Creep closure of channels in deforming subglacial till

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We examine theoretically the creep closure of subglacial tunnels cut into basal till, generalizing Nye's classical analysis of tunnel closure in glacier ice to rheologies in which the creep rate depends on effective pressure (the difference between total pressure and pore-water pressure). The solutions depend critically on a dimensionless permeability parameter. For the appealingly simple Boulton–Hindmarsh rheology in which strain rate depends on powers of applied stress and effective pressure, solutions to the closure problem may not exist; this is related to the existence of a 'failed' zone next to the channel, where piping occurs, and also to a non-physical degeneracy of the assumed rheology, whereby the viscosity is indeterminate at zero effective pressure. Consideration of the failed zone allows solutions to be obtained and shows that the closure characteristics of high permeability tills and low permeability tills are very different.

1. Introduction

The 'classical' theory of glacier sliding (Weertman 1957; Lliboutry 1968; Nye 1969; Kamb 1970) considers the flow of ice over an undeformable bedrock, and aims to derive a relation between the basal velocity u_b and the basal shear stress τ_b. In seeking this relation, one finds (Lliboutry 1968) that a third variable enters the equation: the basal water pressure. Therefore, to specify fully a realistic boundary condition for large-scale problems of ice dynamics, one must describe the subglacial drainage hydraulics, with a view to determining the basal water pressure.

The 'classical' theory of subglacial drainage was expounded by Röthlisberger (1972), who suggested that basal water would flow (arterially) in channels (much like river networks) which were cut into the ice. The flow in such 'Röthlisberger channels' is at a lower pressure (p_c) than the ice overburden pressure (p_i), and the resultant 'effective' pressure $N = p_i - p_c$ tends to cause closure of the channels by means of viscous creep of the ice. The flow is maintained, however, by a compensating melt-back of the channel walls, due to the heat released by viscous dissipation of the channel flow. It is the dynamics of this process which determines the 'effective' pressure N, and thus (in principle) the sliding law. The theory has been used with some success by Bindschadler (1983) for sliding of the Variegated Glacier.

These classical theories have the appeal of simplicity, but it is a well-known fact that glaciers erode their beds, and the consequent erosion products (ranging from large boulders to clay-sized particles) form a layer at the base of a glacier. There is direct observation of such subglacial 'till' layers (Engelhardt *et al.* 1978; Clarke *et al.* 1984), and similar layers have been inferred to exist at the base of ice stream B in the

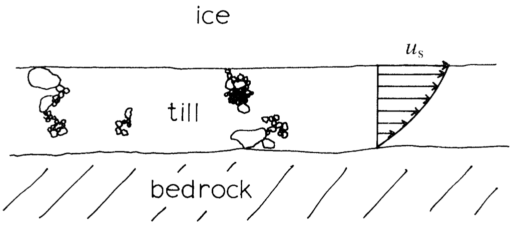

Figure 1. Glacier ice overlying a layer of till that deforms in response to shear stress imposed by the ice. The sliding velocity u_s is here defined as the velocity of the ice relative to the bedrock, which is non-deformable.

Ross Ice Shelf (Blankenship *et al.* 1986, 1987). Moreover, such till layers can deform, and the resultant displacement (integrated over the entire till thickness) may form a significant part of the apparent sliding motion of the overlying ice, particularly under some ice streams.

Thus we are faced with the problem of determining a sliding law for ice over a deformable till. Ultimately, the thickness of the till itself should be a dynamic variable (being produced by erosion of the bedrock, for example), but as a first approach, one would write a sliding law in terms of simple shearing of a till layer of thickness h and viscosity η: $\tau_b = \eta u_b/h$. Here, however, we have the same problem as before: the viscosity of till is likely to depend on water pressure p_w through the *effective pressure* $p_e = p_i - p_w$, since a decrease in effective pressure allows increased mobility of the till constituents. Therefore, we need to determine water pressures in the till, and as before, this involves a description of the subglacial drainage pattern.

A model along the lines of Röthlisberger's will form the basis of a subsequent paper. One possibility is that drainage occurs through channels incised in the ice, as before. However, because the till is itself both erodible and deformable, another possibility is that water will flow in channels cut into the till. If the channel pressure is lower than the overburden (as we expect), there will be a consequent creep closure, just as for Röthlisberger channels. The dynamical equilibrium of these channels is then controlled by erosion of the channel bed, which must balance the inward creep of the till. Which of the two possibilities occurs will depend on the relative viscosities and erosion/melt rates, and will be considered in a subsequent paper. Our purpose in the present paper is to consider only the problem of determining the creep closure of channels cut into till. This is the analogue of the problem considered by Nye (1953), that is, the closure of a conduit in ice. We shall find that the extension to a creeping two-phase medium (water and solid particles) is not as straightforward as was assumed by Boulton & Hindmarsh (1987), in their discussion of tunnel-valley formation. The problem analysed in this paper may also have applications to problems of bore hole closure in sedimentary basins.

Just as in Nye (1953), the particular problem we choose to study is that of the closure of a cylindrical void at pressure p_c in an infinite, porous, viscous medium. The closure is driven by an external pressure P_∞, where we assume $P_\infty > p_c$. The hope is that this idealized problem may give the correct qualitative dynamics for the closure of real till channels. In the next section, we discuss the relevant mechanical properties of till, and then turn to the particular problem we are concerned with in §3. Conclusions follow in §4.

2. Rheology, permeability and compressibility of subglacial till

(*a*) *Rheology*

There is very little data on the creep behaviour of subglacial till. Blake & Clarke (1988) measured an apparent viscosity of 10^{10} Pa s at Trapridge Glacier by *in situ* measurements of basal till deformation, and Boulton & Hindmarsh (1987) used similar measurements at Breidamerkurjokull to infer a flow law of the form

$$\dot{\epsilon} = A_{\mathrm{v}}\, \tau^{a} p_{\mathrm{e}}^{-b}, \tag{2.1}$$

where τ is shear stress, $\dot{\epsilon}$ is shear strain rate, $p_{\mathrm{e}} = p_{\mathrm{i}} - p_{\mathrm{w}}$ is the effective pressure, p_{i} is the ice overburden pressure, and p_{w} is the pore-water pressure in the till. Their fitted values for the parameters A_{v}, a and b based on seven data points were

$$A_{\mathrm{v}} = 3.99\ \mathrm{bar}^{b-a}\ \mathrm{a}^{-1}, \quad a = 1.33, \quad b = 1.8, \tag{2.2}$$

