n is the local normal coordinate at the cell boundary, then the vorticity-buoyancy equation $(5.31)_4$ is given by

$$\text{(5.34)}\qquad \frac{\partial}{\partial N}\left(\frac{\partial w}{\partial n}\right) = T,$$

where $n = -\delta N$, and thus the core vorticity at the plume is determined by

$$\frac{\partial w}{\partial n} = \int_0^\infty T\,dN, \tag{5.35}$$

if, for example, we consider a single cell with a stationary plume structure. The energy equation can be written

$$\mathbf{u}\cdot\nabla T = T_{NN}; \tag{5.36}$$

it follows on integrating across the layer that (using (5.35))

$$\frac{\partial}{\partial \ell}\left[U_\ell \frac{\partial w}{\partial n}\right] + \frac{\partial}{\partial z}\left[w\frac{\partial w}{\partial n}\right] = 0, \tag{5.37}$$

where U_ℓ is the tangential component of $\mathbf{U}$ at the boundary. This gives the boundary equation for the core vorticity; it must be supplemented by a prescription of $\partial w/\partial n = \int_0^\infty T\,dN$ at the base of upwelling plumes, (or at the top of downwelling plumes), which is determined by solving the thermal boundary layer equation.

In the thermal boundary layer (for example, at the base) we have $z \sim \delta$, $w \sim \delta$, and thus (with rescaled z and w)

$$\begin{aligned}\mathbf{u}\cdot\nabla T &= T_{ZZ},\\ W &= -Z\nabla_H\cdot\mathbf{U}.\end{aligned} \tag{5.38}$$

We introduce a pseudo-stream function

$$\Psi = ZU_s(x,y), \tag{5.39}$$

where U_s is the speed (determined from the core flow) at the boundary, and let s be a coordinate measuring distance along flow lines at the base. Then introduction of von Mises coordinates s and Ψ (and a coordinate normal to both) gives the diffusion equation

$$\frac{\partial T}{\partial s} = U_s\frac{\partial^2 T}{\partial \Psi^2}, \tag{5.40}$$

and in fact the same equation can be used to describe T in the plumes.

A general numerical strategy to solve for the flow in a single cell might now go as follows. Guess a value for $\partial w/\partial n$ in the plumes, and solve for the core flow. The flow lines on the boundary form a one-parameter family of closed loops, along which

$$\frac{\partial T}{\partial s} = u_s\frac{\partial^2 T}{\partial \Psi^2} \tag{5.41}$$

gives the temperature field (where s is arc length and u_s is speed). From the (periodic) solution of this equation on each path, we determine $\int_0^\infty T\,d\Psi$. It is required

that this equal $u_s\partial w/\partial n$, and an iterative procedure can now be implemented to adjust the original value of $\partial w/\partial n$. This is a feasible (but major) computational strategy, and its adaptation for a flow consisting of many cells will normally require the use of parallel processing.

By analogy with the two-dimensional simplification, let us now suppose the cells are tall. Rescaling variables with $\mathbf{r}_H \sim \epsilon << 1$, $\mathbf{U}_H \sim \epsilon$, $p \sim 1/\epsilon^2$, we find $p \approx p(z) + \epsilon^2\tilde{p}$, where

$$\begin{aligned}p' &\simeq \nabla_H^2 w,\\ \nabla_H\tilde{p} &= \nabla_H^2\mathbf{U},\\ \nabla_H\cdot\mathbf{U} + \frac{\partial w}{\partial z} &= 0,\end{aligned} \tag{5.42}$$

with appropriate boundary conditions. The simple solution is that $p' =$ constant, $w = w(x,y)$, $\mathbf{U} = \mathbf{0}$, thus

$$\nabla^2 w = p', \quad w\frac{\partial w}{\partial n} = C(\ell), \tag{5.43}$$

where $C(\ell)$ is a function of position on the perimeter. Given C, p' is determined so that $\iint w\,dS = 0$, where the integral is over the cell interior. As before C is determined from the thermal boundary layer equation, now with similarity boundary conditions. Thus at the base

$$T = \frac{1}{2}\operatorname{erfc}\left[\frac{\Psi}{2\left\{\int_0^s u_s\,ds\right\}^{1/2}}\right], \tag{5.44}$$

where s is measured along flow paths in the direction of flow. As before, this leads to an injection of buoyancy equal to $(t^*/\pi)^{1/2}$ (cf. (5.23)), where t^* is the transit time across the base, which depends on the position of the flow path. Further investigation is necessary to see whether any simple parameterisation of t^* in terms of ℓ is possible, since its form will in general depend on the cell shape as well as the aspect ratio and plume vorticity.

If such a parameterisation can be provided , then a complete description of the flow field is possible, and the extension of this to time-dependent problems is straightforward, at least in principle.

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