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Boundary Layer Theory and Subduction

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Numerical models of thermally activated convective flow in Earth's mantle do not resemble active plate tectonics because of their inability to model successfully the process of subduction, other than by the inclusion of artificial weak zones. Here we show, using a boundary layer argument, how the "rigid lid" style of convection favored by thermoviscous fluids leads to lithospheric stresses which may realistically exceed the yield stress and thus cause subduction to occur through the visoc-plastic failure of lithospheric rock. An explicit criterion for the failure of the lid is given, which is sensitive to the internal viscosity η_a below the lid. For numbers appropriate to Earth's mantle, this criterion is approximately $\eta_a > 10^{21}$ Pa s.

INTRODUCTION

Boundary layer theory has been very successful in explaining many of the observed features of mantle convection. *Turcotte and Oxburgh* [1967] were able to explain lithospheric plate velocities and oceanic heat flux variations using a high Rayleigh number flow of a constant viscosity fluid, but as we shall see, there are problems with this theory.

One of these is the fact that the viscosity of mantle rocks is likely [*Kirby*, 1983] to be strongly temperature (and pressure) dependent, as well as being non-Newtonian. Early numerical work on temperature dependent rheology [*Torrance and Turcotte*, 1971; *Kopitzke*, 1979] showed that "thermoviscous" convection develops a rigid lid, because the upper cold boundary layer is also highly viscous, and this was also demonstrated experimentally [*Nataf and Richter*, 1982]. Non-Newtonian effects seem to be less significant [*Parmentier et al.*, 1976].

Thus a problem with variable viscosity convection is how to get the lithosphere to subduct. In the past, various authors have simulated subduction by the means of marginal "weak zones" [e.g. *Gurnis*, 1989], but this is an arbitrary and unsatisfactory procedure. In general, numerical modelers have paid relatively little attention to this question, and there is little coherent prediction that can be made concerning the convective style of Venus, for example.

On the other hand, quite a lot of work has been done on the rather different problem of initiating subduction on an already tectonically active planet. *Turcotte et al.* [1977] put forward an ingenious mechanism based on lithospheric loading at hot spots by volcanic islands (and subsequent buckling of the lithosphere). *McKenzie* [1977] suggested that subduction could occur if two oceanic plates collided rapidly enough, through creation of an overthrust fault. Quite how two plates can collide is unclear. *Cloetingh et al.* [1982, 1989] suggest that subduction might occur at passive margins through the plastic failure of the lithosphere due to the weight of overlying sediments, although it seems that this will not normally occur. This mechanism is similar to that which we propose below, the difference lying in that we find the stresses to be generated by the internal convective process rather than any external forcing, and also we consider the initiation of subduction on a subduction-free mantle. As pointed out by *Mueller and Phillips* [1991], the initiation of primitive subduction complexes is an altogether different problem to that of initiating subduction on a plate-tectonically active planet.

CONVECTION AND STRESS

Turcotte and Oxburgh's [1967] boundary layer theory was based on a constant viscosity mantle. However, it is well recognized [e.g.,

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Kirby, 1983] by experimental studies that any realistic rheology for mantle rock gives a viscosity which depends very strongly on temperature (and pressure). It was not until 1984 that the corresponding boundary layer theory was carried out [*Morris and Canright*, 1984; *Fowler*, 1985], and as would be expected, the top cold boundary layer is very stiff and is essentially immobile.

In Earth's mantle, we take the viscosity η to be the thermally activated form

$$\eta = \eta_0 \exp\left[\frac{E^* + pV^*}{RT}\right], \qquad (1)$$

where η_0 is a viscosity, p is pressure, T is temperature, E^* is the activation energy, V^* is the activation volume, and R is the gas constant. For nonlinear rheology, one often takes $\eta_0 \propto \tau^{-(n-1)}$, with a value of $n = 3$ for dislocation creep or $n = 1$ for diffusion creep. The possible nonlinearity of the flow law will not affect the tenor of the discussion, and we therefore take η_0 constant for convenience. (This is because the viscosity does not vary strongly with stress.)