where τ and p_{e} are measured in bars†. It can be argued that a law such as (2.1) can hardly be justified from such a small sample; however, (2.1) is the simplest nonlinear flow law which incorporates the idea that the viscosity decreases as effective pressure decreases. It seems to us that this qualitative behaviour is the minimal requirement for a realistic flow law, and so we consider (2.1) as the simplest feasible relation, consistent with our physical expectations. For example, Alley *et al.* (1987) treat the till as a linear viscous material, whereas Clarke (1987) uses a viscoplastic model, with the viscosity taken as a function of porosity n, itself a function of τ and p_{e}. Kamb (1991) has recently argued, on the basis of both experiments on till and soil-mechanics experience, that (2.1) may be feasible but only if $a \approx b \gg 1$. If this is correct, then, as Kamb points out, the flow law (2.1) is very nearly equivalent to a perfectly plastic failure criterion, with a yield strength τ_{f} given by

$$\tau_{\mathrm{f}} = c + \psi p_{\mathrm{e}}. \tag{2.3}$$

An unrealistic aspect of (2.1) is when p_{e} tends to zero. (2.1) predicts an infinite shear rate, consistent with Roscoe's (1952) inference of viscosity as a function of porosity. The implication is that when $p_{\mathrm{e}} \to 0$, then arbitrary large dilation can occur to allow the till grains to move freely past each other. For a confined flow, where a confining pressure represses excess dilatancy, this will not be realistic; one approach to resolving this problem might be explicitly to consider a 'dilatancy law' for the till, i.e.

$$n = n(\dot{\epsilon}, p_{\mathrm{e}}), \tag{2.4}$$

with $\partial n/\partial \dot{\epsilon} > 0$ (dilatant) and $\partial n/\partial p_{e} < 0$. However, in this paper we will adopt a much simpler assumption, which, in the absence of useful data, is perhaps as useful, and which is also physically plausible.

(*b*) *Permeability and compressibility*

Till is a permeable medium, and therefore we suppose that Darcy's law may be applied:

$$\boldsymbol{q} = -(k/\mu_{\mathrm{w}})[\nabla p_{\mathrm{w}} + \rho_{\mathrm{w}} g \hat{\boldsymbol{z}}], \tag{2.5}$$

where $\boldsymbol{q}$ is the water flux, μ_{w} the water viscosity, p_{w} the pore water pressure, ρ_{w} the density of water, g the acceleration of gravity, and $\hat{\boldsymbol{z}}$ is a unit vector directed vertically upwards. The permeability is generally an increasing function of n, e.g. the Kozeny–Carman relation

$$k \propto n^{3}/(1-n)^{2} \tag{2.6}$$

† 1 bar = 10^{5} Pa.

(Clarke 1987), or the empirical

$$k \propto \exp(\gamma n) \tag{2.7}$$

(Boulton & Hindmarsh 1987). In any event, measurements indicate a range for k of 10^{-13}–10^{-19} m^2, depending on the clay fraction (Freeze & Cherry 1979; Boulton *et al.* 1974), with a typical variation of one or two orders of magnitude in a given till as effective pressure increases. Permeability also varies as the till is deformed, i.e. $k = k(\dot{\epsilon}, p_e)$ (Clarke & Murray 1991). The form of these constitutive relations obviously has a bearing on the analysis of water flow, but we shall make the simpler assumption that the precise functional forms are not important.

3. Tunnel closure with power law rheology

We now examine the creep closure of a cylindrical void in an infinite medium (till) with a nonlinear viscous rheology, which is saturated with water (figure 3). Although such channels are unlikely to exist subglacially, results of this analysis should give a reasonable estimate of the creep rate of till into a sediment-floored channel, as depicted in figure 2, as long as some characteristic dimension of the channel, such as its depth, is small compared to the total till thickness. This methodology is the same as that of Röthlisberger (1972), who applied Nye's (1953) results for closure of cylindrical tunnels in ice to study the hydraulics of semicircular tunnels at the glacier bed.

To proceed, we must generalize the shear deformation law (2.1) to the case of more general deformations of a compressible two-phase medium. We make the simplest assumption, which is that the deviatoric stress tensor τ_{ij} is related to the strain rate tensor $\dot{\epsilon}_{ij}$ by the relation

$$\dot{\epsilon}_{ij} - \tfrac{1}{3}(\nabla \cdot \boldsymbol{u})\,\delta_{ij} = \tau_{ij}/2\eta, \tag{3.1}$$

where $\dot{\epsilon}_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, and the viscosity η for the power law (2.1) is defined by

$$\eta^{-1} = A_v \tau^{a-1} p_e^{-b}, \tag{3.2}$$

and where τ is the second stress invariant ($2\tau^2 = \tau_{ij}\tau_{ij}$). It is natural in considering deformation of the till, to take $\boldsymbol{u}$ in (3.1) to be the (averaged) velocity of the till particles and $\dot{\epsilon}_{ij}$ as the corresponding strain rate tensor; we may take τ_{ij} as the stress applied to the solids.

In what follows, we neglect gravitational effects. This should be an accurate assumption, provided $\rho_w g d \ll N$, where d is a typical length scale, and N a typical differential (effective) pressure. For example, if $d = 1$ m, $g = 10$ m s^{-2}, $\rho_w = 10^3$ kg m^{-3}, then $\rho_w g d \approx 10^4$ Pa, and is negligible for $N > 10^5$ Pa. We do not consider that gravity will have a significant part to play in the *closure* dynamics.

Radial contraction of a cylindrical hole induces axial strain rates or stresses, or both. We shall choose a flow in which there is zero axial deviatoric stress, although in the event this assumption has little bearing on the results, at least provided the till permeability is quite 'low'.

