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MODELLING ICE SHEET DYNAMICS

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This paper surveys the problem of modelling the dynamics of large ice sheets. A simplified model for two-dimensional plane ice sheets is derived, and both isothermal and non-isothermal cases are considered. The model is not uniformly asymptotically valid at a divide or at a margin, and we suggest local (isothermal) analyses which give order of magnitude estimates for divide curvature and margin slope. We also give a uniformly valid description for small perturbations to an isothermal ice sheet, which decay diffusively. For the more interesting non-isothermal case, we are able to provide explicit approximate solutions for the surface profile, based on Lliboutry's heuristic boundary layer analysis, and give an approximate description of the temperature field.

KEY WORDS: Ice sheets, mathematical model, large activation energy asymptotics.

1. INTRODUCTION

A recent paper by Hutter *et al.* (1986) summarises various efforts directed towards solving the problem of "shallow" ice sheet flow. By this is meant the recognition of the fact that "typical" ice sheets (Antarctica and Greenland, for example) have thicknesses much less than their horizontal extent. Consequently, it is sensible to neglect horizontal thermal conduction in comparison with vertical conduction. With this assumption, the energy equation is parabolic, and it is natural to consider numerical schemes which march outwards from ice divides. This causes some difficulties.

More recently, various more advanced computational strategies for time-dependent two and three-dimensional models have been developed (e.g. Herterich, 1988; Huybrechts and Oerlemans, 1988; Hindmarsh *et al.*, 1989; Budd and Jensen, 1989) which seems to provide a satisfactory basis for the solution of the flow of *cold* ice sheets.

Other than numerical approaches, the question arises whether there are any useful analytic approaches which can provide corroboration for numerical methods, or indeed give any simpler results than a direct numerical approach. Approximate results for ice sheets can be obtained (Bodvardsson, 1955; Nye, 1959) using the idea that most shearing takes place near the base. This is predicated on the fact that (a) thermal conductivity is small, so that the temperature rises towards the base, and hence the viscosity decreases suddenly there [a temperature change from 0°C to −10°C causes a tenfold increase in effective viscosity (Paterson, 1981)]; (b) the stress is largest at the base, and hence the viscosity (for ice, considered as a power-law fluid) decreases sharply there. While this idea has been around for a long time, it has never been fully worked out in its natural language–matched asymptotic expansions. An attempt to do so is made here.

Before proceeding to such an asymptotic analysis, the governing equations must be scaled, using appropriate depth, length, stress and velocity scales. This has been exhaustively done in the literature (see Hutter *et al.*, 1986; Morland, 1984, and literature cited therein). The procedure is reviewed here in Section 2, in a way that is perhaps simpler than previous treatments.

The equations to be discussed are those of a slow, gravity-driven, viscous flow as appropriate to the solid state creep of ice. Ice sheets are "cold" (i.e. below their melting point), except at parts of their base, where the temperature may reach the melting point, as evidenced by radio-echo sounding of sub-Antarctic lakes (Oswald and Robin, 1973). The usual fluid no-slip boundary conditions apply when basal ice is cold, but at the melting temperature, ice can slide over its base (Paterson, 1981). There a dependence results between basal shear stress and basal slip velocity known as the sliding velocity. One assumption of some recent models is that the sliding velocity is non-zero, and in some cases this is *required* in order to obtain a solution (Morland and Johnson, 1980; Hutter *et al.*, 1986). On the other hand, basal topography in Antarctica (see Paterson, 1981) is so rough that the sliding law we should expect is $u \approx 0$ (Richardson, 1973). One aim of this paper is to examine how a reasonable solution can be obtained if the basal velocity is taken as zero.

2. MATHEMATICAL MODEL

We consider plane flow of an ice sheet, as shown in Figure 1. The coordinates are (x, y), with corresponding velocity components (u, v), and the usual equations of slow flow can be written (Hutter, 1983)

$$u_x + v_y = 0,$$

$$0 = -p_x + \tau_{1x} + \tau_{2y},$$

$$0 = -p_y + \tau_{2x} - \tau_{1y} - \rho g,$$

$$\rho c_p \, dT/dt = k\nabla^2 T + \tau_{ij} e_{ij},$$

$$e_{ij} = A(T) g(\tau) \tau_{ij}/\tau. \tag{2.1}$$

In these equations, representing conservation of mass, momentum and energy, and the viscous flow law, letter suffixes represent partial derivatives, p is pressure, τ_1 and τ_2 are longitudinal and transverse stress deviators ($\tau_1 = \tau_{11}$, $\tau_2 = \tau_{12}$), ρ is density, g is gravity, c_p is specific heat, $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$ is the material derivative, T is absolute temperature, k is thermal conductivity. The stress and strain rate invariants e and τ are defined by

$$2e^2 = e_{ij} e_{ij}, \qquad 2\tau^2 = \tau_{ij}\tau_{ij}, \tag{2.2}$$

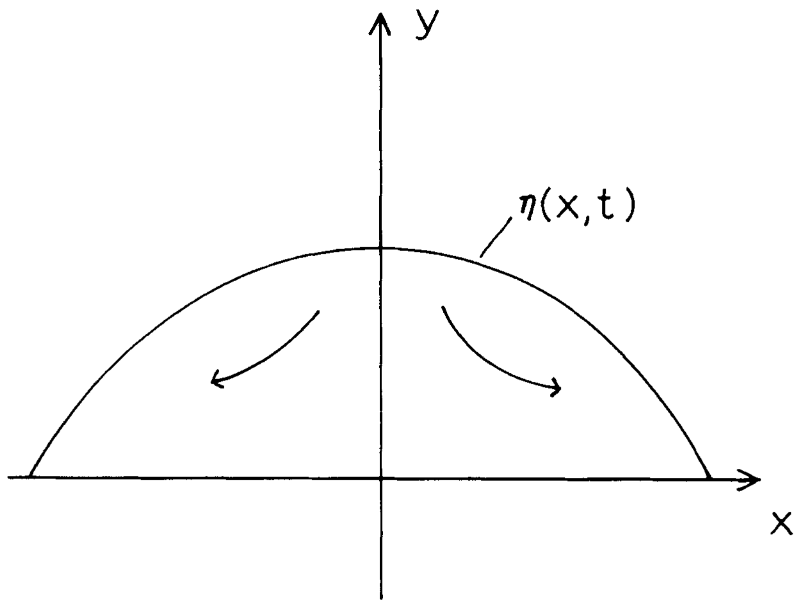

Figure 1 Ice sheet geometry.

and are related by

$$e = A(T)g(\tau), \tag{2.3}$$

where Glen's flow law corresponds to

$$g(\tau) = \tau^n, \qquad n \approx 3, \tag{2.4}$$

and Arrhenius' rate law gives

$$A(T) = A_0 \exp(-Q/RT), \tag{2.5}$$

(Paterson, 1981). Paterson suggests that different values of Q are appropriate above and below $-10°C$. For our analysis, it will only be the rate near the maximum temperature which is important, so that (2.5) will in fact suffice. We shall largely use (2.4) and (2.5) in what follows.

The notation used here follows Fowler and Larson (1980a), and we shall also follow their style of scaling these equations. Scaling of these equations was first done by Morland and Johnson (1980, 1982) and see also Hutter (1983). The present scaling procedure is similar to that of, e.g. Hutter *et al.* (1986), with perhaps some clarification in the meaning of the various parameters.

The boundary conditions for these equations have been given many times before (e.g. Hutter *et al.*, 1986). At this point we simply emphasise that $T \leqslant T_m \approx 273$ K (the melting temperature), and with T prescribed on the surface, there is a natural temperature scale. For the Antarctic this is $\approx 50\,\mathrm{K}$. Similarly, surface accumulation and ablation provides a natural measure of the vertical velocity. For the two major ice sheets, a representative value is about $3\times10^{-9}\,\mathrm{m\,s^{-1}}$ ($10\,\mathrm{cm\,y^{-1}}$) (Paterson, 1981). Its horizontal scale of variation provides a natural horizontal length scale.

To non-dimensionalise, we write

$$x=lx^*, \qquad y=dy^*, \qquad \tau_2=[\tau]\tau_2^*, \qquad \tau_1=\varepsilon[\tau]\tau_1^*,$$

$$T=T_m+(\Delta T)T^*, \qquad u=[u]u^*, \qquad v=\varepsilon[u]v^*,$$

$$A=[A]A^*, \qquad g=[g]g^*, \qquad t=(l/[u])t^*,$$

$$p-\rho g(\eta-y)=\varepsilon[\tau]p^*, \tag{2.6}$$

where $y=\eta(x,t)$ is the top surface of the ice sheet, and ε is defined as the aspect ratio

$$\varepsilon=d/l, \tag{2.7}$$

which we anticipate to be small.

The scales here are l, d, $[\tau]$, ΔT, $[u]$, $[A]$ and $[g]$. Of these seven, l, ΔT are prescribed from the boundary data, as is the "typical" accumulation rate

$$[a]=\varepsilon[u]. \tag{2.8}$$

The choice of the other four is to be such that the asterisked dimensionless variables are of order one. The choice of $[A]$ and $[g]$ is so that the flow law functions a^* and g^* are of O(1). Thus, for the Glen/Arrhenius choice (2.4) and (2.5), we choose

$$[g]=[\tau]^n, \qquad [A]=A_0\exp[-Q/RT_m]. \tag{2.9}$$

The (dominant) shear stress is determined through the balance in $(2.1)_2$ of shear stress and cryostatic pressure gradient. To this end, we choose

$$[\tau]=\rho g d\varepsilon. \tag{2.10}$$

Lastly, the velocity scale $[u]$ is determined through balance of shear strain rate with stress in the flow law.

$$[u]/d=\nu[A][g], \tag{2.11}$$

where we have, with foresight, introduced a free dimensionless parameter ν to take later account of the possibility that most of the shearing takes place near the base [in a zone of thickness o(d)], rather than throughout the ice thickness. This complication now saves us needless complexity later; it is an indication of the fact that the depth scale is unknown *a priori*, and we can therefore choose v so that the dimensionless depth is O(1).

The extra scales d, $[\tau]$, $[u]$, $[A]$, $[g]$ are determined through the relations (2.8) to (2.11). It is worth pointing out that the velocity $[u]$, depth d, and shear stress $[\tau]$ are determined self consistently from the boundary data. This is in contrast to Morland (1984) and Hutter *et al.* (1986), where the stress scale $[\tau]$ and depth d are prescribed arbitrarily. This leaves two extra parameters to play with. The main point of the present discussion is to show that there are essentially only two dimensionless parameters which occur as coefficients in the equations.

Substituting the variables (2.6) into the equations, we are led to the following problem for the velocity and temperature fields, where now and henceforth we omit asterisks on the dimensionless variables:

$$
\begin{aligned}
&u_x+v_y=0,\\
&0=-\eta_x+\tau_{2y}+\varepsilon^2(-p_x+\tau_{1x}),\\
&0=-p_y+\tau_{2x}-\tau_{1y},\\
&dT/dt=(\alpha/v)\tau A(T)g(\tau)+\beta(T_{yy}+\varepsilon^2 T_{xx}),\\
&u_y+\varepsilon^2 v_x=A(T)g(\tau)\tau_2/v\tau,\\
&2vu_x=A(T)g(\tau)\tau_1/\tau,\\
&\tau^2=\tau_2^2+\varepsilon^2\tau_1^2.
\end{aligned}
\tag{2.12}
$$

These equations are those of Hutter *et al.* (1986), if $v=1$, providing we choose $s=\varepsilon=\delta$, $\theta=1$ in their equations (3e,f), and redefine their $\sigma_{xz}=\varepsilon\sigma_{xz}$ (our τ_2). The parameters α and β are the only ones to appear, and are given by

$$\alpha=2gd/c_p\Delta T,\qquad \beta=\kappa/d[a]. \tag{2.13}$$

For values $[a]=3\times10^{-9}\,\mathrm{m\,s^{-1}}$ ($0.1\,\mathrm{m\,y^{-1}}$), $d=3000\,\mathrm{m}$, $\kappa=1.2\times10^{-6}\,\mathrm{m^2\,s^{-1}}$ ($38\,\mathrm{m^2\,y^{-1}}$), $c_p=2\times10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$, $g=10\,\mathrm{m\,s^{-2}}$, $\Delta T=50\,\mathrm{K}$, we compute $\alpha\approx0.6$, $\beta\approx1/8$. Thus we expect α to be $O(1)$, β to be moderately small. As we have mentioned, the parameter v is a free choice. We can take it as equal to one, but it will be convenient to choose it to be less than one later.