In analyzing the equations of convection, one first nondimensionalises the variables. Although V^* may be significant over the depth of the mantle, it has little effect over the lithosphere, and we can ignore it. Suppose the ambient sublithospheric temperature is T_a, with corresponding viscosity

$$\eta_a = \eta_0 \exp\left[\frac{E^*}{RT_a}\right]. \qquad (2)$$

We write T in terms of T_a, and η in terms of η_a, thus

$$\eta = \eta_a \eta^*, \quad T = T_a T^*, \qquad (3)$$

and the dimensionless viscosity η^* is then

$$\eta^* = \exp\left[\frac{1 - T^*}{\epsilon T^*}\right], \qquad (4)$$

where

$$\epsilon = RT_a/E^*. \qquad (5)$$

The parameter ϵ is a measure of the strength of viscosity variation; if $\epsilon \ll 1$, the viscosity is strongly variable. Putting $E^* = 125$ kcal mol^{-1} [*Kirby*, 1983], $R = 8.3$ J mol^{-1} K^{-1}, $T_a = 1500$ K [*Turcotte and Schubert*, 1982], we find

$$\epsilon \approx 0.024, \qquad (6)$$

or $\epsilon = 1/42$. The viscosity ratio between surface and asthenosphere is then

$$\Delta\eta = \exp\left[\frac{1 - T_0^*}{\epsilon T_0^*}\right], \qquad (7)$$

where $T_0^* = T_0/T_a$. Taking $T_0 = 300$ K, $T_0^* = 0.2$, then $\Delta\eta \approx 1.8 \times 10^{73}$. We see that the viscosity is strongly variable.

Of course, the rheology becomes plastic and then elastic in the shallow lithosphere; nevertheless, this does not alter the fact that the lithosphere is very stiff when cold. If we analyze the equations of convection at large Rayleigh number Ra, which we can define as

$$Ra = \frac{\alpha \rho_0 g l^3 T_a}{\eta_a \kappa}, \qquad (8)$$

where l is the convective cell width (or depth), α is the thermal expansion coefficient, ρ_0 is the density, g is gravity, and κ is the thermal diffusivity, then we find that provided $Ra > \epsilon^{-5}$, the convection is characterized by vigorous boundary layer motion beneath a thick, virtually stagnant lid [*Fowler*, 1985].

The large negative buoyancy associated with this lid causes large stresses to exist in the lid, and in order to force the shear stress down to zero at the top, there is a thin skin at the surface where longitudinal stresses increase dramatically. If we use values relevant to Earth, we find that these stresses can be significantly in excess of an appropriate plastic yield stress, of perhaps 10 kbar.

The existence of these large stresses necessitates a rethinking in our analysis. We modify the rheology to be a viscoplastic one, such that if the second invariant of the stress tensor τ reaches the yield stress τ_c, then we have a plastic region where $\tau = \tau_c$ and η is indeterminate. In the following section, we modify our boundary layer analysis to include this plastic region. So long as it lies within the rigid lid, the stagnant lid solution remains valid. We find that as the yield stress is reduced, the plastic zone increases in thickness, until at some critical yield stress, it reaches the lid base. At this point, the lid can become mobile, and we expect it to partake of the circulatory motion. This, then, will be our subduction mechanism, and it will provide us with an explicit criterion for its onset.

VISCOPLASTIC BOUNDARY LAYER THEORY

The following analysis relies heavily on work by *Fowler* [1985, 1986], and to save space, we will refer the reader extensively to those papers for details of the analysis.

We consider two-dimensional Boussinesq equations of variable viscosity convection at infinite Prandtl number. They are given by (2.1) of *Fowler* [1985]. We use z as a vertically downward coordinate, x as a horizontal coordinate, with v and u being the relevant velocity components. We introduce scales

$$\begin{gathered} x \sim l, \quad z \sim d, \quad u \sim U, \\ v \sim Ud/l, \quad t \sim l/U, \\ \tau_2 \sim [\tau], \quad \tau_1 \sim (d[\tau]/l), \\ p - \rho_0 g z \sim l[\tau]/d, \end{gathered} \tag{9}$$

where l is given, but $d, U, [\tau]$ are to be chosen. Here p is pressure, τ_1, τ_2 are longitudinal and shear components of the stress tensor. Introduction of a stream function ψ then leads to the following dimensionless set [cf. *Fowler*, 1986, equations (3.5)]