More importantly, we neglect the effect of axial shear stress. Though potentially important, this may in fact be quite a reasonable approximation. It will be so if $\tau_b \ll \Delta p$, where τ_b is the basal shear stress, and Δp is the difference between the overburden pressure and the channel pressure. For glaciers, values of $\tau_b \approx 10^5$ Pa are common, but we find that then $\Delta p > 10^6$ Pa, typically, for steady channel flow. For ice sheets, we will find values of $\Delta p \approx 10^5$ Pa, but then shear stresses are likely to be

Figure 2

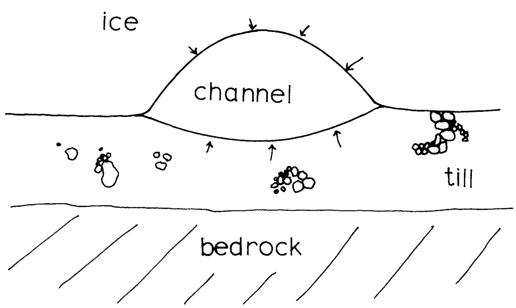

Figure 3

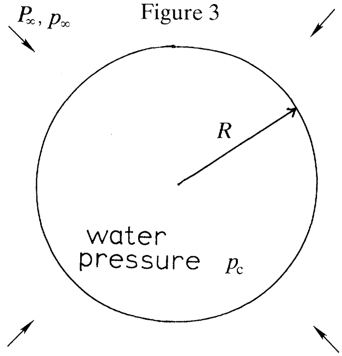

Figure 2. Schematic drawing of a channel at the bed of a till-floored glacier. In general, the channel will be cut into both ice and till. Arrows indicate creep of ice and till into the channel. Melting of the ice roof and erosion of the sediment flow must balance this inward creep if the channel is to remain open.

Figure 3. Idealized picture of a channel through till. Water pressure p_c acts inside the channel; total pressure P_∞ and water pressure p_∞ are applied far from the channel.

lower, e.g. *ca.* 10^4 Pa if 1 km of ice has a surface slope of 10^{-3}. Thus the neglect of shear stress should represent, if only crudely, a useful approximation (see Humphrey 1987).

Let $\boldsymbol{v}$ be the water velocity relative to the till matrix. Darcy's law may then be written (neglecting gravity) as

$$n\boldsymbol{v} = -(k/\mu_w)\nabla p_w. \tag{3.3}$$

Conservation of mass for each phase gives (cf. Fowler 1985)

$$-\partial n/\partial t + \nabla\cdot[(1-n)\boldsymbol{u}] = 0, \quad \partial n/\partial t + \nabla\cdot[n(\boldsymbol{u}+\boldsymbol{v})] = 0, \tag{3.4}$$

where we ignore erosional processes of comminution, and losses due to washing out of fines.

In the case at hand, we use cylindrical polar coordinates, and we seek a solution in which the till velocity is $(u, 0, w)$, where u is the radial component and w is the axial component, and the water velocity is purely radial, of magnitude v. In other words, all water within the till is evacuated by first draining to a channel, thence along the channel towards the glacier terminus. This is in fact a rather restrictive assumption. Beneath an actual glacier, water within the basal till might instead tend to drain downward to relatively permeable strata underlying the till (cf. Boulton & Hindmarsh 1987; Clarke 1987). The present analysis is not applicable in that case, on which we will remark in §4. We suppose that $u = u(r,t)$, $v = v(r,t)$, but $w = w(r,z,t)$. Then

$$\dot{e}_{rr} = \partial u/\partial r, \quad \dot{e}_{\theta\theta} = u/r, \quad \dot{e}_{zz} = \partial w/\partial z, \tag{3.5}$$

and all other components of $\dot{e}_{ij}$ vanish. We seek a solution in which

$$\tau_{zz} = 0, \tag{3.6}$$

and thus from (3.1), $$\dot{e}_{zz} = \partial w/\partial z = \tfrac{1}{3}[\partial u/\partial r + u/r + \partial w/\partial z], \tag{3.7}$$

whence $$w = \tfrac{1}{2}z(\partial u/\partial r + u/r). \tag{3.8}$$

It follows that $$\nabla\cdot\boldsymbol{u} = \tfrac{3}{2}(\partial u/\partial r + u/r), \tag{3.9}$$

and so $$\dot{e}_{rr} - \tfrac{1}{3}\nabla\cdot\boldsymbol{u} = \tfrac{1}{2}(\partial u/\partial r - u/r) = -[\dot{e}_{\theta\theta} - \tfrac{1}{3}\nabla\cdot\boldsymbol{u}]; \tag{3.10}$$

hence $$\tau_{rr} = -\tau_{\theta\theta} = \tau, \tag{3.11}$$

where τ is the second stress invariant (if $\tau_{rr} > 0$) since $2\tau^2 = \tau_{rr}^2 + \tau_{\theta\theta}^2$. Thus the constitutive relation between stress and strain rates is, using (3.2),

$$\partial u/\partial r - u/r = A_v \tau^a p_e^{-b}. \tag{3.12}$$

Consider now the momentum equation for the mixture, which (neglecting inertial terms) may be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) - \frac{\tau_{\theta\theta}}{r} - \frac{\partial P}{\partial r} = 0, \tag{3.13}$$

where P is the total pressure. The effective pressure in (3.2) is defined by

$$p_e = P - p_w. \tag{3.14}$$

Suppose the void is at $r \leqslant R$, thus the till occupies the region $r > R$ (figure 3). In general, if the channel contains water at pressure p_c, then denoting

$$P_\infty = \lim_{r\to\infty} P(r),$$

we expect closure of the channel if $P_\infty > p_c$; hence $R = R(t)$, and the kinematic condition at the channel wall (neglecting spallation) is

$$\dot{R} = u \quad \text{at} \quad r = R, \tag{3.15}$$

where $\dot{R} = \mathrm{d}R/\mathrm{d}t$. Other boundary conditions for the flow are a normal force balance at $r = R$, and as $r \to \infty$; thus

$$-P + \tau_{rr} = -p_c \text{ at } r = R, \quad -P + \tau_{rr} \to -P_\infty \text{ as } r = \infty. \tag{3.16}$$

In addition, we expect the water pressure to be continuous, whence

$$p_w = p_c \text{ at } r = R, \quad p_w \to p_\infty \text{ as } r \to \infty. \tag{3.17}$$

The effective pressure in till far from the channel is thus $P_\infty - p_\infty$.