The following boundary conditions are appropriate. At the top surface $y=\eta$, zero stress requires

$$
\begin{aligned}
&\tau_2+\varepsilon^2(p-\tau_1)\eta_x=0,\\
&\tau_1+p+\tau_2\eta_x=0,
\end{aligned}
\tag{2.14}
$$

and we suppose temperature is prescribed,

$$T = T_A(x); \tag{2.15}$$

in addition, the kinematic boundary condition is

$$\eta_t + u\eta_x - v = a, \tag{2.16}$$

where a is the (dimensionless) accumulation rate ($a < 0$ signifies ablation). At the base $y = h$, we prescribe no slip:

$$u = v = 0, \tag{2.17}$$

and there is a prescribed heat flux,

$$\frac{\partial T}{\partial y} - \varepsilon^2 h' \frac{\partial T}{\partial x} = -(1 + \varepsilon^2 h'^2)\Gamma, \tag{2.18}$$

where Γ is a dimensionless geothermal heat flux; it is defined as

$$\Gamma = Gd/k\Delta T, \tag{2.19}$$

where G is the dimensional geothermal heat flux. Taking $G = 5 \times 10^{-2}\,\mathrm{Wm}^{-2}$, $k = 2.1\,\mathrm{J\,m^{-1}\,s^{-1}}$, we have $\Gamma \approx 1.5$: geothermal heating is therefore significant. If T reaches the melting point at the base, (2.19) is replaced by

$$T = 0 \quad \text{at} \quad y = h. \tag{2.20}$$

In this case, there is a net release of heat at the interface, which melts the basal ice, and hence causes a basal drainage. The net downward flux due to melting is (neglecting ε)

$$-v \approx (\beta/St)[\Gamma + \partial T/\partial y], \tag{2.21}$$

if $-\partial T/\partial y|_{y=h+} < \Gamma$, where the Stefan number is

$$St = L/c_p\Delta T.$$

With $L = 3.3 \times 10^5\,\mathrm{J\,kg^{-1}}$, $St \approx 3.4$, so that together with the small value of β, it is consistent to neglect basal drainage.

We now consider the reduced model, which is obtained from the equations by neglecting $O(\varepsilon^2)$. We then find that

$$\tau \approx |\tau_2|,$$

$$\tau_2 \approx -(\eta - y)\eta_x; \tag{2.22}$$

we adopt a stream function ψ defined by

$$u=\psi_y, \qquad v=-\psi_x; \tag{2.23}$$

then

$$\psi_{yy}=A(T)g(\tau)\,\mathrm{sgn}(\tau_2)/v,$$
$$dT/dt=(\alpha/v)\tau A(T)g(T)+\beta T_{yy}, \tag{2.24}$$

together with (2.22), give the equations for ψ and T. The boundary conditions are

$$\text{on} \quad y=h, \qquad \psi=\psi_y=0, \qquad T(\Gamma+\partial T/\partial y)=0;$$

$$\text{on} \quad y=\eta, \qquad \eta_t+\frac{\partial}{\partial x}\{\psi[x,\eta(x,t)]\}=a, \qquad T=T_A. \tag{2.25}$$

The first condition on $y=\eta$ is simply the kinematic wave equation for η. As specific examples of the functions A and G, we shall assume Glen's law

$$g=\tau^n \tag{2.26}$$

and the Frank-Kamenetskii approximation (Frank-Kamenetskii, 1955) to the Arrhenius term (2.5):

$$A=e^{\gamma T}, \tag{2.27}$$

where

$$\gamma=Q\Delta T/RT_m^2. \tag{2.28}$$

Although Q varies with T below T_m (Paterson, 1981), the main point is that A decreases rapidly: a factor of ten between 0°C and −10°C. Numerical analysis of the equations is not dependent on (2.27), and asymptotic analysis focusses on the creep rate near the maximum temperature. For our purposes, (2.27) is no limitation: the value of Q at T_m is about 140 kJ mole^{-1} (Paterson, 1981), so that with $R=8.3$ J mole^{-1} K^{-1}, we find

$$\gamma\approx 11.3. \tag{2.29}$$

It is because of this large value of γ that "large activation-energy asymptotics" is suggested, and this lies behind the approximate analysis suggested in Section 4.

There is nothing particularly sacrosanct about the rheology, the no-slip condition or the temperature-dependent rate factor. We choose Glen's law simply through popularity, but it has been questioned (Smith and Morland, 1981; Doake and Wolff, 1985). However, it is debatable whether laboratory data at the melting point is best used for cold ice sheets, since it is feasible that fluid-assisted diffusion creep may be more prominent in such data (Stocker and Ashby, 1973). Nor do we necessarily concur with the view that an infinite viscosity is necessarily strictly inapplicable at zero stress; we consider that a successful numerical or analytical model should be able to deal with such a rheology, and part of our purpose here is to show how this may be done.

Sliding (basal slip), as opposed to fabric or temperature-enhanced basal shear (Morland *et al.*, 1984), certainly occurs on ice sheets, particularly at the margins. We consider it unlikely to occur where basal ice is cold, and where shear stresses are close to zero. In addition large scale basal roughness may mean that any sliding which does occur inland will be very small. Therefore we pose zero sliding velocity, and consider that the model must be solvable with this condition. This necessarily leads us to singular behaviour in the reduced model at divides and at margins, and we will show how such singular behaviour can be removed.

3. ISOTHERMAL FLOW

The advantage of an isothermal situation is that A does not depend on T. Thus the flow problem uncouples from the energy equation. In fact, "isothermal" can be relaxed to mean "uncoupled" by formally assuming $A=1$, e.g. $\gamma=0$ in (2.27). This would be a consequence of $\Delta T\approx 0$, or more generally if $\gamma \ll 1$. In the sequel we use Glen's law (2.26) for illustration. We choose $\nu=1$ in (2.24) and will show that $\eta\sim 1$ with this choice. Two integrations using $(2.25)_1$ yield

$$\psi|_\eta = -|\eta_x|^{n-1}\eta_x(\eta-h)^{n+2}/(n+2). \tag{3.1}$$

Denoting ice thickness $\eta-h=H$, the kinematic condition in $(2.25)_2$ yields

$$H_t = -\frac{\partial}{\partial x}\{[-|H_x+h_x|^{n-1}(H_x+h_x)]H^{n+2}/(n+2)\}+a. \tag{3.2}$$

For simplicity we now suppose $h=0$. Then (3.2) is a nonlinear diffusion equation for H and with $h=0$, it is

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[\frac{H^{n+2}|H_x|^{n-1}H_x}{n+2}\right]+a. \tag{3.3}$$

Steady state solutions of equations like (3.2) were studied by Vialov (see Paterson, 1981), and compared with some success to Antarctic profiles. Bodvardsson (1955) derived an equation similar to (3.3), and studied both steady solutions and their stability. If we assume $a=a(x)$, and a symmetric solution about $x=0$ (so $H_x=0$ there), then in the steady state

$$Q = -|H_x|^{n-1}H_xH^{n+2}/(n+2) = \int_0^x a\,dx = s(x), \tag{3.4}$$

say, where Q is the horizontal flux across a vertical section. The above equation states that in a steady state, this flux is balanced by the net accumulation between the divide and the section. Then

$$H^{2(n+1)/n}=\frac{2(n+1)}{n}\int_{x}^{x_m}[(n+2)s]^{1/n}\,dx, \tag{3.5}$$

where x_m gives the margin. Notice that $H=O(1)$ at $x=0$ since $s(x)=O(1)$, as we anticipated. We include this old result here partly for completeness, and partly to point out several features. Firstly, we require a boundary condition to prescribe x_m. For a land-based ice sheet (such as that which covered North America in the last ice age), mass balance requires

$$s(x_m)=0, \tag{3.6}$$

which defines x_m: no explicit boundary condition is necessary (or: we require the flux $Q=-H^{n+2}|H_x|^{n-1}H_x/(n+2)$ to be zero when $H=0$). For an ice sheet bounded by the ocean, a simple condition is that x_m is prescribed (and equals the coast position), but $s(x_m)$ is not necessarily zero. Then $H=0$ when $x=x_m$, but the flux Q does not tend to zero. One visualises this flux as occurring through iceberg calving and outlet glacier drainage. A more realistic condition might be that $H=H_m>0$ at x_m, $Q=Q_m$, with Q_m a function of H_m. Such thoughts are not pursued here.

The next point is that H is singular at x_m (specifically, $H_x\to\infty$ as $x\to x_m$). For the land-based margin (which we henceforth consider), $s(x_m)=0$, and so (3.5) implies $H\sim(x_m-x)^{1/2}$ as $x\to x_m$ (independently of n). This singularity is due to the neglect of longitudinal stress components in the governing equation (Weertman, 1961). We will show how a local rescaling near the margin re-introduces the neglected terms, so that we can then expect that the exact solution has a finite but large slope. This is basically Weertman's (1961) idea, recast here in an appropriate asymptotic procedure.

From (2.12), one can show that when $x_m-x\sim\delta\ll1$, the reduced model has $x_m-x\sim s\sim\delta$, $H, y\sim\delta^{1/2}$, $\tau_2\sim1$, $\tau_1\sim\delta^{-1/2}$, $u\sim\delta^{1/2}$, $v\sim1$, $p\sim\delta^{-1/2}$. A distinguished limit occurs when

$$\delta=\varepsilon^2, \tag{3.7}$$

when the neglected terms of $O(\varepsilon^2)$ in (2.12) and (2.14) become comparable to the others. Thus with $x_m-x\sim\varepsilon^2$, the full Stokes flow problem must be solved. If we suppose this can be done, then (3.7) suggests that

$$H_x\sim O(1/\varepsilon), \qquad H_{xx}\sim O(1/\varepsilon^2). \tag{3.8}$$

at the margin, [There are other issues as to whether the complete Stokes equation does have a solution with finite slope (see, e.g. Benney and Timson, 1980).] Note that in the original *dimensional* variables, this corresponds to a slope of $O(1)$.

The singularity here is not due to Glen's law, but to the use of the lubrication approximation $\varepsilon \ll 1$. The local analysis suggests that this singularity is a real physical effect, which needs to be analysed. If sliding is non-zero at the margin (Morland and Johnson, 1980), then the problem is alleviated naturally, although normal sliding laws (e.g. Weertman, 1957) which have the velocity tending to zero with the applied stress would still yield a singularity. Morland and Johnson, 1980 (cf. Morland *et al.*, 1984) propose a "sliding law" which avoids this problem, but is less clearly based on known physics. If one wishes to prescribe no slip, then the implication of (3.8) for numerical computations is that a local coordinate stretching should be used, as finite difference approximations (for example) will be inaccurate.

A related question arises if one considers time-dependent perturbations. Nye (1960) studied small perturbations to glacier surfaces, and found unbounded growth near the snout. Fowler and Larson (1980a) showed that this non-uniformity arose due to expansion about the basic state $H_0=0$ at the snout. A similar problem can be expected for ice sheets, but the comparable theory (Bodvardsson, 1955) did not emphasise this point (see also Hutter, 1983, pp. 403–404). Here we show how the use of the method of strained coordinates (Van Dyke, 1975) can give a uniformly valid perturbation method for small deviations, and that these decay diffusively, as might be expected.

