THE ONSET OF FRECKLING IN THE SOLIDIFICATION OF BINARY ALLOYS

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ABSTRACT. A mathematical model is presented for the solidification of a binary alloy when cooled from below. When light fluid is released, compositional convection can occur in the dendritic, mushy zone as well as in the fluid above. Here we summarise efforts to study the onset of convection in the mush, by parameterising the vigorous convection in the liquid above, with a view to prediction of the circumstances under which flow channels (freckling) can occur.

1. Introduction

Since the early work on the formation of freckles in alloy castings (McDonald and Hunt 1969, 1970, Copley et al. 1970) there has been much interest in the dynamics of convective fluid motion within and above a 'mushy' region of partially solidified alloy material. Descriptions of the process are of interest in metallurgy (e.g., Sarazin and Hellawell 1988), in studies of convective motions in the earth's core (Loper and Roberts 1981), and in magma chambers (Tait and Jaupart 1992). Theoretical studies of convective motion have been undertaken by Worster (1991, 1992): in the second paper, he studies the onset of convective motion in a combined mush/liquid system, and distinguishes between two types of motion, one in the liquid, and one in the mush, with very different length scales. At high values of the Lewis number $Le = \kappa/D$, it is found that convection is initiated first in the liquid, consistent with observations of Sample and Hellawell (1984). Double-diffusive, 'finger' convection develops, and then convection in the mush is observed through the formation of channelised flow, which is initiated when the fluid flow in the mush exceeds the solidification rate. Worster (1991) studies this fully developed flow.

In this note, we summarise recent efforts to develop a pseudo-linear theory for the onset of convection in a mushy region, where we suppose the amplitude of motion in the mush is 'small', but that convection in the liquid has already been initiated, and is in fact vigorous. Thus we attempt a boundary layer analysis of the convection in the liquid, which is then parameterised to allow a description of the motion in the mush. We offer no apologies for using every approximation available, in the hope of producing a tractable theory. In particular, we assume the melt composition is close to eutectic, which enables the model for the mush to be reduced to that of convection in a porous medium.

2. Mathematical Model

The model presented here is similar to that of Fowler (1985), Worster (1986) or Hills et al. (1983). We consider solidification of a binary melt, with a liquid region $z > z_l$, a dendritic, or 'mushy' region $z_s < z < z_l$, and a fully solid region $z < z_s$. In the mush, we neglect density variations (specifically, $(\rho_s - \rho_l) \ll \rho_s$): the equations are then

$$\begin{gathered}
\nabla.\mathbf{u} = 0, \\
\rho c_p \frac{dT}{dt} - L\frac{\partial}{\partial t}[\rho(1-\chi)] = k\nabla^2 T, \\
\rho\frac{d}{dt}[\chi c + (1-\chi)s] + \nabla.[\rho(1-\chi)(c-s)\mathbf{u}] = \nabla.[\rho\chi D\nabla c], \\
\frac{\partial}{\partial t}[\rho(1-\chi)s] = \lambda c\frac{\partial}{\partial t}[\rho(1-\chi)], \\
\mathbf{u} = -(k_p/\mu_l)[\nabla p + \rho g\mathbf{k}], \\
\rho = \rho_0[1 - \alpha(T - T_0) - \beta c], \qquad (1)
\end{gathered}$$

where c and s are liquid and solid compositions, χ is liquid mass fraction, T is temperature, Γ is the liquidus slope, $\mathbf{u}$ is the liquid flux, d/dt is the material derivative, $\mathbf{k}$ is a unit vertical vector and other symbols have their usual meanings (k_p is the permeability).

Equations of motion in the liquid are

$$\begin{gathered}
\nabla.\mathbf{u} = 0, \\
\frac{dT}{dt} = \kappa\nabla^2 T, \\
\frac{dc}{dt} = D\nabla^2 c, \\
\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \mu_l\nabla^2\mathbf{u} - \rho g\mathbf{k}, \qquad (2)
\end{gathered}$$

and these are coupled across the mush by the usual jump-derived boundary conditions, which we omit. A point of note is the necessity to pose an 'extra' condition for χ on z_l, and we take this to be $\chi = 0$. The justification for this choice is not clear, however, and Worster (1986) prefers a condition of marginal supercooling.