$$p_x = \nu^2\tau_{1x} - \tau_{2z}, \tag{10a}$$

$$-p_z = \nu^2(\tau_{2x} + \tau_{1z}) + T - 1, \tag{10b}$$

$$\frac{1}{\epsilon^2}\tau_1 = 2\eta\psi_{xz}, \tag{10c}$$

$$\frac{1}{\epsilon^2}\tau_2 = -\eta(\psi_{zz} - \nu^2\psi_{xx}), \tag{10d}$$

$$\eta = \exp[-\theta/\epsilon], \quad \theta = 1 - (1/T), \tag{10e}$$

$$\psi_z T_x - \psi_x T_z = \epsilon^3(T_{zz} + \nu^2 T_{xx}), \tag{10f}$$

where

$$\nu = d/l, \tag{11}$$

and we have chosen

$$\begin{gathered} U = Ra^{2/5}U_0, \quad [\tau] = (Ra^{3/5}/\epsilon^2)\tau_0, \\ \nu = 1/\epsilon Ra^{1/5}, \end{gathered} \tag{12}$$

where

$$U_0 = \kappa/l, \quad \tau_0 = \eta_a\kappa/l^2. \tag{13}$$

The choice of scales is relevant for the lithospheric slab, where all the variables (except ψ) are $O(1)$. Thus d is the (order of) the slab thickness. We assume $\nu \ll 1$ (thin slab) and also that $\epsilon \ll 1$. We now solve the equations under the assumption that the slab has a plastic zone overlying a viscous, sticky zone.

Viscous Lid

We suppose the lid base is at $z = s(x)$, and the plastic/viscous transition zone is $z = q(x)$. Thus $0 < z < q$ is plastic, and $q < z < s$ is viscous. In this latter region, we put

$$\psi = \frac{\Lambda z u(x)}{\eta_q} + \frac{\Psi}{\eta}, \tag{14}$$

where $\eta_q = \eta|_q$, and the choice (14) is dictated by considerations of *Fowler* [1985], as we shall require a "stress skin" near $z = q$. Λ is a parameter to be chosen, and it is anticipated that it is only algebraically large in ϵ, whence in $q < z < s$, $\Lambda \ll \eta_q/\eta$, so that $\psi \sim \Psi/\eta$.

It then follows, exactly as in *Fowler* [1985], that, to leading order in ϵ and ν,

$$T \sim T_0 + (1 - T_0)z/s, \tag{15a}$$

$$\tau_2 \sim -\frac{(1-T_0)s'}{2s^2}\left[\frac{1}{3}z^3 - s^2 z + \frac{2}{3}s^3\right] \tag{15b}$$

$$p \sim -\frac{(1-T_0)}{2s}(s-z)^2, \tag{15c}$$

where p, τ_2 go quadratically to zero at $z = s$ in order to match to the shear layer at $z = s$ (see below). Also

$$\tau_2 \sim -\Psi\theta_z^2, \quad \tau_1 \sim 2\Psi\theta_x\theta_z, \tag{16}$$

which determine τ_1 and Ψ.

Plastic Lid

Here $0 < z < q(x)$; the equations are still (10), but we replace the definition of η by the Von Mises yield criterion, which can be written

$$\nu^2\tau_1^2 + \tau_2^2 = c^2, \tag{17}$$

where

$$c = \tau_c/[\tau], \tag{18}$$

and τ_c is the yield stress. A realistic determination of the yield envelope [*Cloetingh et al.*, 1982] gives yield stress as a function of depth, in the range 2-5 kbar. This could be accommodated here by choosing c as a function of z, but we forgo this complication: τ_c should be thought of as some average value.

We rescale the longitudinal deviatoric stress by putting

$$\tau_1 = T_1/\nu^2; \tag{19}$$

then the (rescaled) stress equations are, to leading order (neglecting $O(\nu^2)$),

$$\begin{aligned} p_x &= T_{1x} - T_{2z}, \\ -p_z &= T_{1z} - (1-T), \\ T_1^2 + T_2^2 &= c^2. \end{aligned} \tag{20}$$

This hyperbolic set of equations [*Hill*, 1989] has a first integral

$$p + T_1 = \Delta + \frac{(1-T_0)s}{2} - \frac{(1-T_0)}{2s}(s-z)^2, \tag{21}$$

where Δ is determined from the topographic uplift h at the surface (assumed $\ll d$) as

$$h = \frac{[\tau]}{\rho_0 g \nu}\Delta. \tag{22}$$

We put

$$\begin{aligned} T_1 &= -c\cos\phi, \\ T_2 &= c\sin\phi, \end{aligned} \tag{23}$$

so that (19) reduces to

$$\begin{aligned} &-2\phi_x \sin\phi + \phi_z \cos\phi \\ &\quad = -\frac{1}{c}\left[\Delta' + \frac{(1-T_0)s'}{2s^2}z^2\right], \end{aligned} \tag{24}$$