For simplicity, consider the Newtonian case $n=1$. Denote the steady state given by (3.5) as H_0, and write

$$H=H_0+H_1\ldots, \qquad H_1 \ll H_0. \tag{3.9}$$

Linearisation about H_0 of (3.3) with $n=1$ yields

$$\frac{\partial H_1}{\partial t}=\frac{1}{3}\frac{\partial^2}{\partial x^2}(H_0^3 H_1). \tag{3.10}$$

We put

$$H_1=e^{-\lambda t}w(x)/H_0^3 \tag{3.11}$$

to obtain

$$\frac{d^2 w}{dx^2}+\frac{3\lambda}{H_0^3}w=0. \tag{3.12}$$

With suitable boundary conditions at the margins, this is a linear second-order Sturm-Liouville problem. We can expect a denumerable increasing sequence of λ's satisfying (3.12) together with appropriate boundary conditions. Now, as $x \to x_m$, we have $H_0 \sim c\zeta^{1/2}$, where $\zeta = x_m - x$. Thus as $\zeta \to 0$, w satisfies (at leading order)

$$\frac{d^2 w}{d\zeta^2}+\frac{3\lambda}{c^3}\zeta^{-3/2}w=0. \tag{3.13}$$

Linearly independent solutions are

$$w_1=\zeta^{1/2}J_2(\mu\zeta^{1/4}), \qquad w_2=\zeta^{1/2}Y_2(\mu\zeta^{1/4}) \tag{3.14}$$

with

$$\mu=4(3\lambda/c^3)^{1/2}. \tag{3.15}$$

Thus as $\zeta\to 0$, the two independent solutions have behaviour

$$w_1\sim\alpha_0\zeta+\alpha_1\zeta^{3/2}+O(\zeta^2), \qquad w_2\sim\beta_0+O(\zeta^{1/2}), \tag{3.16}$$

where α_0, α_1. β_0 are known from the definitions of J_2, Y_2; w_1 and w_2 correspond to solutions

$$H_2\sim\zeta^{-1/2}, \qquad H_1\sim\zeta^{-3/2}. \tag{3.17}$$

That is, *neither* solution has bounded H_1 at $\zeta=0$, and thus the expansion cannot be uniformly valid at x_m. To resolve this problem we introduce strained coordinates (Van Dyke, 1975) and write

$$x=\xi+\delta e^{-\lambda\tau}\sigma(\xi),$$
$$t=\tau, \tag{3.18}$$

where δ is the scale of the perturbation. We expand the thickness as

$$H=H_0(\xi)+\delta e^{-\lambda\tau}h(\xi)+\ldots, \tag{3.19}$$

and substitute (3.19) into (3.3), written in terms of independent variables ξ and τ. At leading order [i.e., O(1)], H_0 is again given by (3.5):

$$H_0=\left[12\int_{\xi}^{\xi_m}s(\xi)d\xi\right]^{1/4}. \tag{3.20}$$

The linearised equation for h can be written, after careful simplification, as

$$\frac{d^2w}{d\xi^2}+\frac{3\lambda w}{H_0^3}=0 \tag{3.21}$$

[compare (3.12)], where now

$$w=H_0^3[h-\sigma H_0']. \tag{3.22}$$

Putting $\zeta=\xi_m-\xi$, where $s(\xi_m)=0$, we have that $H_0\sim c\zeta^{1/2}$ as $\zeta\to 0$. The two independent solutions of (3.21) therefore have

$$w_{1,2} \sim c^3\zeta^{3/2}\left[h + \frac{1}{2}c\zeta^{-1/2}\sigma\right] \qquad \text{as} \quad \zeta \to 0, \tag{3.23}$$

where, as in (3.16), $w_1 \sim \alpha_0\zeta + \alpha_1\zeta^{3/2} + O(\zeta^2)$, and $w_2 \sim \beta_0 + O(\zeta^{1/2})$, where α_0, α_1, β_0, are known. The extra flexibility in the expansion is now used to ensure uniformity. We require (3.18) and (3.19) to be uniformly valid as $\xi \to \xi_m$: Thus we require $\sigma = O(1)$ (so that $\delta e^{-\lambda\tau}\sigma \ll \xi$), $h = O(\zeta^{1/2})$ [so that $h = O(H_0)$] as $\xi \to \xi_m$, i.e. as $\zeta \to 0$. Evidently, this can be done by suppressing the Y_2 solution ($\sim \beta_0 + \ldots$), and choosing

$$\sigma = 2(\alpha_0 + \alpha_1\zeta^{1/2})/c^4, \tag{3.24}$$

and then $h = O(\zeta^{1/2})$ as $\zeta \to 0$. The choice is not unique, but this leads to no difference in the solution. The procedure can be continued to higher order. The boundary conditions on (3.21) are thus that $w = 0$ at $\pm\xi_m$: this forms a sensible eigenvalue problem. The eigenvalues are given by multiplication of (3.21) by w and integrating between $\pm\xi_m$. We find

$$\lambda = \left[\int_{-\xi_m}^{\xi_m} w'^2\, d\xi\right] \Bigg/ \left[\int_{-\xi_m}^{\xi_m} 3w^2/H_0^3\, d\xi\right]. \tag{3.25}$$

In fact, minimisation of (3.25) subject to $w = 0$ at the boundaries yields the minimum eigenvalue (Courant and Hilbert, 1953). It is positive, implying stability, and the essential structure of the transient will be like $H_1 \approx e^{-\lambda t}\cos kx$, where $k = 2/\pi\xi_m$ (i.e. the x dependence of H_1 has a similar shape).

The perturbation analysis above was developed by Peter Gillott, and is discussed further in his M.Sc. dissertation (Gillott, 1985). It serves to show that ice sheets are stable (as one would imagine) if thermal effects are uncoupled. It also gives us some idea how the mechanical part of the problem tends to behave.

A different kind of singularity occurs at a divide. Raymond (1983) found numerical evidence of a discontinuity in surface slope at a divide, and Hindmarsh (1989) has shown that the reduced model does in fact have infinite curvature, if Glen's law is used with $n > 1$. In addition, he shows how the inclusion of longitudinal stresses in the model gives finite but large curvature at the divide. Hutter *et al.* (1986) show that curvature is finite if sliding is included or if $n \to 1$ at the divide. Szidarowsky *et al.* (1989) showed how finite curvature is obtained even with no sliding, if longitudinal stretching is (numerically) included. A similar idea was advanced by Weertman (1961). Here we sketch this analysis (but note that temperature dependence has a radical effect on the divide curvature, as shown in the subsequent section), but recast it in the language of matched asymptotic expansions.

As $x \to 0$ in (3.5) or (3.4), we have

$$|H_x| \sim x^{1/n}, \qquad u \sim x, \qquad \tau \sim \tau_2 \sim x^{1/n}, \qquad p \sim \tau_1 \sim x^{(1/n)-1}; \tag{3.26}$$

the reduced model becomes invalid when either $\tau_{2y} \sim \varepsilon^2 \tau_{1x}$ (mechanical boundary layer) or $\tau_2 \sim \varepsilon\tau_1$ (rheological boundary layer). In view of (3.26), these both happen when

$$x \sim \varepsilon, \qquad \tau_1 \sim p \sim \varepsilon^{(1/n)-1}, \quad \tau_2 \sim \varepsilon^{1/n}, \qquad H_x \sim \varepsilon^{1/n}, \tag{3.27}$$

at which point a local rescaling corresponding to (3.27) reproduces the full equations. Supposing a solution exists, we infer that

$$H_{xx} \sim \varepsilon^{(1/n)-1} \tag{3.28}$$

at the divide. This kind of idea has been applied by Johnson and McMeeking (1984) to a weak boundary layer near the ice surface.

Again, the reduced model is uniformly valid for the primary variables τ_2, H, u, and the local analysis is only of concern if we wish to compute the curvature. In practice, it may be best to adapt numerical computations to cope with the singularity by using a local straining of the coordinates.

4. STEADY, NON-ISOTHERMAL FLOW

We now revert to the coupled equations (2.24), where we adopt the thermally activated Glen's law (2.26) and (2.27). We suppose the flow to be in a steady state. The equations may be written

$$u_y = \frac{1}{v}\tau^n e^{\gamma T},$$

$$\tau = -(\eta - y)\eta_x,$$

$$\frac{dT}{dt} = \frac{\alpha}{v}\tau^{n+1} e^{\gamma T} + \beta T_{yy}, \tag{4.1}$$

where d/dt is the material derivative $u\partial/\partial x + v\partial/\partial y$, and the boundary conditions (with $u = \psi_y$) are

$$u = \psi = 0, \qquad \partial T/\partial y = -\Gamma \qquad \text{at} \quad y = 0, \tag{4.2}$$

until $T = 0$, when we replace the last of (4.2) by $T = 0$. On the top surface $y = \eta$, we have, in a steady state,

$$T = T_A, \qquad \int_0^\eta u\,dy = s(x) = \int_0^x a(x)dx. \tag{4.3}$$

In general, T_A will depend both on distance inland and on altitude. In this study, we limit ourselves to an explicit prescription of T_A, that is, we assume $T_A = T_A(x)$. Similar remarks apply to $a(x)$. For a land-based margin, the ice sheet boundary is defined by $s(x)=0$ $(x \neq 0)$. Let us recollect that v is an arbitrary parameter, to be chosen such that the ice sheet thickness is $O(1)$.

The parameters (other than v) which are prescribed here are α, β, γ and Γ. We have seen in Section 2 that we may expect $\alpha \approx 0.6$, $\beta \approx 1/8$, $\gamma \approx 11$, $\Gamma \approx 1.5$ as typical values of these, and, although these are not extreme, it does suggest that some idea of how solutions may behave can be gained from asymptotic solutions when either $\beta \ll 1$ or $\gamma \gg 1$, or both. This is particularly true when $\gamma \gg 1$, since the effect of this coefficient is exponentially exaggerated. Moreover, we shall find thermal boundary layers of thickness $O(\beta^{1/2})$ when $\beta \ll 1$; since $\beta^{1/2} \approx 0.3$, the accuracy of this is only a qualitative one. Thus our main concern shall formally be with "large activation energy asymptotics".

The primary source material for this technique is combustion theory, for which see, for example, Buckmaster and Ludford (1982). Thermally activated fluid shear flows have been considered by Ockendon and Ockendon (1977), Pearson (1977) and Ockendon (1979). Their techniques have been extended to treat buoyancy-driven shear flows (Fowler, 1986) and thermally activated (thermoviscous) Rayleigh-Bénard convection (Morris and Canright, 1984; Fowler, 1985).

A relevant question which arises in this context is whether thermal runaway can occur. In the glaciological context, the implicit meaning of such thermal runaway is not simply that the temperature reaches melting point, but that, subsequent to this occurring, melting occurs on a massive, self-sustaining scale, presumably leading to collapse of the ice sheet, as envisaged by Wilson (1964); see also Hollin (1965, 1969, 1977). Such a phenomenon has been studied by Schubert and Yuen (1982) and Yuen *et al.* (1986), although not with as realistic a model as that presented here. We shall have little to say on this topic, but will be interested in the circumstances under which the basal ice can reach melting point. Since sub-Antarctic lakes have been documented, this is known to occur. Such evidence as there is tends to suggest that thermal runaway will not occur (Fowler and Larson, 1980b), but a proper discussion of this is beyond the scope of the present paper. Models of ice flow which include moisture generation have been developed by Hutter, Blatter and Funk (1988) and Blatter and Hutter (1991), but a discussion of possible runaway needs the incorporation of realistic coupling between basal sliding and moisture generation.