These equations represent a significant complication over Nye's (1953) closure problem for ice. The equations (3.3) and (3.4) can be written

$$\left.\begin{aligned} nv = -\frac{k}{\mu_w}\frac{\partial p_w}{\partial r}, \quad -\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}[(1-n)\,ru] + \frac{\partial}{\partial z}[(1-n)\,w] = 0, \\ \frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}[nr(u+v)] + \frac{\partial}{\partial z}(nw) = 0; \end{aligned}\right\} \tag{3.18}$$

supposing that $n = n(r,t)$, and using (3.8), we find

$$u = -C/r - \tfrac{2}{3}nv, \tag{3.19}$$

and thus

$$\frac{\partial n}{\partial t} + u\frac{\partial n}{\partial r} = -\frac{(1-n)}{r}\frac{\partial}{\partial r}(nrv). \tag{3.20}$$

In effect, this is a diffusion equation for p_e (or n), on account of the dilatancy relation (2.7) between n and p_e $(= P - p_w)$, with $\partial n/\partial p_e < 0$. For purposes of illustration, let us suppose $n = n(p_e)$, and write $|n'(p_e)| = \beta_v$, which is essentially a measure of sediment compressibility. (The normal compression index C_c would be defined in soil mechanics practice as $-p_e n'(p_e)/(1-n)^2$, hence $\beta_v = (1-n)^2 C_c/p_e$.) We non-dimensionalize the equations by writing

$$\left.\begin{aligned} P = p_\infty + NP^*, \quad \tau = N\tau^*, \quad p_w = p_\infty + Np_w^*, \quad r = R_0 r^*, \\ R = R_0 R^*, \quad [\eta] = (A_v N^{a-b-1})^{-1}, \quad t = t_v t^*, \quad u = (R_0/t_v)\,u^*, \\ v = (R_0/t_v)\,v^*, \quad p_e = Np_e^*, \quad C = (R_0^2/t_v)\,C^*, \quad t_v = [\eta]/N, \end{aligned}\right\} \tag{3.21}$$

where R_0 is the initial channel radius and now $N = P_\infty - p_\infty$. The model equations become

$$\left.\begin{aligned} &u = -C/r - \tfrac{2}{3}nv, \quad \frac{\partial u}{\partial r} - \frac{u}{r} = \tau^a p_e^{-b}, \quad -\frac{\partial P}{\partial r} + \frac{\partial \tau}{\partial r} + \frac{2\tau}{r} = 0, \\ &nv = -\kappa \Lambda\, \partial p_w/\partial r, \quad p_e = P - p_w, \quad -\frac{\partial p_e}{\partial t} - u\frac{\partial p_e}{\partial r} = \Lambda(1-n)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_w}{\partial r}\right), \end{aligned}\right\} \tag{3.22}$$

where the dimensionless parameters κ and Λ are defined as

$$\kappa = \beta_v N, \quad \Lambda = t_v/t_D = k[\eta]/\beta_v R_0^2 \mu_w N, \tag{3.23}$$

and t_D is the Darcy timescale,

$$t_D = \beta_v R_0^2 \mu_w / k. \tag{3.24}$$

In writing $(3.22)_6$, we have assumed that β_v is constant, but this will not affect our principal conclusions. In addition, we have omitted the asterisks for convenience.

The boundary conditions for (3.22) are that

$$\left.\begin{aligned} \text{as } r \to \infty, \quad & -P + \tau \to -1, \\ & p_w - 0, \\ \text{on } r = R, \quad & \dot{R} = u, \\ & -P + \tau = -\Delta, \\ & p_w = \Delta, \end{aligned}\right\} \tag{3.25}$$

where

$$\Delta = (p_c - p_\infty)/N \tag{3.26}$$

is the scaled excess channel pressure.

Values of compression index C_c are usually quite low, in the range 0.01–0.1 (Lambe & Whitman 1979). Since $\beta_v \sim C_c/N$, this suggests that a plausible magnitude of κ is

$$\kappa \sim 0.1. \tag{3.27}$$

To estimate Λ, take a range of permeability from $k \approx 10^{-13}$ m^2 (coarse gravelly till) to $k \approx 10^{-19}$ m^2 (clay-rich till), $[\eta] \approx 10^{10}$ Pa s (a typical inferred value), $\kappa = 0.1$, $R = 1$ m, $\mu_w = 10^{-3}$ kg m^{-1} s^{-1}. We find that

$$10^{-5} < \Lambda < 10. \tag{3.28}$$

Evidently, Λ can be large or small, and its size depends primarily on the till permeability. Since our goal is to obtain parameterizable information which can be incorporated into subglacial drainage theories, we will choose to study solutions using simplifications based on asymptotic limits. In this case, two approximations may be appropriate, when Λ is large or small, and we now give approximations for these limits. We have also solved the problem numerically, and the results confirm the nature of the asymptotic behaviour. From our estimates, it would seem that $\Lambda \ll 1$ is the more common case, and we consider this first.

(*a*) *Small permeability: the case* $\Lambda \ll 1$

In general, the channel pressure p_c will not be equal to the far field pore pressure (just as for river flow). We may, for example, expect p_∞ to be some sort of long term average of p_c through the diffusion due to Darcy flow. If $\Lambda \ll 1$, then the limit

$\Lambda \to 0$ is a regular limit for the till deformation. However, it is a singular limit for the diffusion equation $(3.22)_6$, and a boundary layer is necessary to solve for p_w. We anticipate that $C = O(1)$; then the far field solution of $(3.22)_6$ is just

$$p_e \approx 1 \quad (r - R = O(1)). \tag{3.29}$$

Two cases now arise. Firstly, suppose that, as indicated in the problem formulation, the channel closes at a rate $-\dot{R}$. If we seek a boundary layer for p_w near $r = R$, we put

$$r = R(t) + \Lambda^{\frac{1}{2}} s, \tag{3.30}$$

and then (anticipating that P hardly changes in the boundary layer)

$$\frac{\partial p_w}{\partial t} + \frac{(u - \dot{R})}{\Lambda^{\frac{1}{2}}} \frac{\partial p_w}{\partial s} \approx (1-n) \frac{\partial^2 p_w}{\partial s^2}, \tag{3.31}$$

where we take n = constant for simplicity. Expanding $(3.22)_1$ near $r = R$, we have

$$u = -C/R + \Lambda^{\frac{1}{2}}[(C/R^2) s + \tfrac{2}{3}\kappa n\, \partial p_w/\partial s], \tag{3.32}$$

thus

$$\dot{R} = -C/R + \tfrac{2}{3}\kappa n \Lambda^{\frac{1}{2}} p_0', \quad p_0' = |\partial p_w/\partial s|_{s=0}, \tag{3.33}$$

and thus

$$(u - \dot{R})/\Lambda^{\frac{1}{2}} \approx (C/R^2) s + \tfrac{2}{3}\kappa n (\partial p_w/\partial s - p_0'). \tag{3.34}$$

Now for the case of closure, $C > 0$, and (3.31) does not have a solution with a boundary layer attached to $r = R$, which decays as $s \to \infty$. In fact, $u - \dot{R} > 0$ for large s, so that we expect a moving front to develop, which propagates into the till, and which separates two regions of constant, but different, effective pressures.