with the zero shear stress condition $\tau_2 = 0$ corresponding to $\phi = 0$ on $z = 0$ (the alternative, $\phi = \pi$, leads to a solution with negative viscosity in the plastic lid). The solution of (24) must be obtained numerically, using the method of characteristics. Thus we solve

$$\begin{aligned} \dot{x} &= -2\sin\phi, \\ \dot{z} &= \cos\phi, \\ \dot{\phi} &= -\frac{1}{c}\left[\Delta' + \frac{(1-T_0)}{2}\frac{s'}{s^2}z^2\right], \end{aligned} \tag{25}$$

where $\dot{x} = dx/d\tau$, etc., with

$$\phi = 0, \quad z = 0, \quad x = \sigma \quad \text{on} \quad \tau = 0. \tag{26}$$

We thus obtain ϕ, x, z as functions of σ and τ, and elimination of σ and τ (in principle) yields $\phi(x,z)$.

The two relations for ϕ, (10c,d), determine both ψ and η. Bearing in mind the rescaled τ_1, we therefore find

$$\psi \sim \Lambda z u(x)/\eta_q, \tag{27}$$

and the lid velocity $(\Lambda u/\eta_q)$ is transcendentally small provided $\eta_q \gg 1$, i.e., if $q < s$. The effective viscosity in the plastic zone is then given by

$$\eta = \frac{T_1}{2\nu^2\epsilon^2\psi_{xz}} = \frac{-c\cos\phi}{2\nu^2\epsilon^2\Lambda(u/\eta_q)'}. \tag{28}$$

If q, Δ, and s are known, then with ϕ determined from (25), the values of T_1 and T_2 at $z = q$ are found from (23). We denote these as T_1^{plas} and T_2^{plas}, respectively. On the other hand, the corresponding values at $z = q$ from the viscous part of the lid are, from (15),

$$T_2^{visc} \sim -\frac{(1-T_0)ss'}{2}\left[\frac{1}{3}\left(\frac{q}{s}\right)^3 - \frac{q}{s} + \frac{2}{3}\right], \tag{29a}$$

$$T_1^{visc} \sim 0, \tag{29b}$$

and also

$$p \sim -\frac{(1-T_0)s}{2}\left[1 - \left(\frac{q}{s}\right)\right]^2. \tag{30}$$

Now at the join between the two regions, we require $p + \tau_1$ and τ_2 to be continuous, and we have yet to choose Δ, q, and s.

We choose Δ so that $p + \tau_1$ is continuous. From (21) and (30), this requires

$$\Delta = -\frac{(1-T_0)s}{2}. \tag{31}$$

The viscous lid base $s(x)$ is determined by the equations in a shear layer at the lid base. Specifically, by putting

$$\begin{aligned} z = s + \epsilon\zeta, \quad T = 1 + \epsilon\phi, \quad \tau_2 = \epsilon^2 T_2, \\ \tau_1 = \epsilon^2\hat{T}_1, \quad p = \epsilon^2 P, \quad \psi = \epsilon^2\Psi, \end{aligned} \tag{32}$$

we regain the boundary layer equations of *Fowler* [1985] (but with $\phi \sim (1-T_0)\zeta/s$ as $\zeta \to -\infty$). These have a similarity solution in which $s(x)$ is self-consistently determined [*Fowler*, 1985] as (we first write $s = (1-T_0)\hat{s}$, $T_2 = (1-T_0)^{1/2}\hat{T}_2$, $\Psi = (1-T_0)^{1/2}\hat{\Psi}$, $x = (1-T_0)^{1/2}\hat{x}$, where the hatted variables are those of *Fowler* [1985])

$$s = k(1-T_0)^{4/5}x^{2/5}, \quad k \approx 0.82. \tag{33}$$

We now see that we cannot have continuity of τ_2 at $z = q$, since ϕ in $z < q$ satisfies

$$\dot{\phi} = -\frac{\Delta'}{c}\left[1 - \left(\frac{z}{s}\right)^2\right] > 0, \tag{34}$$

since $\Delta' < 0$, and thus $\phi > 0$, hence $\tau_2 > 0$ (since $\phi < \pi/2$, otherwise the characteristics intersect to form a shock), while $\tau_2^{visc} < 0$. There is therefore a jump in τ_2 at $z = q$, and this is facilitated by a stress boundary layer, in which also T_1 jumps, and we choose u in order that $|T_1|$ be continuous also (so that the yield stress is approached from below).