The system (4.1) is parabolic for T, and we expect to prescribe

$$\eta_x = T_x = u = 0 \qquad \text{at} \quad x = 0. \tag{4.4}$$

The equations are degenerate at $x=0$, however, as is easily seen on writing those for u and T in von Mises variables x, ψ:

$$uu_\psi = \frac{1}{v}\tau^n e^{\gamma T},$$

$$uT_x = \beta u \frac{\partial}{\partial \psi}\left(u \frac{\partial T}{\partial \psi}\right) + \frac{\alpha}{v} \tau^{n+1} e^{\gamma T}, \tag{4.5}$$

where also $\eta = \int_0^s d\psi/u$. Since $s(0)=0$, the (x, ψ) region of the flow degenerates to a point as $x \to 0$. Since also $u \to 0$ there, both the domain and the diffusion coefficient degenerate, and it is reasonable to expect that the solution of (4.5) has singular behaviour as $x \to 0$. In particular, we do not expect to satisfy $T_x = 0$ at $x=0$, but this would be accommodated by a boundary layer of thickness $\varepsilon\beta^{1/2}$, in which the temperature jumps by a corresponding amount. Just as for the isothermal case, we can however, satisfy the other two conditions.

Following Pearson (1977) and Ockendon (1979), it is attractive to march out in x, and thus we first seek a similarity solution for (4.1) when x is small. By inspection, we find that a suitable ansatz is

$$\psi = xF(y),$$

$$T = G(y),$$

$$\eta \approx \eta_0, \qquad \eta_x \approx -bx^{1/n}, \tag{4.6}$$

when x is small. Substituting these into (4.1), we find

$$F'' = \frac{1}{v}(b\eta_0)^n e^{\gamma G},$$

$$\tau = b(\eta_0 - y)x^{1/n},$$

$$\beta G'' + FG' = -\frac{\alpha}{v}(b\eta_0)^{n+1} x^{1+1/n} e^{\gamma G} \tag{4.7}$$

with boundary conditions

$$F = F' = 0, \qquad G' = -\Gamma, \qquad \text{on} \quad y=0,$$

$$G = -1 \qquad \text{on} \quad y = \eta_0, \tag{4.8}$$

where we may choose $T_A(0) = -1$. Additionally, if $a(0) = s'(0) = 1$, then we have

$$F(\eta_0) = 1. \tag{4.9}$$

The above system of equations describes an approximation to the solution for all y, and sufficiently small values of x. The parameters b and η_0 are unknown. We can expect to determine b in terms of η_0, which we expect to be determined from the margin condition $\eta = 0$ at x_m, where $s(x_m) = 0$. We can also pre-suppose that v will be chosen so that $\eta_0 = O(1)$. It is not at this stage obvious that $b = O(1)$, but in fact it will turn out that it is. We anticipate this by proceeding on the formal assumption that $b = O(1)$. An exact similarity solution of (4.7) does not exist, but a local similarity solution exists for sufficiently small x, if we neglect the term of $O(x^{1+1/n})$.

Now suppose $\gamma \gg 1$. In this case the solution of (4.7) [neglecting $O(x^{1+1/n})$ in $(4.7)_3$] develops a boundary layer structure wherein a basal shear layer lies below an outer region of plug flow. If, in addition, $\beta \ll 1$ (and $\beta^{1/2}\gamma \gg 1$), there is a thermal boundary layer outside the shear layer where G matches to its outer value. This scheme is pictured in Figure 2. The solution of (4.7) is as follows. G will take its maximum value G_0 at $y = 0$. Away from the base, $G < G_0$, and the exponential terms are negligible. Thus, we have

$$F \sim y/\eta_0, \tag{4.10}$$

and this will in fact be a uniform approximation, although F' will jump in a basal shear layer. Thus G satisfies

$$\beta\eta_0 G'' + yG' = 0, \tag{4.11}$$

and hence

$$G' = -\Gamma e^{-y^2/2\beta\eta_0}, \tag{4.12}$$

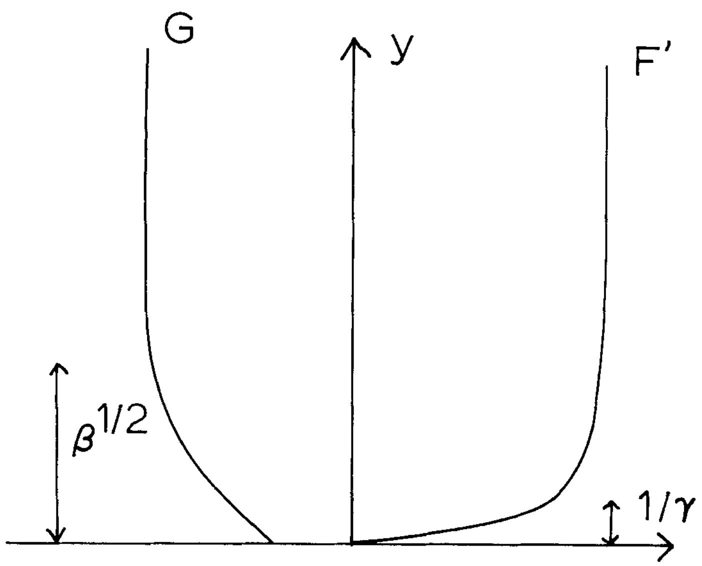

Figure 2 Structure of the solutions for temperature (G) and horizontal flow (F') for the local similarity solution near $x = 0$. There is a boundary layer for G of thickness $O(\beta^{1/2})$, and a shear layer for F' of thickness $O(1/\gamma)$, in the case that basal heat flux is prescribed. If local temperature is prescribed, then the basal shear layer is of thickness $O(\beta^{1/2}/\gamma)$.

and

$$G=-1+\Gamma\int_{y}^{\eta_0} e^{-y^2/2\beta\eta_0}\,dy. \tag{4.13}$$

This satisfies both boundary conditions, in particular, at $y=0$,

$$G=G_0=-1+\Gamma(2\beta\eta_0)^{1/2}\int_0^{(\eta_0/2\beta)^{1/2}} e^{-\xi^2}\,d\xi. \tag{4.14}$$

Notice that, when $\beta\ll 1$, there is an $O(\beta^{1/2})$ jump in T in a thermal boundary layer of thickness $O(\beta^{1/2})$.

We require a shear layer in which u (and hence F') decreases to zero. In order to balance $(4.7)_1$, we write $S=F'$ (note the horizontal velocity is xS); then S jumps by O(1), and we use stretched variables θ and Y in the shear layer, defined by

$$G=G_0+\theta/\gamma, \qquad y=Y/\gamma, \tag{4.15}$$

in order to match to (4.13); it follows that, to leading order, (4.11) is just $\theta_{YY}=0$, so that to satisfy $(4.8)_1$, θ is given in the shear layer by

$$\theta\sim -\Gamma Y, \tag{4.16}$$

providing $\beta\gamma^2\gg 1$ [since the advective term is $O(1/\beta\gamma^2)$ in the shear layer]. Substituting for y and $S=F'$ into $(4.7)_1$, we obtain

$$\frac{dS}{dY}=e^{\theta}, \tag{4.17}$$

providing we choose

$$v=\frac{(b\eta_0)^n}{\gamma}e^{\gamma G_0}. \tag{4.18}$$

Remember that $G_0<0$, so that we can expect $v\ll 1$. Notice that the case $\beta\ll 1$ does not render the shear layer analysis invalid providing $\beta^{1/2}\gg 1/\gamma$, consistent with (4.16). Integrating (4.17) with (4.16) and applying the no slip condition $S(=F')=0$ at $Y=0$ gives

$$S=(1-e^{-\Gamma Y})/\Gamma, \tag{4.19}$$

and matching this to (4.10) requires

$$\eta_0=\Gamma, \tag{4.20}$$

which is O(1) as assumed. Actually, the depth is not really prescribed from this local analysis, and this is manifested by the fact that b is not yet determined, or equivalently, (4.18) determines b, and v has yet to be found. The above description gives the vertical boundary layer structure of the local similarity solution for small x. In order to determine b, we extend the boundary layer structure exhibited above to the whole ice sheet.

4.1 Shear layer: freezing base

The idea of the analysis is that the shear layer structure will occur across the bed. This is no new idea, and stems from Nye (1959). Lliboutry (1979, 1987) used it to give a heuristic account of the boundary layer structure, although he stopped short of determining the surface profile. Essentially, the present analysis is based on his work, but formalises it in asymptotic terms.

The basic idea is that the shear layer of thickness $1/\gamma$, which was found to exist for small x above, exists for all x; across this layer u jumps from zero to a far field value. As the base of the ice warms, it becomes more "slippery", and thus the ice accelerates. It is not therefore *a priori* obvious that η or η_x are O(1) for all $x>0$. In order to try and analyse the basal shear-dominated flow, we will for simplicity consider two particular cases. The first, which we treat in this sub-section, is where the base temperature is always below the melting point. (In the next section, we consider the particular case where the temperature is always at the melting point; in general, neither situation may be completely true.) To be specific, suppose $T=T_0(x)$ at $y=0$. For the moment, T_0 is unknown, and it will be determined in the course of the analysis. The melting temperature (neglecting the small variation due to the Clapeyron effect) is zero, so that we suppose $T_0<0$.

We anticipate that $T<T_0$ for $y>0$; then (if $\gamma\gg 1$) $(4.1)_1$ implies that u_y is exponentially small away from $y=0$. Hence, for $y=O(1)$, $u\approx u_0(x)$, and t satisfies

$$\frac{dT}{dt}=\beta T_{yy}, \tag{4.21}$$

($d/dt=u\partial/\partial x+v\partial/\partial y$) (this will apply even if viscous heating is significant near the base), with boundary conditions (providing α is small enough)

$$T=T_A \quad \text{on} \quad y=\eta, \qquad T_y=-\Gamma \quad \text{on} \quad y=0. \tag{4.22}$$

Mass conservation gives

$$\eta u_0=s(x). \tag{4.23}$$

In the shear layer, we put, as before,

$$y=Y/\gamma, \qquad T=T_0+\theta/\gamma, \tag{4.24}$$

so $Y, \theta \sim O(1)$. The basal temperature T_0 is to be determined later, and we expect that $T_0 \to G_0$ as $x \to 0$, in keeping with the local similarity solution. We have

$$u_Y = \frac{\tau^n}{v\gamma} e^{\gamma T_0} e^{\theta},$$

$$\tau \approx -\eta\eta_x,$$

$$\frac{1}{\beta\gamma} u T_0' + \frac{1}{\beta\gamma^2} \frac{d\theta}{dt} = \left(\frac{\alpha}{\beta}\right) \tau u_Y + \theta_{YY}. \tag{4.25}$$

Let us suppose that $u \ll \beta\gamma$, $\beta/\alpha\tau$ (as well as $\beta\gamma^2 \gg 1$). Then

$$\theta \sim -\Gamma Y, \tag{4.26}$$

where as before we can take $\theta = 0$ on $Y = 0$, since T_0 is as yet undetermined, and so, integrating $(4.25)_1$ [with $\tau = \tau(x)$],

$$u = \frac{\tau^n}{v\gamma\Gamma} e^{\gamma T_0}(1 - e^{-\Gamma Y}), \tag{4.27}$$

whence, letting $Y \to \infty$, we obtain (by matching) an expression for the outer velocity $u_0(x)$:

$$u_0 \approx \frac{\tau^n}{v\gamma\Gamma} e^{\gamma T_0}. \tag{4.28}$$

Together with $(4.25)_2$ and (4.23), this determines η, u_0 and τ in terms of the still unknown T_0. We find

$$\eta^{2+(1/n)} \approx \left(\frac{2n+1}{n}\right) \int_x^{x_m} [v\gamma\Gamma s(x) e^{-\gamma T_0(x)}]^{1/n}\, dx, \tag{4.29}$$

and thus v is determined (since $\eta_0 = \Gamma$) by

$$(v\gamma)^{1/n} \frac{(2n+1)}{n} = \Gamma^2 \Big/ \int_0^{x_m} [s(x) e^{-\gamma T_0(x)}]^{1/n}\, dx, \tag{4.30}$$

and so (4.29) may be rewritten as

$$(\eta/\eta_0)^{2+(1/n)} = \frac{\int_x^{x_m} (s e^{-\gamma T_0})^{1/n}\, dx}{\int_0^{x_m} (s e^{-\gamma T_0})^{1/n}\, dx}. \tag{4.31}$$

If $T_0 < G_0$, then $\eta \ll \eta_0$, and $\eta_x \neq O(1)$. The surface profile given by (4.31) matches automatically to the local solution given earlier.