Actually, this case is of less physical interest, since we expect erosion of the walls to keep R essentially constant. In that case, we put

$$r = R + \Lambda s, \tag{3.35}$$

and approximately, ignoring transients,

$$\left[-\frac{C}{R} + \tfrac{2}{3}\kappa \frac{\partial p_w}{\partial s} \right] \frac{\partial p_w}{\partial s} \approx (1-n) \frac{\partial^2 p_w}{\partial s^2}, \tag{3.36}$$

which can be solved explicitly.

With $\partial p_w/\partial s|_{s=0} = p_0'$, the closure rate is then given by

$$-\dot{R} = -u = C/R - \tfrac{2}{3}\kappa p_0'. \tag{3.37}$$

A convenient alternative gives the (dimensional) strain rate, based on the dimensional cross-sectional area S, as

$$\tilde{C} = -\dot{S}/S = 2A_v N^{a-b} (\dot{R}/R), \tag{3.38}$$

with $\dot{R}$ and R still dimensionless. The outer approximations to $(3.22)_{2,3}$ are thus

$$2C/r^2 = \tau^a, \tag{3.39}$$

$$-\frac{\partial P}{\partial r} + \frac{\partial \tau}{\partial r} + \frac{2\tau}{r} = 0, \tag{3.40}$$

since $p_e = 1$. Now the boundary condition on $-P + \tau$ at $r = 1$ cannot necessarily be applied, since P and τ will generally change within the boundary layer. However, $(3.22)_3$ indicates that

$$\frac{\partial}{\partial r}(-P + \tau) = -\frac{2\tau}{r}, \tag{3.41}$$

so that the change in $-P+\tau$ through the boundary layer will be $O(\Delta)$, providing $\tau = O(1)$. It follows that the boundary condition $-P+\tau = -\Delta$ can be applied to the far-field stress. Integration is straightforward (Nye 1953), and we find that C is given by

$$C = \tfrac{1}{2}R^2((1-\Delta)/a)^a, \tag{3.42}$$

providing $\Delta < 1$, i.e. $P_\infty > p_c$ as we assume for closure. The wall stress τ_w, i.e. the value of τ at $r = R$, is

$$\tau_w = (1-\Delta)/a.$$

Thus across the pore pressure boundary layer, p_e satisfies

$$-\left[\frac{C}{R}+\tfrac{2}{3}\kappa\frac{\partial p_e}{\partial s}\right]\frac{\partial p_e}{\partial s} = (1-n)\frac{\partial^2 p_e}{\partial s^2}, \tag{3.43}$$

with

$$p_e = \tau_w \text{ on } s = 0, \quad p_e \to 1 \text{ as } s \to \infty, \tag{3.44}$$

and with p_0' defined by $-\partial p_e/\partial s$ at $s = 0$, $\dot{R}$ is given by (3.37). Explicit integration of (3.43) gives

$$\frac{2\kappa}{3}p_0' = \frac{C}{R}\left[1-\exp\left\{\frac{2\kappa(1-\tau_w)}{3(1-n)}\right\}\right], \tag{3.45}$$

whence (3.37) is

$$-\dot{R} = \frac{C}{R}\exp\left\{\frac{2\kappa(1-\tau_w)}{3(1-n)}\right\}, \tag{3.46}$$

and the dimensional closure rate $\tilde{C}$ from (3.38) is, using (3.42),

$$\tilde{C} = A_v N^{a-b}\tau_w^a \exp\left[2\kappa(1-\tau_w)/3(1-n)\right], \tag{3.47}$$

where $\tau_w = (1-\Delta)/a = (P_\infty - p_c)/\{a(P_\infty - p_\infty)\}$.

Notice that $\tilde{C}$ is a non-monotonic function of τ_w. Because $\kappa \ll 1$, however, the effect due to sediment compressibility is small, and

$$\tilde{C} \approx \{A_v/a^a\}(P_\infty - p_c)^a(P_\infty - p_\infty)^{-b}. \tag{3.48}$$

The above results are to be compared with Boulton & Hindmarsh's heuristic result $\tilde{C} \propto (P_\infty - p_\infty)^{a-b}$. The results are the same if $p_c = p_\infty$, but not otherwise.

(*b*) *Large permeability: the case $\Lambda \gg 1$*

The analysis in this case is less transparent, and was in fact finally only resolved by examining numerical solutions of the problem. We now sketch the asymptotic structure of the solutions, applicable specifically to the case $1 < a < b$ of interest here. The details are given in Appendix A.

We find that the pore pressure p_w decays to zero over a long range $r \sim \exp[O(\Lambda^{1/\delta})]$, while the effective pressure p_e and stress τ are very small for $r < r^* \sim \Lambda^{(\delta-1)/2\delta}$, increase sharply near r^*, and then in $r > r^*$, τ declines algebraically towards zero, while p_e is approximately constant. A solution can only be found for $0 < \Delta < 1$, and the closure rate $\tilde{C}$ given by (3.38) is found to be, using (A 24) together with the definition of Λ in (3.21) and (3.23),

$$\tilde{C} \approx k_2 A_v^{1/\delta} t_D^{-(\delta-1)/\delta}(p_c - p_\infty)^{-(\delta-1)/\delta}, \tag{3.49}$$

independent of N, where $\delta = 1+b-a$.

(*c*) *Piping*

The two principal results we have are (3.48) and (3.49). The second of these (for very permeable tills) gives $\tilde{C} \sim (p_c - p_\infty)^{-(\delta-1)/\delta}$, and only exists for $p_c > p_\infty$. As $p_c \to p_\infty$, the closure rate becomes very large. What happens physically in this case?

A condition that the till maintains its integrity is that the minimum effective compressive stress be positive. This is $\sigma_m = P - \tau - p_w$, thus the condition of integrity is that $p_e > \tau$. When $\Lambda \gg 1$, we see from (A 25) that close to the channel, $\sigma_m \sim \Delta^{1/\delta}$ and goes to zero as $p_c \to p_\infty$ (it is always zero at $r = R$ because of the boundary conditions). Thus we see that as $p_c \to p_\infty$, $\sigma_m \to 0$, and we pose the hypothesis that piping occurs: that is, channels or pipes can spontaneously form to alleviate the negative stress condition.