The analysis of this stress layer follows that of *Fowler* [1985]. We put

$$z = q + \epsilon\zeta. \tag{35}$$

With $\theta = \theta_q$ at $z = q$, $\eta_q = e^{-\theta_q/\epsilon}$, we find, to leading order,

$$\begin{aligned} -q'p_\zeta &= -q'T_{1\zeta} - \tau_{2\zeta}, \\ -p_\zeta &= T_{1\zeta}, \\ T_1 &= 2\eta_q e^{\theta_{zq}\zeta}(\epsilon u/\eta_q)', \end{aligned} \tag{36}$$

where $\theta_{zq} = \theta_z|_q$, and we have chosen

$$\Lambda = \frac{1}{\epsilon\nu^2}. \tag{37}$$

Also $(\epsilon u/\eta_q)' \sim u\theta_q'/\eta_q$, thus

$$T_1 \sim 2u\theta_q' e^{-\theta_{zq}\zeta}. \tag{38}$$

Straightforward integration yields $p + T_1$ constant through the layer, confirming our choice of Δ in (31). Then

$$\tau_{2\zeta} = -2q'T_{1\zeta}, \tag{39}$$

thus, if τ_2^{skin} and T_1^{skin} denote the values of τ_2 and T_1 at $z = q$, then

$$\tau_2^{skin} = \tau_2^{visc} - 2q'\tau_1^{skin}, \tag{40a}$$
$$T_1^{skin} = 2\theta_q' u. \tag{40b}$$

With s given by (33), Δ by (31), we have to choose q and u to satisfy the two relations

$$\tau_2^{plas} = \tau_2^{skin}, \tag{41a}$$
$$\pm T_1^{plas} = T_1^{skin}, \tag{41b}$$

where these four variables are defined through (23), (40), and (29a). The minus sign allows for a possible tangential stress discontinuity at $z = q$, of dubious plausibility, since it is not a slip line.

Determination of $q(x)$. As we require $0 < \phi < \pi/2$, then $\tau_2^{plas} > 0$, $T_1^{plas} < 0$. Also $\tau_2^{visc} < 0$, thus (40a) necessitates that $q'T_1^{skin} < 0$, and from (41b), $\pm q'T_1^{plas} < 0$; thus $\pm = \operatorname{sgn} q'$ in (41b). From (40a) and (41),

$$\tau_2^{plas} = \tau_2^{visc} - 2|q'|T_1^{plas}, \tag{42}$$

i.e. (using (29a))

$$2|q'|\cos\phi - \sin\phi = -\frac{\Delta' s}{c}\left[\frac{2}{3} - \left(\frac{q}{s}\right) + \frac{1}{3}\left(\frac{q}{s}\right)^3\right], \tag{43}$$

where $\phi = \phi|_q$ is determined through the solution of (25). This is a first order differential equation for q which can be solved numerically, with $q(0) = 0$ (since $s(0) = 0$). Hence $q' > 0$, and we can replace $|q'|$ by q' in (43). Thus we select the positive sign in (41b), and there is no stress discontinuity. Having found q, u is then determined explicitly from (40) and (41); thus

$$u = T_1^{plas}/2\theta_q', \tag{44}$$

and $u > 0$ if $\theta_q' < 0$, which will be the case if $(q/s)' > 0$, which it in fact is.

The solution for q depends on the parameter $c = \tau_c/[\tau]$. For very large values of c, there will be no plastic region at all, except perhaps near the surface, and the stress skin will occur at the surface, as in the study by *Fowler* [1985]. For lower values of c (but still large), an approximate solution is possible. The characteristic equations are, from (25) and (34),

$$\begin{aligned}\dot{x} &= -2\sin\phi,\\ \dot{z} &= \cos\phi,\\ \dot{\phi} &= -\frac{\Delta'}{c}\left[1-\left(\frac{z}{s}\right)^2\right],\end{aligned} \qquad (45)$$

with $\phi = 0$, $z = 0$, $x = \sigma$ on $\tau = 0$. If $c \gg 1$, then $\phi \ll 1$, so that, approximately,

$$x \sim \sigma,\quad z \sim \tau,\quad \phi \sim -\frac{\Delta'}{c}\left[\tau - \frac{\tau^3}{3s^2}\right], \qquad (46)$$

and thus

$$\phi \sim -\frac{\Delta' s}{c}\left[\frac{q}{s} - \frac{1}{3}\left(\frac{q}{s}\right)^3\right] \qquad (47)$$