In order to completely solve for η, we have to calculate T_0 by solving for the temperature field, which satisfies in the outer region, $y = O(1)$,

$$\frac{dT}{dt} = \beta T_{yy}, \tag{4.32}$$

and where we may take $u = u_0$, $v = -yu_0'$ outside the basal shear layer, which does not affect the temperature field if $\beta^{1/2}y \gg 1$, as explained previously; additionally, T satisfies

$$T = T_A \quad \text{on} \quad y = \eta, \qquad T_y = -\Gamma \quad \text{on} \quad y = 0. \tag{4.33}$$

Since u_0 given by (4.23) depends on η, the solution of (4.31) and (4.32) represents a complicated free boundary problem. Nevertheless, some further simplification is possible, using the idea that β is small.

We first write the temperature problem in von Mises coordinates ξ, ψ, which gives

$$T_\xi = \beta T_{\psi\psi}, \qquad \xi = \int_0^x u_0\, dx, \qquad \psi = u_0 y \tag{4.34}$$

with

$$T = T_A \quad \text{on} \quad \psi = s, \qquad T_\psi = -\Gamma/u_0 \quad \text{on} \quad \psi = 0. \tag{4.35}$$

If $\beta \ll 1$ the outer solution ($\psi \sim 1$) is just $T = T_0(\psi)$ where T_0 is defined, for $s < s_{\text{max}}$, by

$$T_0[s(x)] \equiv T_A(x); \tag{4.36}$$

(for $s > s_{\text{max}}$, a further thermal boundary layer at the surface will exist: we ignore that in this analysis). In the thermal boundary layer, we write

$$T = -1 + \beta^{1/2}\Delta, \qquad \psi = \beta^{1/2}\Psi; \tag{4.37}$$

then Δ satisfies

$$\Delta_\xi = \Delta_{\Psi\Psi} \tag{4.38}$$

with

$$\Delta \to -1 = T_A(0) \quad \text{as} \quad \Psi \to \infty, \tag{4.39}$$

and on $\Psi = 0$, the boundary condition $(4.35)_2$ for (4.38) is

$$\Delta_\Psi = -\Gamma\eta/s,$$

$$\left(\frac{\eta}{\eta_0}\right)^{2+(1/n)}=\frac{\int_x^{x_m} s^{1/n}e^{-\lambda\Delta}\,dx}{\int_0^{x_m} s^{1/n}e^{-\lambda\Delta}\,dx},$$

$$\lambda=\gamma\beta^{1/2}/n,$$

$$\xi=\int_0^x (s/\eta)dx, \tag{4.40}$$

where we have used (4.23) in the first equation above, and appended the definitions of η in (4.31) [using (4.37)] and ξ in (4.34). Then (4.40) defines both the boundary condition for Δ and also η. As $\xi\to 0$, it follows by matching Δ to the local similarity solution (4.13) for small x, that

$$\Delta\sim(\pi/2)^{1/2}\Gamma^{3/2}\,\mathrm{erfc}(\Psi/2\xi^{1/2}). \tag{4.41}$$

The diffusion equation (4.38) for Δ has boundary conditions (4.39) as $\Psi\to\infty$ and (4.40) on $\Psi=0$, and an "initial" condition (4.41) as $\xi\to 0$. In fact, (4.40) is an extremely complicated "boundary" condition of integro-differential type. It depends on the parameter λ; since we have previously supposed $\gamma^2\beta\gg 1$, it is logically consistent (and, in fact, necessary) to suppose $\lambda\gg 1$ as well. The numerical validity of this assumption is unsound, since if $\gamma=11$, $\beta^{1/2}=1/3$, $n=3$, then $\lambda\approx 1.2$, and we may expect the numerical consequences to be inaccurate; nevertheless, the assumption $\lambda\gg 1$ is a formal part of our analysis, and we suppose it to be formally valid.

Now (4.40) gives Δ_Ψ on $\Psi=0$ in terms of the surface elevation $\eta(x)$. Let us define $G_0=-1+\beta^{1/2}\Delta_0$, so that from (4.37) and (4.14), Δ_0 is the value of Δ on $\Psi=0$ as $\xi\to 0$. Suppose that $\Delta>\Delta_0$ for $x>0$ ($\Delta<\Delta_0$ is not possible), and that $\Delta-\Delta_0=O(1)$. Then $(4.40)_2$ implies that $\eta\ll 1$ for $x>0$, and $(4.40)_4$ implies $\xi\sim 1/\eta\gg 1$. Thus in the solution of the diffusion equation (4.38), Δ diffuses out to a distance $\Psi\sim 1/\eta^{1/2}$, and therefore $\Delta_\Psi\sim\eta^{1/2}$. But $(4.40)_1$ implies $\Delta_\Psi\sim\eta$ at $\Psi=0$, and this is inconsistent, as the maximum value of $|\Delta_\Psi|$ must be at the boundary. The only other possibility is that $\Delta\approx\Delta_0$ at $\Psi=0$, so that (4.41) is a uniform approximate solution for all ξ [as it satisfies (4.38), (4.39), $(4.41)_1$ and $\Delta=\Delta_0$ on $\Psi=0$]. The rôle of $(4.40)_1$ is thus to determine η at leading order, and $(4.40)_2$ provides the boundary condition for the correction to Δ. That is, we write

$$\Delta=\Delta^{(0)}+\frac{1}{\lambda}\Delta^{(1)}\ldots,$$

$$\eta=\eta^{(0)}+\frac{1}{\lambda}\eta^{(1)}+\ldots; \tag{4.42}$$

each $\Delta^{(i)}$ satisfies (4.38), $\Delta^{(0)}$ is given by (4.41), and then (on $\Psi=0$) (4.40) implies

$$\eta^{(0)}=-s\Delta_\Psi^{(0)}/\Gamma,$$

$$\int_x^{x_m} s^{1/n} e^{-\Delta^{(1)}} dx = \left[\int_0^{x_m} s^{1/n} e^{-\Delta^{(1)}} dx \right] \Big/ [\eta^{(0)}/\eta_0]^{2+(1/n)},$$

$$\eta^{(1)} = -s\Delta^{(1)}_{\Psi}/\Gamma \tag{4.43}$$

and so on. In particular, (4.41) implies that

$$-\Delta^{(0)}_{\Psi} = \Gamma^{3/2}/(2/\xi)^{1/2} \tag{4.44}$$

on $\Psi = 0$; substituting this into $(4.40)_1$ [or $(4.43)_1$] and using $(4.40)_4$, we obtain a differential equation for ξ whose solution is $\xi \sim x^2/2\Gamma$, and then we can deduce

$$\eta \sim \Gamma s(x)/x, \qquad u_0 \sim x/\Gamma. \tag{4.45}$$

A uniformly valid approximation (Van Dyke, 1975) to the temperature profile is given by combining the inner and outer expansions for T and subtracting the common part; we obtain

$$T = T_0(xy/\Gamma) + (\pi\beta/2)^{1/2}\Gamma^{3/2}\,\mathrm{erfc}[y/(2\Gamma)^{1/2}]; \tag{4.46}$$

in particular, the basal temperature is approximately uniform,

$$T_0 \approx -1 + (\pi\beta/2)^{1/2}\Gamma^{3/2}, \tag{4.47}$$

and the base remains below freezing if

$$\Gamma < (2/\pi\beta)^{1/3}. \tag{4.48}$$

If $\beta = 1/8$, then this critical value is about 1.7. Since a realistic value for Antarctica is $\Gamma \sim 1.5$, this ice sheet is quite close to the critical value, even without the inclusion of viscous heating. Figure 3 sketches the locations of the shear layer and the upper and lower thermal boundary layers.

3.2 Shear layer: base at melting point

We can now examine the validity of the assumptions made in deriving the above approximate results. Away from the divide, we have from (4.45) u, $\eta \sim 1$, $T_0' \sim \beta^{1/2}\partial\Delta/\partial x|_{y=0} = (\beta^{1/2}/\lambda)\partial\Delta^{(1)}/\partial x|_{y=0}$ from $(4.42)_1 \sim n/\gamma$ [using $(4.40)_3$], and thus the advective term in (4.25) is negligible so long as $\beta\gamma^2 \gg 1$. However, the viscous heating term is only negligible so long as $\tau \ll \beta/\alpha$. Since $\tau \sim O(1)$ [from $(4.1)_2$, with $\eta = O(1)$, $\eta_x = O(1)$] for $x \sim O(1)$, we can expect viscous heating to be significant. In particular, since the basal temperature is close to melting at the divide, it is plausible that viscous heating raises the temperature to the melting point over much of the base. We therefore now assume that the basal thermal boundary condition is

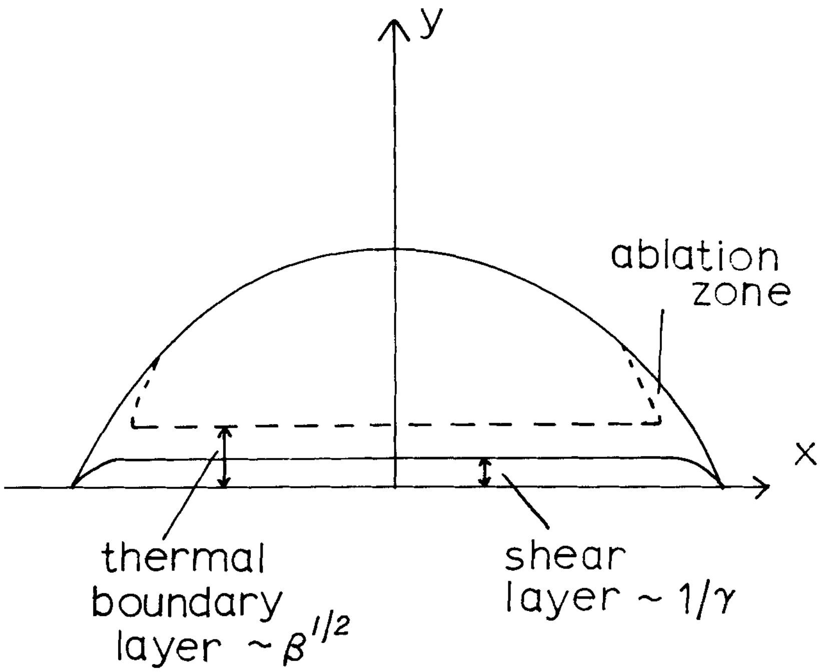

Figure 3 Location of the thermal and shear layers, for prescribed heat flux. The surface thermal boundary layers would be situated where ablation occurs, and are of thickness $O(\beta)$ (since the normal velocity is non-zero); but they are of little relevance if ablation is confined to the margins.

(neglecting the relatively small dependence of melting temperature on pressure)

$$T=0 \quad \text{at} \quad y=0, \tag{4.49}$$

and note that we require $T_y<0$ in order that a temperate zone be not formed. Lliboutry (1987) has suggested that in this case sliding (on the average) does not occur, although Fowler and Larson (1980) consider that sliding will gradually increase over a small (average) temperature interval. Here we suppose that any sliding which does occur when T reaches the melting point is small, due to the significant roughness of the bed. This is obviously not likely to be true at the margin, e.g. under ice streams, but may be reasonable in the main part of the ice.