Piping can also occur if $\Lambda \ll 1$. We have $p_e = \tau = \tau_w$ at $r = R$, and p_e jumps to the value one through the pore pressure boundary layer, hence piping will occur if $\tau_w > 1$, i.e. if $\Delta < 1 - a$. Although a mathematical solution exists in this case, it is unphysical, and we expect piping and till failure to occur near the wall.

There are various modifications one can make to the model to alleviate the piping. We have carried out such modifications, but they simply help in justifying one's intuition. Firstly, it is reasonable to expect piping to occur near the wall if

$$\Delta < -\Delta^*, \tag{3.50}$$

where Δ^* depends on Λ, $\Delta^*(0) = a - 1$, $\Delta^*(\infty) = 0$; in general, if the channel pressure is too low. There are two consequences of piping failure we must address: the effect on the till rheology, and the effect on the till permeability. Piping is sometimes associated with slope failure in dams for example, and thus one possible consequence of the onset of piping is that the till loses integrity entirely. In this case we would suppose that the failed zone would be unable to maintain any shear stress, and we would expect that in practice a rapid collapse of the tunnel structure would occur (as sometimes occurs when river banks are destabilized by groundwater seepage). Failure in dams occurs through the progressive enlargement of flow 'pipes' due to the existence of large driving pressures. When driving pressures are lower, or are varied more slowly, it is feasible that the till may retain its integrity, and in this case the momentum balance equation still makes sense.

We consider the possibility of tunnel collapse to be a real one, either when p_c is sufficiently low (during the winter, for example) or if p_c is lowered suddenly (as would be the case when boreholes are drilled, or water is pumped out, as in Boulton & Hindmarsh's experiment), but the issue as to whether piping necessarily leads to collapse is not one that we are able to address here.

The other effect of piping is to increase the permeability dramatically. In fact, we may legitimately assume that the failed region increases its permeability sufficiently by the creation of interconnected cracks and pipes so that the water pressure does not exceed the minimum compressive stress.

Another feature manifested by the solution for $\Lambda \gg 1$ is that $\tilde{C} \to \infty$ as $\Delta \to 0$. This is a consequence of the singular form of the rheological law, which allows infinite strain rate at zero effective pressure. This is unrealistic, since even at zero effective pressure, a saturated till must dilate when sheared: a given sample of till must then suck in water to increase the pore space, implying an increase in p_e. If one makes the simplest modification to the rheology which circumvents this difficulty, i.e. replace (2.1) by $\dot{\epsilon} = A_v \tau^a (p_e + \sigma_c)^{-b}$, where σ_c is a kind of cohesion term, then one can obtain large but finite closure rates when $\Lambda \gg 1$ and $\Delta = 0$. Introduction of 'plastic' failed zones having $\tau = p_c = D$, $\partial p_w / \partial r = 2D/r$ (consistent with a large permeability) then allows one to obtain solutions with $\Delta < 0$: the failed zone is extensive, and the closure rate is large.

Based on considerations of this type, we posit the following scenarios. If p_c is too low, and $\Lambda \gg 1$, then till failure will be followed by rapid tunnel collapse. The practical effect of this is to squeeze the channels until the channel pressure is increased again to p_∞. On the other hand, if $\Lambda \ll 1$, i.e. permeability is low, then a sufficiently low p_c can induce piping. As low permeability tills are likely to be more cohesive, we might then expect a failed zone which reduces τ to p_c, but little further effect.

4. Discussion

In seeking to develop a theory describing the flow in subglacial channels incised into basal till (which will form the subject of a subsequent paper), we are led to study the problem of the closure of a cylindrical void in a saturated, deformable, two-phase medium. Glacial till is an example of such a medium, but other relevant examples are soil, and consolidated or cemented sediments. The study of this problem may also be of interest in the oil industry, for example.

In studying this problem, a crucial parameter is the dimensionless permeability parameter Λ, defined by (3.23) as

$$\Lambda = k[\eta]/\beta_v N R_0^2 \mu_w, \tag{4.1}$$

where the symbols are as defined in §3. This parameter is the ratio of the viscous creep time scale t_v and the Darcy flow time scale t_D, and can be plausibly either large or small for different till types. If the till is coarse and gravel-rich, then $\Lambda \gg 1$, whereas if it is clay-rich, then $\Lambda \ll 1$. In fact, we suspect that the lower value is more typically appropriate. If we take a more precise estimate based on the Boulton–Hindmarsh rheology, using $R_0 = 5$ m (corresponding to the anticipated result that streams will be wide), $N = 1$ bar, $\kappa = \beta_v N = 0.1$, $A_v = 4$ bar^{b-a} a^{-1}, $\mu_w = 10^{-3}$ kg m^{-1} s^{-1}, then

$$\Lambda \sim [k] \times 3 \times 10^{14}, \tag{4.2}$$

where $k = [k]$ m^3. Thus for $k < 10^{-15}$ m^2, Λ is small, and only the most porous tills allow sufficient drainage to have Λ significant (with this choice of parameters). Of course we might also expect $[\eta]$ to depend significantly on porosity and clay fraction.

We have solved the closure problem in the two approximate limits $\Lambda \ll 1$ and $\Lambda \gg 1$, for the case in which all pore water drains into channels rather than into a subtill aquifer. Providing compressibility is relatively small, we obtain the closure rate $\tilde{C} = -\dot{S}/S$ in the form of (3.48) and (3.49):

$$\left.\begin{aligned} \tilde{C} &\approx \{A_v/a^a\}(P_\infty - p_c)^a (P_\infty - p_\infty)^{-b}, \quad & \Lambda \ll 1, \\ \tilde{C} &\sim \{A_v/t_D^{\delta-1}\}^{1/\delta}(p_c - p_\infty)^{-(\delta-1)/\delta}, \quad & \Lambda \gg 1, \end{aligned}\right\} \tag{4.3}$$

where $\delta = 1 + b - a$. These results are principally distinguished by exhibiting the dependence on p_c as well as p_∞.

We have found that piping failure is likely to occur if

$$p_c - p_\infty < -\Delta^*(\Lambda)(P_\infty - p_\infty), \tag{4.4}$$

where $\Delta^*(0) = a - 1$, $\Delta^*(\infty) = 0$, and we consider that the high Λ response will be rapid tunnel closure, while the effect will be less dramatic if $\Lambda \ll 1$, and possibly irrelevant.