at $z = q$, and (43) gives

$$q' \sim -\frac{\Delta' s}{3c}, \qquad (48)$$

and thus, using (31) and (33),

$$q \sim \left[\frac{(1-T_0)^{13/5}k^2}{12c}\right] x^{4/5}. \qquad (49)$$

We see that $q < s$ for small x, but as c decreases, we find that q reaches s first at $x = 1$, when

$$c = c^* \approx \frac{(1-T_0)^{9/5}k}{12} \approx .046 \qquad (50)$$

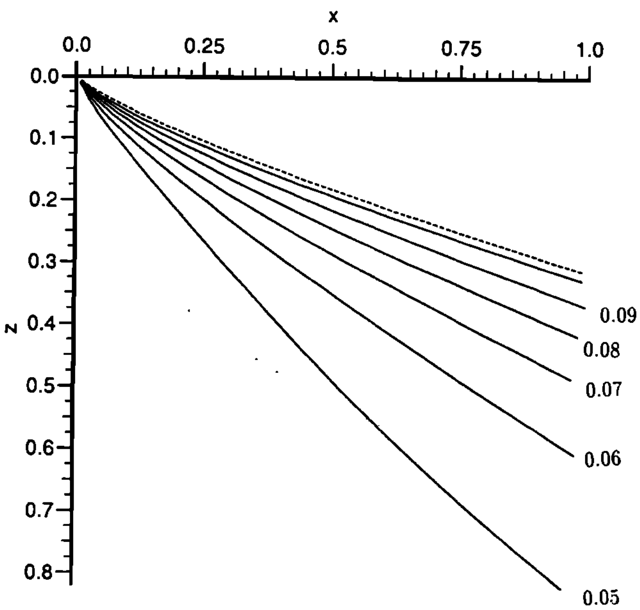

Fig. 1. The dotted line indicates the base of the lid $s(x)$. The base of the plastic zone, $q(x)$, is shown for various values of c between 0.05 and 0.1. Also shown (dashed) is the estimate for q in (49), valid for $x \ll 1$ or $c \gg 1$.

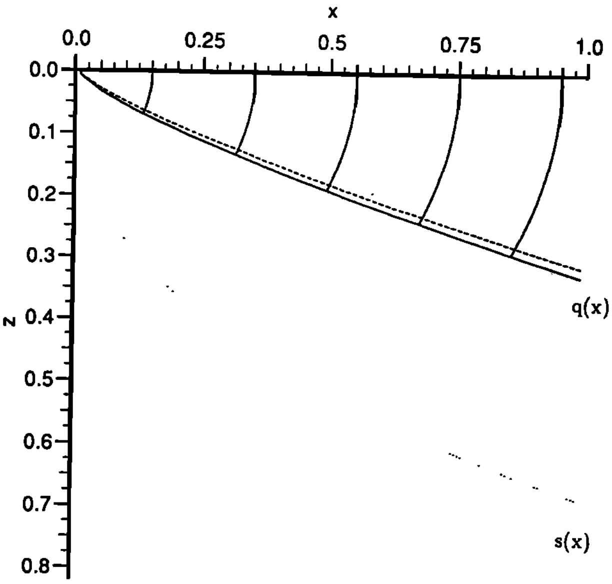

Fig. 2. The arcuate subvertical curves are the characteristics of (45). They terminate on the plastic base $q(x)$, here computed using $c = 0.1$. The dashed line just above is the estimate (49) for q, valid for $c \gg 1$ or $x \ll 1$. The lowest (dotted) curve is the lid base $s(x)$.

for values $T_0 = 0.2$, $k = 0.82$. Of course, the assumption that $c \gg 1$ is then invalid. A numerical solution of (45) in $z < q$, together with (43) for q, is straightforward, and for values $T_0 = 0.2$, $k = 0.82$, we find that q first reaches s at $x = 1$, when $c = c^* \approx 0.056$. Since ϕ increases as z increases, (45) suggests possible problems if ϕ reaches $\pi/2$, for then the characteristics turn round and a shock would form. However, it is easy to see from (43) that q must reach s before ϕ reaches $-\pi/2$, for otherwise, (43) implies q' and hence $q \to \infty$ as $\phi \to \pi/2$, so that q must have reached s first.

In solving the problem numerically, we need the small x solution for q. It can be shown that this is also given by (49) and that is then used to seed the solution at a small initial value of x. In fact, this solution is a useful approximation even for small c. Figure 1 shows a set of results for various values of c, and Figure 2 shows a typical set of characteristics for $c = 0.1$.