The local flow for small x is now given by (4.7) and (4.8), but with $G=0$ on $y=0$ rather than $G'=-\Gamma$. As before, we suppose that $\gamma \gg 1$, so that the outer solution for F in (4.7) is still $F \approx y/\eta_0$, and there is a shear layer near $y=0$. Beyond this layer, G still satisfies (4.11), and provided β is small, the solution is approximately

$$G=-1+\text{erfc}[y/(2\beta\eta_0)^{1/2}], \tag{4.50}$$

and

$$G \approx -(2/\pi\beta\eta_0)^{1/2} y \tag{4.51}$$

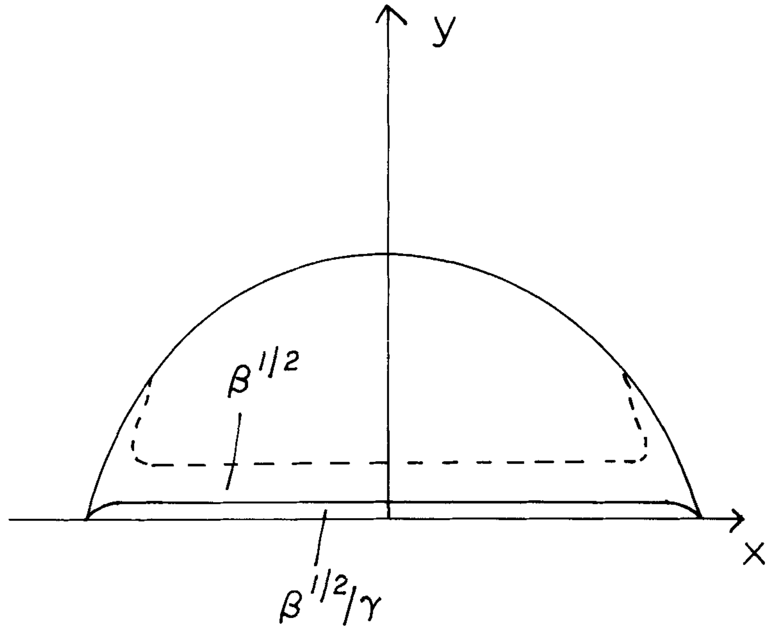

Figure 4 As for Figure 3, but for prescribed basal temperature. The shear layer is thinner.

for small $y(\ll \beta^{1/2})$. Whereas in the previous case G jumped by $O(\beta^{1/2})$ in the thermal boundary layer, now the jump is $O(1)$ over the thermal boundary layer of thickness $O(\beta^{1/2})$. Now the thickness of the shear layer is such that the variation of G is $O(1/\gamma)$ (since a larger variation causes the shear rate to become exponentially small). Since G changes by $O(1)$ across the thermal boundary layer, it follows that the appropriate shear layer thickness is $O(\beta^{1/2}/\gamma)$. Therefore we put

$$G=\theta/\gamma, \qquad y=(\beta^{1/2}/\gamma)Y, \tag{4.52}$$

so that, solving (4.7) to leading order [with $F \sim (\beta^{1/2}/\gamma)$ to match to the outer flow]

$$\theta \sim -(2/\pi\eta_0)^{1/2}Y, \tag{4.53}$$

and $S=F'$ satisfies, from $(4.7)_1$,

$$\frac{dS}{dY}=e^{\theta}, \tag{4.54}$$

provided we now choose

$$v=\beta^{1/2}(b\eta_0)^n/\gamma. \tag{4.55}$$

Thus, integrating (4.54) using (4.53) and the no slip condition $S=0$ on $Y=0$,

$$S=(\pi\eta_0/2)^{1/2}[1-\exp\{-(2/\pi\eta_0)^{1/2}Y\}], \tag{4.56}$$

and so matching to the outer flow (4.10) requires

$$\eta_0=(2/\pi)^{1/3}. \tag{4.57}$$

Away from $x=0$, we expect a shear layer in which $T\sim 1/\gamma$. We thus write, for the same reasons as above,

$$T=\theta/\gamma, \qquad y=\beta^{1/2}Y/\gamma, \tag{4.58}$$

so that, approximately,

$$u_Y=\frac{\beta^{1/2}\tau^n}{\nu\gamma}e^{\theta},$$

$$\tau\sim-\eta\eta_x,$$

$$\frac{1}{\gamma^2}\frac{d\theta}{dt}=\left(\frac{\alpha}{\beta^{1/2}}\right)\tau u_Y+\theta_{YY}; \tag{4.59}$$

we suppose α sufficiently small that $\theta_Y<0$ at $Y=0$. (Otherwise a temperate zone appears at the base, for which a separate model is necessary to determine water content.) We write $\theta_Y=-g$, $g=g_0$ at $Y=0$, $g\to g_\infty$ as $Y\to\infty$. Integrating (4.59) subject to the boundary conditions $u=\theta=0$ on $Y=0$, $u\to u_0$ as $Y\to\infty$, we therefore obtain

$$g_0=\left[g_\infty^2-\frac{2\alpha\tau^{n+1}}{\nu\gamma}\right]^{1/2}, \tag{4.60}$$

$$u_0=\frac{\beta^{1/2}}{\alpha\tau}(g_\infty-g_0); \tag{4.61}$$

we have neglected $O(1/\gamma^2)$ in $(4.59)_3$, so that $[\tau=\tau(x)]$ the θ equation can be integrated twice: (4.61) is the first integral, and then (4.60) [using $(4.39)_1$] is the second. Also

$$\eta u_0\sim s(x), \qquad \tau\sim-\eta\eta_x, \tag{4.62}$$

as before. The first of these is conservation of mass; in (4.60) and (4.62), τ is the *basal* shear stress. Given g_∞, these equations provide four relations for the unknowns g_0, u_0, η, τ. Notice that we require $g_0>0$, i.e. $\nu\gamma g_\infty^2>2\alpha\tau^{n+1}$, otherwise a temperate layer forms. The heat flux to the main ice sheet, g_∞, is determined by the solution of the outer temperature field, which, as before, satisfies

$$T_\xi = \beta T_{\psi\psi},$$

$$T = T_A \quad \text{on} \quad \psi = s,$$

$$T = 0 \quad \text{on} \quad \psi = 0, \tag{4.63}$$

where $\xi = \int_0^x u_0\, dx$, $g_\infty = -\beta^{1/2} u_0 T_\psi|_{\psi=0}$, and T matches to the small x similarity solution. If $\beta \ll 1$, then with

$$\psi = \beta^{1/2}\Psi, \tag{4.64}$$

the outer solution is again $T_0(\psi)$, and for $\Psi \sim O(1)$, we have

$$T_\xi = T_{\Psi\Psi} \tag{4.65}$$

with

$$T = 0 \quad \text{on} \quad \Psi = 0, \qquad T \to -1 \quad \text{as} \quad \Psi \to \infty,$$

$$T \sim -1 + \operatorname{erfc}(\Psi/2\xi^{1/2}) \quad \text{as} \quad \xi \to 0. \tag{4.66}$$

The similarity solution is valid for all ξ, thus

$$g_\infty = u_0/(\pi\xi)^{1/2}, \tag{4.67}$$

and this provides the extra relation required to solve for η; one easily checks that this solution matches to the local solution at the divide.

In general, these equations must be solved numerically. We define K by

$$v = \beta^{1/2}K/\gamma, \qquad K = (b\eta_0)^n, \tag{4.68}$$

and then the set is

$$g_\infty = u_0/(\pi\xi)^{1/2}, \qquad \xi = \int_0^x u_0\, dx,$$

$$g_0 = [g_\infty^2 - 2(a/K)\tau^{n+1}]^{1/2},$$

$$u_0 = (g_\infty - g_0)/a\tau,$$

$$\eta u_0 = s(x),$$

$$\tau = -\eta\eta_x, \tag{4.69}$$

where

$$a = \alpha/\beta^{1/2}. \tag{4.70}$$

K must be determined from the solution, and if $a = O(1)$, we expect $K = O(1)$, thus $v \sim \beta^{1/2}/\gamma$. In practice, if $\alpha = 0.6$ and $\beta^{1/2} = 0.3$, then $a = 2$ is of O(1). We solve (4.69) subject to

$$\eta \to \eta_0, \qquad \eta_x \sim -bx^{1/n}, \qquad b = K^{1/n}/\eta_0, \qquad \eta_0 = (2/\pi)^{1/3} \tag{4.71}$$

as $x \to 0$, and choose the unknown K so that $\eta = 0$ at $x = x_m$ (where $s = 0$). We see that, just as before, η, u_0, τ are all O(1).

When a is small, then (4.69) can be approximated by the simpler set

$$\begin{aligned} \tau &= -\eta\eta_x, \\ \eta u &= s, \\ u^2/(\pi\xi)^{1/2} &= \tau^n/K, \end{aligned} \tag{4.72}$$

or as a single equation

$$(-\eta\eta_x)^n = Ks\left[\pi^{1/2}\eta^2\left(\int_0^x \eta^{-1}s\,dx\right)^{1/2}\right]^{-1} \tag{4.73}$$

however, neither this nor (4.69) is susceptible to analytic solution, except (possibly) for special choices of s. "Massive" melting (i.e. the onset of a basal temperate zone) occurs if

$$2a\tau^{n+1} > Kg_\infty^2 \tag{4.74}$$

in (4.69).

The equations (4.69) were solved numerically using Heun's method for the differential equations for ξ and η, and using a binary search to compute the correct value of b (and hence K). The accumulation rate is chosen so that the balance function $s(x)$ is given by

$$s(x) = x[1 - x^A]^{1/B} \tag{4.75}$$

(see the discussion in Section 5). Figure 5 shows the depth profile for values $A = 1$, $B = 1$, $a = 2$, corresponding to an accumulation rate $s'(x) = 1 - 2x$ (which is very unrealistic). It has the typical concave shape that one expects. For this accumulation rate, the onset of a basal melting zone occurs when $a = a_c \approx 3.036\ldots$ Figure 6 shows the basal heat flux g_0 as a function of x at this value. The depth profile is virtually identical to Figure 2, and indeed, hardly varies for varying a.

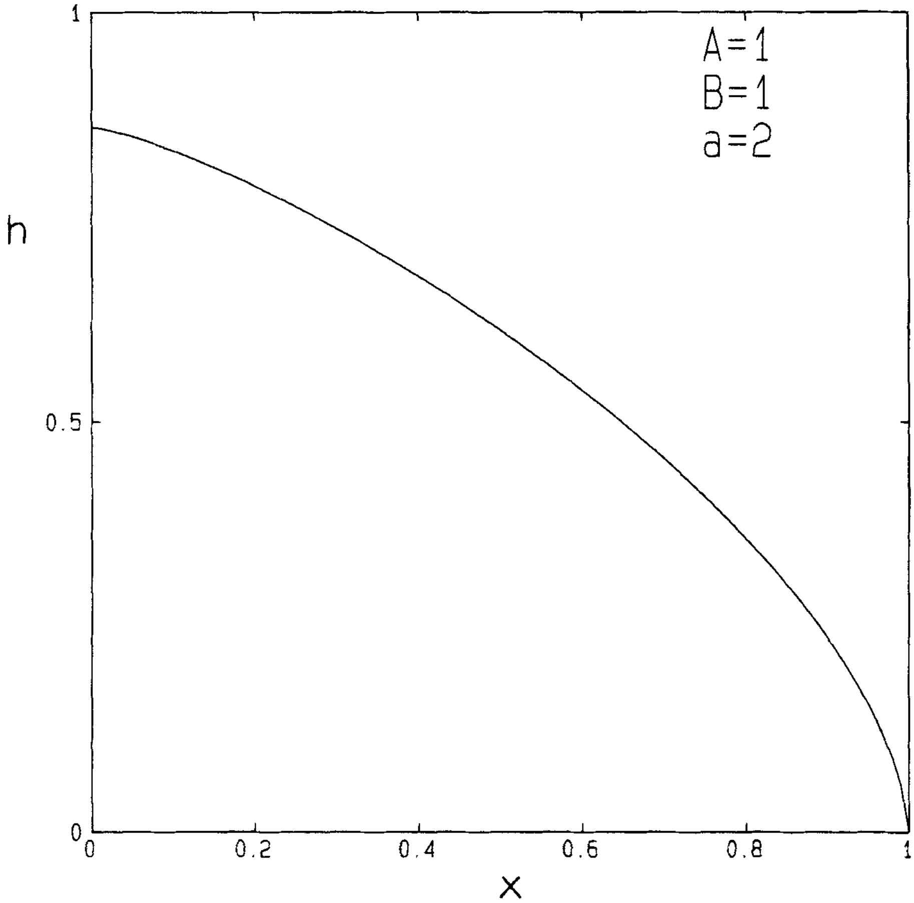

Figure 5 η versus x for $A=1=B$, $a=2$.