Since our preferred value is $\Lambda \ll 1$, and since we can expect p_c to take a value near p_∞ as a long term average, it seems reasonable to suppose that in this case, a suitable average of $(4.3)_1$ is approximately

$$\tilde{C} \sim \{A_v/a^a\}(P_\infty - p_\infty)^{a-b}, \quad \Lambda \ll 1, \tag{4.5}$$

as suggested by Boulton & Hindmarsh (1987).

For gravel-rich till, the result may be altogether different. $(4.3)_2$ suggests that the closure rate is independent of confining pressure. Moreover, if $p_c < p_\infty$, we expect rapid collapse to occur, squeezing the channels to force p_c up to p_∞, while in flood conditions, steady closure can be maintained with $p_c > p_\infty$. It seems that any drainage theory would have to take into account the fluctuations in the water supply, and could be altogether different from the classic Röthlisberger theory.

Finally, we emphasize that the results obtained in the present paper are intended to apply specifically to the case where the sub-till medium is an aquitard, either bedrock or clay rich sediments. In many (perhaps most) cases, this will not be the case, and water may drain vertically to a layer of permeable sediments or fractured bedrock. Some modification of the present theory is necessary in that case, and has been considered by Clarke (1987) and Boulton & Hindmarsh (1987).

In a subsequent paper, we will develop a drainage theory based on the results of the present paper.

Appendix A. Tunnel closure analysis when $\Lambda \gg 1$

In this appendix, we derive a solution for the tunnel closure problem of §3 for the case $\Lambda \gg 1$.

The basic equations are (3.22), which we recast in the form

$$\left.\begin{aligned} &u = -C/r + \tfrac{2}{3}\kappa\Lambda\,\partial p_w/\partial r, \quad \partial u/\partial r - u/r = \tau^a p_e^{-b}, \\ &-\frac{\partial p_e}{\partial r} + \frac{\partial \tau}{\partial r} + \frac{2\tau}{r} = \frac{\partial p_w}{\partial r}, \quad -u\frac{\partial p_e}{\partial r} = \Lambda(1-n)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_w}{\partial r}\right), \end{aligned}\right\} \tag{A 1}$$

where we suppose a quasi-static pore pressure distribution. In our intended application, this will in fact be reasonable, since a channel will be maintained at constant radius by sediment erosion. The boundary conditions for the variables are

$$\left.\begin{aligned} -p_e + \tau = 0, \quad p_w = \Delta \text{ on } r = 1, \\ -p_e + \tau \to -1, \quad p_w \to 0 \text{ as } r \to \infty. \end{aligned}\right\} \tag{A 2}$$

The fourth equation in (A 1) suggests that p_w changes slowly with r, and this is the basis for our approximate solution. Substituting for u, we have

$$\left[-\frac{C}{r} + \tfrac{2}{3}\kappa\Lambda\frac{\partial p_w}{\partial r}\right]\left[\frac{\partial p_w}{\partial r} - \frac{\partial \tau}{\partial r} - \frac{2\tau}{r}\right] = \Lambda(1-n)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_w}{\partial r}\right). \tag{A 3}$$

If we anticipate that τ changes on a scale much less than that on which p_w changes, then a uniformly valid approximation to (A 3) can be found by neglecting the terms in τ. In this case, we can explicitly integrate the equation, and we obtain

$$p_w = \frac{3(1-n)}{2\kappa}\ln\left|\frac{Ar^\nu}{Ar^\nu - \tfrac{2}{3}\kappa\Lambda}\right|, \tag{A 4}$$

where A is given by

$$\Delta = \frac{3(1-n)}{2\kappa}\ln\left[\frac{A}{A - \tfrac{2}{3}\kappa\Lambda}\right], \tag{A 5}$$

and

$$\nu = C/[\Lambda(1-n)]. \tag{A 6}$$

Note that our assumption on p_w will be valid if ν is small enough.

We now rescale the problem by writing

$$C = \Lambda C_1, \quad A = \Lambda A_1, \quad u = \Lambda u_1, \tag{A 7}$$

so that

$$u_1 = -\frac{C_1}{r}[1+\tfrac{2}{3}\kappa g], \quad \Lambda r\frac{\partial}{\partial r}\left(\frac{u_1}{r}\right) = \tau^a p_e^{-b}, \quad -\frac{\partial p_e}{\partial r}+\frac{\partial \tau}{\partial r}+\frac{2\tau}{r} = -\frac{C_1 g}{r}, \quad g = \frac{1}{A_1 r^{\nu}-\frac{2}{3}\kappa}. \tag{A 8}$$

We report this scaling here, since the problem was solved numerically in this form. The numerical strategy involved a further rescaling

$$u_1 = (C_1/\rho)\,\tilde{u}, \quad r = \rho\tilde{r}, \quad \Lambda = (\rho^2/C_1)\,\tilde{\Lambda}, \quad A_1 = \rho^{-\nu}\tilde{A}, \tag{A 9}$$

where ρ is to be determined. This rescaling renders the inner boundary indeterminate, but the equations are independent of ρ, so one can determine the solution explicitly, for given C, Δ and $\tilde{\Lambda}$, by integrating inwards from infinity until $-p_e+\tau = 0$, which then determines ρ, and hence Λ. In this way, we solve implicitly for C, in the form $\Lambda = \Lambda(C, \Delta)$.

The numerical solutions suggest the following further rescaling of the variables:

$$\tau = C_1 T, \quad p_e = C_1 \Pi, \quad C = \Lambda C_1 = \lambda C_1^{a-b}, \tag{A 10}$$

so that the model is given by

$$-\lambda r\frac{\partial}{\partial r}\left[\frac{1}{r^2}\{1+\tfrac{2}{3}\kappa g\}\right] = T^a \Pi^{-b}, \quad -\frac{\partial \Pi}{\partial r}+\frac{\partial T}{\partial r}+\frac{2T}{r} = -\frac{g}{r}, \quad g = \frac{1}{A_1 r^{\nu}-\frac{2}{3}\kappa}, \tag{A 11}$$

and $$-\Pi+T = 0 \text{ on } r = 1; \quad -\Pi+T \to -1/C_1 \text{ as } r\to\infty. \tag{A 12}$$

We suppose $\lambda = O(1)$ (for example, numerical results with $a = 1.33$, $b = 1.8$ suggest $\lambda \approx 0.5$ at $\Delta \approx 0.5$, $\lambda \approx 0.4$ at $\Delta \approx 0.7$); thus (with $a-b < 1$) $C_1 \ll 1$, and so $\nu = C_1/(1-n) \ll 1$. Now with $\Delta = O(1)$, (A 5) and (A 7) suggest $A_1 = O(1)$, so that with $\kappa = 0.1$, we can reasonably neglect the $\frac{2}{3}\kappa g$ terms in (A 11)$_1$; furthermore, since $\nu \ll 1$, g is slowly varying, and it is sufficient to put $g = 1/(A_1-\frac{2}{3}\kappa)$ as constant in (A 11)$_2$.