DISCUSSION

If $c < c^*$, the plastic zone extends to the base of the viscous lid. Furthermore, the effective viscosity in the plastic lid is given by (28) and is

$$\eta = \frac{T_1}{2\epsilon(u/\eta_q)'} \sim \frac{\eta_q T_1}{2u\theta'_q} = \left(\frac{\cos\phi}{\cos\phi_q}\right)\eta_q; \quad (51)$$

that is, $\eta \sim \eta_q$, so that when q reaches s, the plastic viscosity drops to the sub-lithospheric value, and there is then nothing to prevent the lid taking part in the circulation. This is a mechanism for the onset of subduction.

Once subduction is initiated, the style of convection changes, because the lid now partakes in the flow (and will in fact be the dominant driving force). The maintenance of such active plate tectonics then only requires plastic yielding to occur at the trench.

Our explicit criterion for the onset of subduction is that $c < c^*$, or recalling the definition of c via (18),(12),(8), and (5),

$$\tau_c < c^*(Ra^{3/5}/\epsilon^2)\tau_0 = \tau^*. \quad (52)$$

Using the definitions of Ra and ϵ in (5) and (8), as well as η_a in (2) and τ_0 in (13), this criterion can be written as

$$\eta_a > \eta_c = \frac{\tau_c^{5/2} l^{1/2} T_a^{7/2} R^5}{c^{*5/2}(\alpha\rho_0 g)^{3/2}\kappa E^{*5}}. \quad (53)$$

If we use values $\alpha = 4 \times 10^{-5}$ K^{-1}, $T_a = 1500$ K, $\rho_0 = 3 \times 10^3$ kg m^{-3}, $g = 10$ m s^{-2}, $l = 3000$ km, $\eta_a = 10^{19}$ Pa s, $\kappa = 10^{-6}$ m^2 s^{-1}, relevant to present-day oceanic mantle conditions, then $Ra \approx 3.7 \times 10^9$. For a viscosity more appropriate to the lower part of the mantle, $\eta_a = 10^{21}$ Pa s, then $Ra = 3.7 \times 10^7$. For $E^* = 125$ kcal mol^{-1} [*Kirby*, 1983], we have $\epsilon = 0.024$ (see (6)). Also $\tau_0 = 1.1 \times 10^{-5}$ bars (or 10^{-3} with $\eta_a = 10^{21}$ Pa s). With $c^* = 0.06$, we then find $\tau^* \approx 600$ bars for the lower viscosity, while $\tau^* \approx 4$ kbar for the higher value. *Cloetingh et al.* [1982] suggest that τ_c typically lies in the range 2-4 kbar, so that we might expect subduction to be initiated in the latter case.

Other features of the stagnant lid flow can be determined as well; in particular, the lid thickness and the topographic uplift.

The depth of the lid is given through the aspect ratio ν as $\nu = 1/\epsilon Ra^{1/5}$, or more precisely $\nu s(x)$. With $k = 0.82$ and $T_0 = 0.2$, $s(1) = 0.69$, the maximum lid depth is $\bar{d} = 0.69$ νl, and with $\epsilon = 0.024$ and $Ra = 3.7 \times 10^9$, this is $0.35l$, or 1050 km for $l = 3000$ km. For $\eta_a = 10^{21}$ Pa s, $Ra = 3.7 \times 10^7$, we have $Ra^{1/5} < 1/\epsilon$, and the approximations break down (but the lid is thicker still).

Uplift in the absence of convection is determined by (22); thus

$$h = \frac{[\tau]}{\rho_0 g \nu}\Delta = -\frac{[\tau]}{\rho_0 g \nu}\frac{(1-T_0)^{9/5}}{2}kx^{2/5}, \quad (54)$$

and

$$\Delta h = h(0) - h(1) = \frac{[\tau]}{\rho_0 g \nu}\frac{k(1-T_0)^{9/5}}{2}. \quad (55)$$

With $[\tau] \sim 1 - 7 \times 10^4$ bars, corresponding to $\eta_a = 10^{19} - 10^{21}$ Pa s, then $\Delta h \sim 0.1 - 0.6 \times 10^5/\nu$ m; i.e., for $\nu \sim 0.1 - 0.6$, $\Delta h \sim 100$ km. We discuss these values below.