3. Nondimensionalisation

We scale the variables as

$$\mathbf{x} \sim \kappa/V, \quad \mathbf{u} \sim V, \quad t \sim \kappa/V^2, \quad p - p_{hydro} \sim \kappa/k^*, \quad T - T_\infty^L \sim \nu\Gamma, \qquad (3)$$

where V is a 'typical' value of $\dot{z}_l$, p_{hydro} is the hydrostatic pressure, T_∞^L is the far-field liquidus temperature, $\nu = c_E - c_\infty$ is the difference between eutectic and far field concentrations, and we have defined the permeability as $k_p = k^*\chi^2$. Now also rescale by putting

$$\chi = 1 - \nu\phi, \quad c - c_\infty \sim \nu, \qquad (4)$$

and taking $\nu \ll 1$. At leading order, we then have in the mush,

$$\nabla.\mathbf{u} = 0,$$

$$\mathbf{u} = -\nabla p + Rc\mathbf{k},$$

$$(c_\infty + St)\phi_t = [1 + St/c_\infty]\frac{dc}{dt} = \nabla^2 c, \tag{5}$$

just the equations of convection in a porous medium. ϕ uncouples, St is the Stefan number $St = L/c_p\Gamma$, and R is the Rayleigh number

$$R = \Delta\rho g k^* / \mu_l V, \tag{6}$$

where $\Delta\rho$ is the density difference between eutectic and far field liquid.

The main simplification in deriving (5) is the limit $\nu \to 0$, and also we have supposed $Le \gg 1$, so that compositional diffusion can be ignored (an apparently regular approximation).

4. Reduction

The idea to parameterise the correspondingly scaled liquid equations (2) is in parallel with Roberts' (1979) boundary layer theory, generalised for double-diffusive convection. In terms of the scaled variables, we find that the compositional boundary layer above the mush is of thickness $\delta \sim Le^{-1/3}$, while the velocity is of order $\mathbf{u} \sim (R/Le\gamma)^{1/4}$, where $\gamma = k^* V^2/\kappa^2$ ($1/\mathcal{H}$ of Worster 1992). The method of solution is then as follows. In the mush $0 < z < z_l$ (as it is found that $\dot{z}_s \sim \nu/St$),

$$[1 + St/c_\infty]\frac{dc}{dt} = \nabla^2 c,$$

$$c = 1 \text{ on } z = 0, \quad c = 0 \text{ on } z = z_l, \tag{7}$$

so that if z_l is known, one can compute the compositional flux

$$-\frac{\partial c}{\partial n} = g. \tag{8}$$

In the liquid, we hypothesise that the convective finger width is such as to balance thermal advection with conduction. This width is then of order $\epsilon = (\gamma Le/R)^{1/4}$, typically ~ 0.1. We then find that the compositional convection problem uncouples from that of the heat flow, and the temperature satisfies

$$T_t + \frac{1}{\epsilon}\mathbf{u}.\nabla T = \nabla^2 T, \tag{9}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}/\epsilon)$ is the (rescaled) convective flow field, and t and $\mathbf{x}$ in (9) are scaled as for the mush. This is a multiple scale problem, but at leading order, T satisfies

$$T_t = T_{xx} + (1 + D_T)T_{zz}, \quad T = 0 \text{ on } z_l, \quad T \to \Delta_\infty \sim O(1) \text{ at } \infty. \tag{10}$$

The enhanced diffusivity D_T arises through Taylor dispersion, and $D_T \propto \left\langle |\mathbf{u}|^2 \right\rangle$. The extra condition $\overline{\partial T/\partial n} = g$ (the overbar denotes a spatial average) allows determination of z_l. Thus, compositional convection does not affect the diffusional nature of the temperature field, and this is our major conclusion. Before the onset of convection in the mush, a similarity solution is appropriate, and we find that $z_l \sim At^{1/2}$, where A depends on Δ_∞, D_T and St/c_∞. A stability analysis for the mush then predicts that convection is initiated at a critical value of the dimensional mush thickness of

$$z_l > \frac{c_\infty}{(c_\infty + St)} \frac{\mu_l \kappa R_c}{\Delta \rho g k^*}, \tag{11}$$

where for a linear compositional profile and a quasi-static analysis, $R_c = 27.1$, though this is modified in the present situation. This analysis is consistent with results of Tait and Jaupart (1992, fig. 16).

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