Now define (neglecting the term in κ in (A 11)$_1$)

$$\left.\begin{aligned} T = (\alpha r^{\gamma}/V)^{1/(1-\beta)}, \quad \Pi = VT, \\ \alpha = (2\lambda)^{-1/b}, \quad \beta = a/b \approx 0.74, \quad \gamma = 2/b \approx 1.1; \end{aligned}\right\} \tag{A 13}$$

we find that V satisfies

$$V' = \frac{\mathrm{d}V}{\mathrm{d}r} = \frac{V}{(\beta V-1)\,r}[\gamma(V-\delta)-(1-\beta)\,g(V/\alpha r^{\gamma})^{1/(1-\beta)}], \tag{A 14}$$

together with $$(V-1)\,T = 0 \text{ on } r = 1, \quad (V-1)\,T\to 1/C_1 \text{ as } r\to\infty. \tag{A 15}$$

In order to satisfy the second of these, $V \gg 1$ for $r \gg 1$. When r is large, g is small, and we can neglect the second term in square brackets in (A 14). Integrating, we find, for large r,

$$V(V-\delta)^{\zeta} \sim \alpha^{\delta}(r/r^*)^{\gamma\delta}, \tag{A 16}$$

where $$\left.\begin{aligned} \delta = 1-(a-b) \approx 1.47, \quad \zeta = \delta\beta-1 \approx 0.09, \\ r^* = C_1^{-(1-\beta)/\gamma} \gg 1. \end{aligned}\right\} \tag{A 17}$$

The problem for small r thus reduces to that of solving (A 14) together with the boundary condition (A 12), namely $(V-1)(\alpha r^{\gamma}/V)^{1/(1-\beta)} = 0$ on $r = 1$, whence either $V = 1$ or $V \to \infty$ (since $0 < \beta < 1$). However, since $1 < 1/\beta$, the choice of $V = 1$ will cause a singularity in V (at $1/\beta$) on solving (A 14) unless α is chosen so that the square bracketed term vanishes at $V = 1/\beta$. However, this flexibility is not available, since α must be chosen so that V matches to (A 16) when $r \gg 1$. Therefore, we decline the finite boundary condition and specify that

$$V \to \infty \quad \text{on} \quad r = 1, \tag{A 18}$$

and α (hence λ) must be chosen so that V matches to (A 16) for $r \gg 1$ and $r \ll r^*$, thus we choose α so that (to leading order)

$$V \to \delta \quad \text{as} \quad r \to \infty. \tag{A 19}$$

Here, 'infinity' is in the matching region between the solution of (A 14) for $r \sim 1$, and (A 16) for $r \gg 1$; more precisely, we require V to satisfy $V = \delta + o(1)$ when $1 \ll r \ll r^*$. That such a solution exists is attested by the fact that (A 14) has asymptotic solutions $V \sim \delta + kr^{-\gamma/(1-\beta)}$ for $r \gg 1$, with $k > 0$. Because solutions of V will vary monotonically with α, it is obvious that a unique such α will exist.

One can go further if $A_1 \gg \kappa$, since then (A 5) and (A 6) imply

$$\Delta \approx (1-n)/A_1 \tag{A 20}$$

(and this will be accurate for a value such as $\kappa = 0.1$), whence (A 8) gives

$$g \approx 1/A_1 \approx \Delta/(1-n). \tag{A 21}$$

Now it is clear from (A 14) that $g/\alpha^{1/(1-\beta)}$ will be constant, $g^*/(1-n)$ say, and then we find $\alpha \approx (\Delta/g^*)^{1-\beta}$, so that

$$\lambda^{1/\delta} \approx l\Delta^{-\mu}, \tag{A 22}$$

where

$$l = 2^{-1/\delta} g^{*(\delta-1)/\delta}, \quad \mu = (\delta-1)/\delta \approx 0.32. \tag{A 23}$$

Then the closure rate is approximately given (for small κ) by C, and

$$C \approx l\Delta^{-\mu}\Lambda^{\mu}. \tag{A 24}$$

Both p_e and τ are very low for $r < r^* \sim \Lambda^{(1-\beta)/\gamma\delta}$, $(1-\beta)/\delta\gamma \approx 0.16$, and for $1 \ll r \ll r^*$, the solutions have the explicit approximations

$$V \approx \delta, \quad \tau \approx (2l)^{-1/(\delta-1)}\delta^{-1/(1-\beta)}\Lambda^{-1/\delta}\Delta^{1/\delta}r^{\gamma/(1-\beta)}, \quad p_e \approx \delta\tau, \tag{A 25}$$

while for $r \gg r^*$,

$$V \approx [\alpha(r/r^*)^{\gamma}]^{1/\beta}, \quad \tau \approx (2l)^{1/a}(\Lambda/\Delta)^{(\delta-1)/\delta a}r^{-\gamma/\beta}, \quad p_e \approx 1. \tag{A 26}$$

Thus p_e and τ are small for $r < r^*$ and increase suddenly near r^*, with a subsequent slow decline of τ over a scale $r \sim \Lambda^{(\delta-1)/2\delta} \approx \Lambda^{0.16}$. The pore pressure is given by (A 4), and for $A_1 \gg \kappa$, this is (using also (A 20))

$$p_w \approx \Delta/r^{\nu}, \nu \approx [l/(1-n)]\,\Delta^{-\mu}\Lambda^{-1/\delta}, \tag{A 27}$$

and it decays to zero over a space scale

$$r \sim \exp[O(1/\nu)] = \exp[O(\Lambda^{1/\delta})], \tag{A 28}$$

which is much larger than the stress adjustment scale $r^* \sim \Lambda^{(1-\beta)/\gamma\delta}$ for $\Lambda \gg 1$.

Evidently, the solution exists for $\Delta > 0$. As before, it follows directly from integrating (A 1)$_3$ (since $\tau > 0$) that $\Delta < 1$ (consistent with the physical expectation that $P_\infty > p_c$, see remark following (3.50)). Hence the problem has a solution when $0 < \Delta < 1$, but the closure rate tends to infinity as $\Delta \to 0$.

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