More realistic balance functions can be obtained by choosing values $A,B>1$. A typical example is shown by Figure 7, with $A=4/3$, $B=8/3$. The corresponding depth profile at $a=1.2$ is shown in Figure 8. It is similar to that in Figure 5, but rather fuller, as one would indeed expect. The important difference lies in the value of a_c, which for these parameters is around 1.2. [With 100 grid points, it is about 1.218, but this value is very grid size dependent. This seems to be due to the fact that for the accumulation rate shown in Figure 7, a basal melting zone ($g_0=0$) first appears at the margin, and this extra singularity confuses the numerical results.] It does seem that for more realistic balance functions, a value of a of 2 will cause basal melting (if the temperature reaches the melting point in the first place). Thus, it seems likely that the inception of large scale melting is feasible for realistic values of a. From (4.69), we have

$$\tau \sim x^{1/n}, \qquad g_\infty \sim O(1) \quad \text{as} \quad x \to 0, \tag{4.76}$$

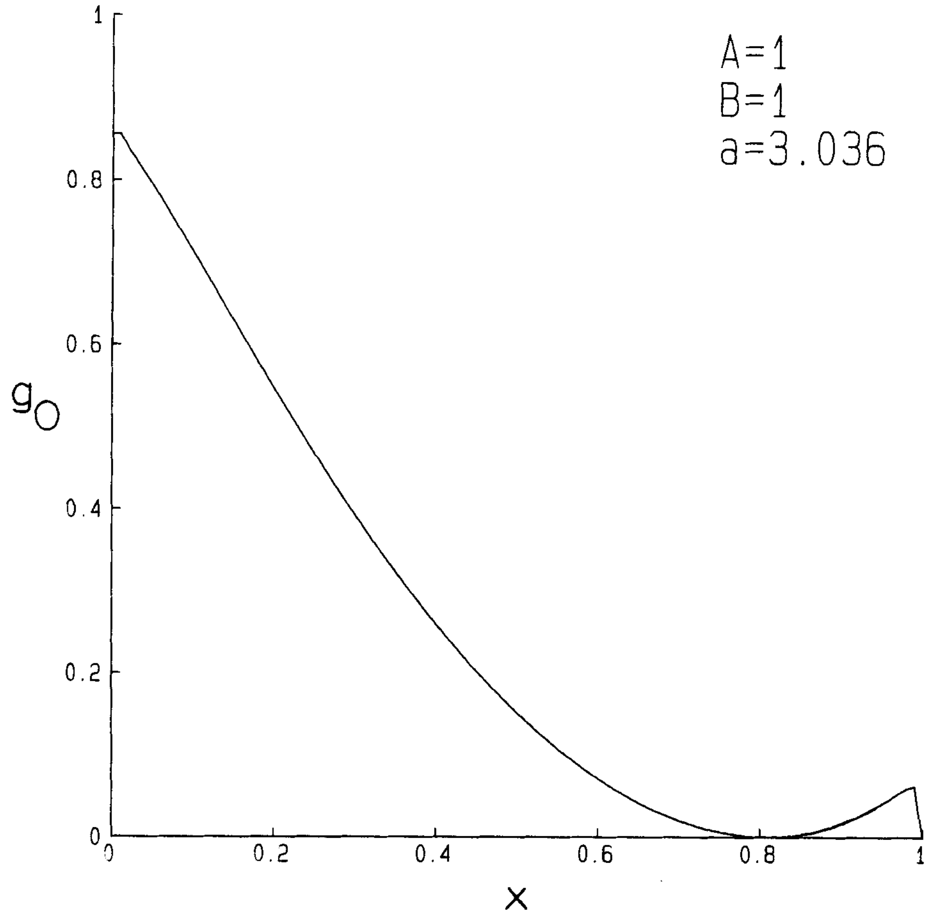

Figure 6 g_0 versus x at $A=1=B$, $a=3.036$; onset of basal melting. (The jump at $x=1$ is a real effect.)

whereas

$$\eta \sim (x_m - x)^{(n+2)/2(n+1)},$$

$$\tau \sim (x_m - x)^{1/(n+1)},$$

$$u, g_\infty \sim (x_m - x)^{n/2(n+1)} \quad \text{as} \quad x \to x_m, \tag{4.77}$$

and therefore $g_0 > 0$ for sufficiently small a. The existence of a critical a_c is therefore a general feature. Notice also that the basal ice can only stay temperate if $g_0 < \beta^{1/2}\Gamma$ (so that ice is melted there). Following from (4.77), this is always true at the margin, and is true at the divide provided $g_0 \approx g_\infty \approx \eta_0 < \beta^{1/2}\Gamma$ there, which is, however, unlikely. The more likely circumstance is that the base is partly freezing and partly at the melting point.

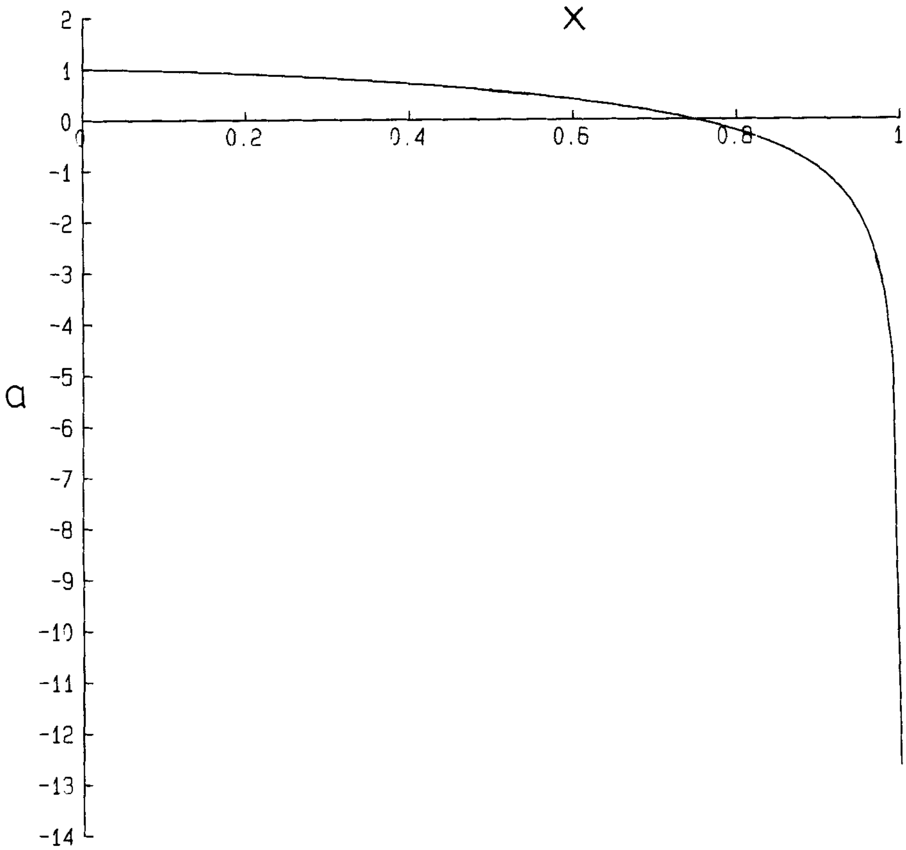

Figure 7 Accumulation rate given by (4.75), $A=4/3$, $B=8/3$.

5. DISCUSSION

5.1 Choice of depth scale

The depth scale is chosen from the relations (2.7)–(2.11), whence we find

$$\nu d^{2(n+1)} = l^{n+1}[a]/[A](\rho g)^n, \tag{5.1}$$

with ν given by (4.30) for the case of a cold base:

$$\nu \sim e^{\gamma G_0}/\gamma, \qquad G_0 \sim -1 + (\pi\beta/2)^{1/2}\Gamma^{3/2}, \tag{5.2}$$

or by (4.61), for a base at the melting point:

$$\nu \sim \beta^{1/2}/\gamma. \tag{5.3}$$

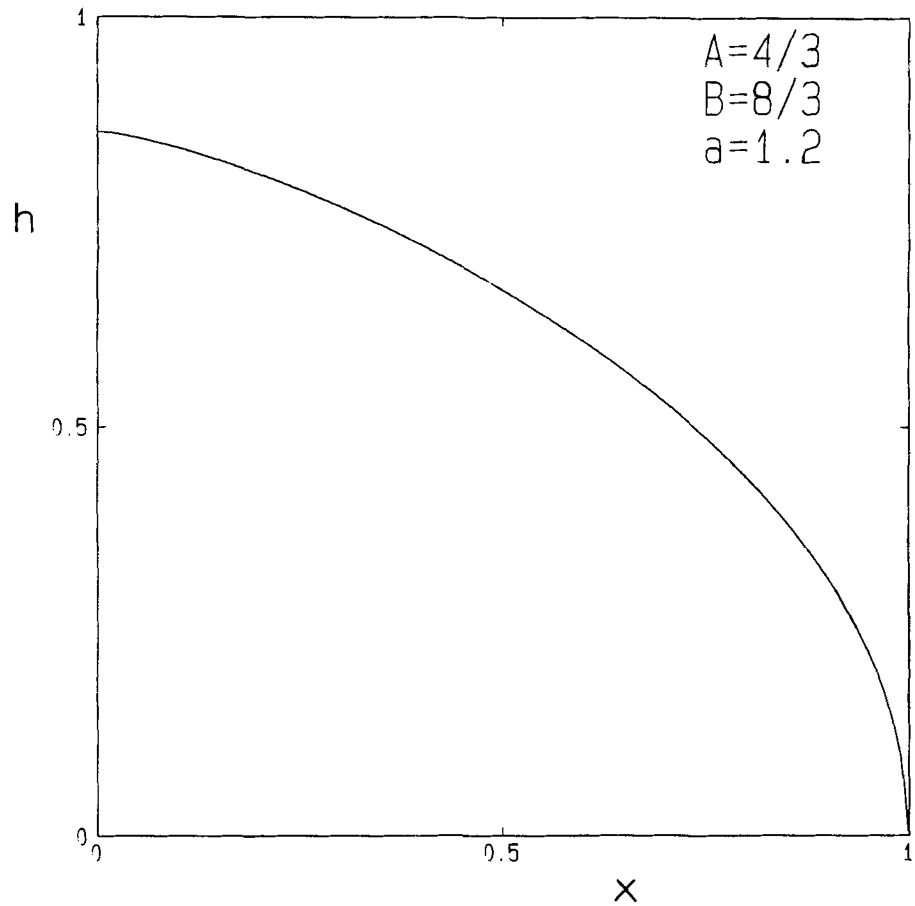

Figure 8 η versus x for $A=4/3$, $B=8/3$, $a=1.2$ (with 100 grid points).

Since $\beta \propto 1/d$, $\Gamma \propto d$, d is uniquely determined in both cases. In practice, the dependence on ν is slight, so that roughly

$$d \approx [l^{n+1}[a]/[A](\rho g)^n]^{1/2(n+1)}. \tag{5.4}$$

If we choose (Paterson, 1981) $l=10^6$ m (10^3 km), $[a]=3\times 10^{-9}\,\mathrm{m\,s^{-1}}$ ($10\,\mathrm{cm\,y^{-1}}$), $[A]=0.65\times 10^{-(5n+7)}\,\mathrm{Pa^{-n}\,s^{-1}}$ ($0.2\,\mathrm{bar^{-n}\,y^{-1}}$), $\rho g=10^4\,\mathrm{Pa\,m^{-1}}$ ($0.1\,\mathrm{bar\,m^{-1}}$), $n=3$, then we find $d\sim 1600$ m (1.6 km). If we take $\nu=10^{-2}$, a reasonable value, then $d\sim 3000$ m (3 km). This seems to accord well with observation.

5.2 Divide curvature

In Section 3, we found that for an isothermal ice sheet, the curvature at a divide was

$O(\varepsilon^{(1/n)-1})$, based on the length scale ($\sim\varepsilon$) over which the full Stokes equations applied. For the non-isothermal ice sheet, the corresponding analysis is less clear. The mechanical boundary layer is where, using (2.12), $x\sim\varepsilon(ve^{-\gamma T})^{1/2}$, and we note that $\varepsilon\tau_1/\tau_2\sim x/\varepsilon$; if we use $T=-1$, then we have that near the surface, longitudinal stresses are important where $x\sim\varepsilon(ve^{\gamma})^{1/2}\gg\varepsilon$, using either definition of v. On the other hand, if we use the basal value of T, then we find $x\sim\varepsilon/\gamma^{1/2}$ ($T_0<0$) or $x\sim\varepsilon\beta^{1/4}/\gamma^{1/2}$ ($T_0=0$), and in either case $x\ll\varepsilon$. Thus in general, rheological and mechanical boundary layers are distinct, and the analysis of the local stress structure is non-trivial. Since is the basal ice which controls deformation, we might hazard the guess that the basal value of T is the relevant one, and the divide curvature is (at least) $O(\gamma^{1/2}\varepsilon^{(1/n)-1})$.

5.3 Model uses

Although the reduction in complexity achieved by use of the limits $\beta\to 0$, $\gamma\to\infty$ is attractive, the approximations which result are only likely to be qualitative; nevertheless, they may be useful, particularly when used in combination with numerical computations. For example, (4.29) gives the surface profile as a function of the basal temperature; but it could also be used to compute the basal temperature from the surface profile. If such a "prediction" was found to be accurate for direct numerical computations, it could (eventually) provide a credible way of inferring basal temperatures from observations of surface profiles. In such ways, the rough predictions here can be used as calibration tools.

5.4 Surface profiles

The predicted profiles given by (4.45), or from (4.69), are rather opaque, and their dependence on the balance function $s(x)$ is obscure. Since $s(x)$ can at best be estimated, it is a reasonable procedure to find forms of $s(x)$ which give particularly simple equations for $\eta(x)$. A particular class of profiles which appear quite realistic are the "hyper-ellipses"

$$\left(\frac{\eta}{\eta_0}\right)^B + x^A = 1. \tag{5.5}$$

where for a nonlinear isothermal flow law (Vialov, 1958; Paterson, 1981; Hutter, 1983), we would have $B=2+2/n$, $A=1+1/n$; for a plastic flow law ($n\to\infty$), $B=2$, $A=1$. The choice $B=8/3$, $A=4/3$ fits the Antarctic profile inland from Mirny quite well, although other choices are also quite good.

The approximation $\eta=\eta_0 s(x)/x$ of (4.45) is consistent with (5.5), provided

$$s(x)=x[1-x^A]^{1/B}, \tag{5.6}$$

which is a perfectly feasible form of the balance function. In particular, for $A>1$, $B>1$, accumulation becomes almost uniform over the interior, and ablation is concentrated at the margin. The warm-based approximation in the form (4.73), for

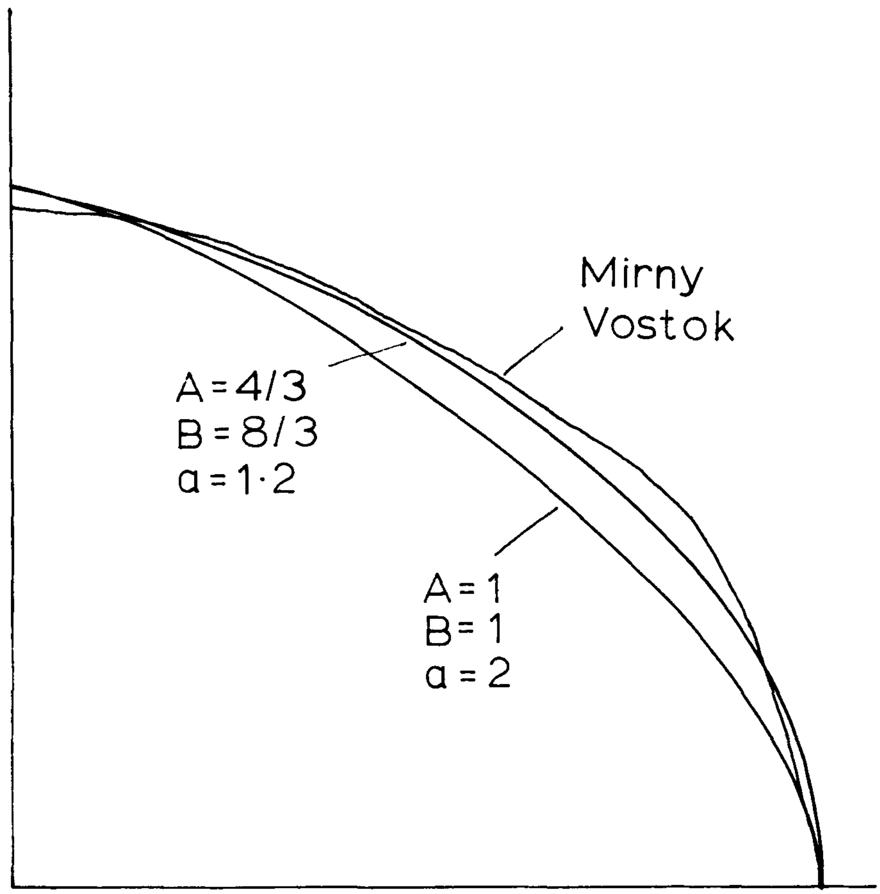

Figure 9 Comparison of Vostok-Mirny profile with the warm based model results with $A=1$, $B=1$, and $A=4/3$, $B=8/3$.

small a, can also be represented in the form (5.5) with $B=2$, provided

$$s=Cx^p(1-x^A)^{1/B}, \tag{5.7}$$

where $C\propto K^{-2/3}$, $p=[2(A-1)n-1]/3$. Choosing $C=1$ and $p=1$ gives $A=1+2/n=5/3$ for $n=3$, and a plausible result. In Figure 4 we showed the accumulation rate corresponding to $A=4/3$, $B=8/3$. We have not plotted the rate for $A=5/3$, $B=2$, as it is very similar, and the function $\eta(x)$ is very similar to Figure 8. Note that Figure 8 is for a warm-based ice sheet. In Figure 9 we compare (crudely) the results in Figures 4 and 8 (for a *warm-based* ice sheet) with the Vostok-Mirny profile in Antarctica [redrawn from Paterson (1980)]. The comparison is crude, but shows that the warm-based (as well as cold-based) models can fit data satisfactorily, with the assumption of realistic accumulation rates.

5.5 *Temperature profile*

For a freezing base, a uniformly approximate temperature profile is approximately given by (4.46):

$$T = T_0(xy/\Gamma) + (\pi\beta/2)^{1/2}\Gamma^{3/2}\,\mathrm{erfc}[y/2\Gamma)^{1/2}]. \tag{5.8}$$

For a warm base, the corresponding result is

$$T = T_0[u_0(x)y] + \mathrm{erfc}\left\{u_0(x)y/2\left[\beta\int_0^x u_0(x)\,dx\right]^{1/2}\right\}. \tag{5.9}$$

Both exemplify the typical observation (Paterson, 1981) of a relatively uniform upper temperature which increases smoothly in the lower parts of the ice sheet. Occasionally an inversion occurs, such as at Mirny Station in Antarctica. This is caused by the upper temperature $T_0(\psi)$ which depends on the temperature along the stream line. When T_0 decreases as the divide is approached, this temperature difference is advected downstream and revealed as an inverted temperature gradient. Downstream of the maximum value of s (i.e., in the ablation area) the "outer" solution $T_0(\psi)$ cannot satisfy any prescribed surface temperature, and thus a boundary layer at the surface is also required. This gives rise to secondary inversion, as shown by the temperature profile of White Glacier (Paterson, 1981, p. 210).

5.6 *Surface velocity*

For a freezing base, we have the simple prediction $u \propto x$. The particular choice of s given by (5.7) gives $u \propto x^p$, so that also $u \propto x$ if $p = 1$. A constant strain rate is a simple prediction that could be used for comparison in numerical tests. Flow of this form (called extending flow) was studied by Nye (1957).

5.7 *Numerical computations*

Hutter *et al.* (1986) describe a numerical method for solving the plane, steady, non-isothermal problem based on solving for u, v, T in the reduced model. Their method is to shoot from a divide using η_0 as an unknown, but they are unable to obtain a solution for a no-slip boundary condition; in particular, their method requires both non-zero velocity and finite viscosity at the divide. The suggestion of the present paper is that the local divide solution should be taken from the local (quasi) similarity solution of (4.7) (neglecting viscous heating). For given η_0, this provides an accurate start-up solution). More generally, shooting from the divide is unlikely to be a good method for solving time dependent problems, where a more direct method [e.g.: time step η from t_j to t_{j+1} using $(2.25)_2$; Crank-Nicolson or other implicit method on $(2.24)_2$ to get T at t_{j+1}; quadrature of $(2.24)_1$ to get ψ at t_{j+1}] might be useful for both stationary and time-dependent problems. Schemes of this sort have been successfully implemented by, for example, Huybrechts and Oerlemans (1988).

6. CONCLUSIONS

A simplified scaling procedure for ice sheet models shows that the aspect ratio ε is determined self consistently, and that only three essential parameters enter the problem. These are α, the viscous dissipation number (Brinkman times Peclet number); β, the thermal conduction number (inverse Peclet number); and γ, the viscosity number. Typical values of these are $\alpha \sim O(1)$, $\beta \sim 10^{-1}$, $\gamma \sim 10$.

Isothermal analysis shows that the reduced model ($\varepsilon \to 0$) has weak singularities at divides and at margins, but the reduced model itself is still valid. A local analysis is necessary to show that curvature is $O(\varepsilon^{(1/n)-1})$ at divides, while the slope is $O(1/\varepsilon)$ at a margin, but these are glosses on the basic solution, and a successful numerical method should be able to incorporate these singularities without having to avoid them artificially. In addition, we have shown how (isothermal) ice sheets respond to small perturbations, i.e. diffusively, and how the perturbations can be made uniform. This local analysis will also have bearing on the implementation of numerical methods.

Our approximate analysis suggests that geothermal heating contributes significant warming at the base of large ice sheets, and may be enough to raise the base to melting point, even under divides. In any case, viscous heating should act as a significant basal heater, so that much of the base of large ice sheets could be at the melting point. Once the base reaches the melting point, the ice above remains cold. Viscous heating acts to reduce the heat flux supplied to this cold ice, and hence warms it. When the heat flux at the bed reaches zero, a layer of temperate ice will form, and this may occur in practice.

If such temperate layers form in the interior of an ice sheet, then viscous heating causes meltwater production, and moreover, this is a runaway effect, since the viscous heating $A\tau^{n+1}$ increases with water content w (τ being controlled by the depth, but A increasing with w, see Duval, 1977). Massive release of meltwater suggests (as is observed) the existence of subglacial lakes (which may provide a clue to the existence of these temperate regions). It does not necessarily mean massive creep instability and surging, for if a basal zone of temperate ice surmounting a subglacial lake does exist, it simply acts as an inviscid bubble overlying a frictionless boundary; this requires only that the surface stress be zero, but the ice can be dammed by colder ice on either side. On the other hand, such isolated temperate regions could be unstable to the formation of finger-like instabilities, whereby a protruding finger of temperate ice causes increased water production, and hence increased flow, which enhances water production further and enables the finger to propagate further. The existence of outlet ice streams is suggestive of such speculation. There are many interesting questions to be examined.

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