CONCLUSIONS

Convection of a strongly variable viscosity fluid in a box of lateral dimension l has rigid lid style convection, with a lid of thickness $\bar{d}$. This lid will typically have a plastic upper part and a very viscous lower part and will be essentially stagnant. The effective plastic viscosity will be of the order of that at the top of the viscous part of the lid. In this case, its topographic uplift is

$$\Delta h \sim \frac{[\tau]}{\rho_0 g \nu} \quad (56)$$

and the lid thickness is

$$\bar{d} \sim l/\epsilon R^{1/5}, \quad (57)$$

where

$$\epsilon = RT_a/E^*, \quad R = \alpha\rho_0 g T_a l^3/\eta_a\kappa, \quad (58)$$

T_a and η_a being the sublithospheric temperature and viscosity, respectively, and the lid stress is of order $[\tau]$ defined by

$$[\tau] = (R^{3/5}/\epsilon^2)(\eta_a\kappa/l^2). \quad (59)$$

The plastic lid becomes thicker as the parameter c decreases, where

$$c = \tau_c/[\tau], \quad (60)$$

τ_c being the yield stress, and reaches the base of the lid if $c = c^* = 0.06$. At this point, subduction must occur; thus our criterion for subduction is that

$$\frac{R^{3/5}}{\epsilon^2} \cdot \frac{\eta_a \kappa}{l^2} > \tau_c / c^*, \qquad (61)$$

and the value of c^* is probably a lower estimate. Subduction occurs if $\eta_a > \eta_c$, where $\eta_c \sim 10^{22}$ P for Earth-type values.

Although we have computed values appropriate to Earth, it is important to put our results in the correct context. We are not calculating lid thickness, topographic uplift, etc., which are relevant to Earth as it is. We are posing the hypothetical question: For a planetary mantle akin to that of Earth, is rigid skin convection possible, or not? Our conclusion is then the following. For an Earth-type mantle without active plate tectonics, and allowing for all the features neglected here, we would propose, if $\eta_a = 10^{19}$ Pa s, that the lid thickness would be $\sim$ 1050 km and topographic uplift would be 100 km.

Now, such an unrealistically large uplift would be dramatically reduced by erosion. Indeed, a millimeter per year is equivalent to 100 km/100 M.Y. Thus the value of this number qua prediction is very little. If we wanted to make a relevant prediction for a proto-earth without subduction, we would necessarily have to include a model for the uplift. A viscous relaxational model (analogous to postglacial rebound) would be $\dot{h} = (h_0 - h)/\tau_v - \dot{E}(h)$, where h_0 is the topography predicted here, τ_v is a viscous relaxation time, and $\dot{E}$ is an erosion rate which can be expected to increase dramatically with h. Such a model could be used to predict equilibrium values of h well below h_0. Note that such a model would also alter the boundary layer analysis somewhat, as it would allow for a vertical velocity at the surface.

In a similar vein, a lithosphere thickness of 1050 km (for $\eta_a = 10^{19}$ Pa s) seems much thicker than present estimates for Earth; but again, this is not a statement about the present Earth but only an estimate for a possible proto-Earth without subduction. Even with this, it is not the best estimate one could make in view of the simplifications in the model.

To give one example, lid thickness will be affected by the presence of radioactive elements in the lithosphere. If heat is released at a rate q (heat per unit mass per unit time), then the scaled temperature in the lid satisfies (compare (15a)) $T_{zz} + 2\beta = 0$, where $\beta = \rho q d^2 / 2KT_a$, K being the thermal conductivity. If we take β as constant, then

$$T \sim T_0 + (1 - T_0 + \beta s^2) z/s - \beta z^2 \qquad (62)$$

replaces (15a), so that

$$T_z \approx \frac{1 - T_0 - \beta s^2}{s} \qquad (63)$$

at $z = s$. Now this will have some effect on the slab stresses, but the lid thickness can be calculated directly. Equation (33) relies on a similarity solution, which is no longer available for (63), but the sense of the effect of $\beta \neq 0$ is to replace (33) by

$$s = k[1 - T_0 - \beta s^2]^{4/5} x^{2/5}. \qquad (64)$$

If we adopt this and put $k = 0.82$, $x = 1$, then the maximum depth of the lid, $\bar{d} = \nu l s = ds$, is computed from

$$\bar{d} = 0.82 \nu l \left[1 - T_0 - \left(\frac{\rho q}{2KT_a} \right) \bar{d}^2 \right]^{4/5}. \qquad (65)$$

If we take $\eta_a = 10^{21}$ Pa s, then (with $T_0 = 1/7$ corresponding to $T_a = 1900$ K) $Ra = 4.7 \times 10^7$, $\epsilon = 0.03$, hence $\nu = 0.97$ and $0.82\nu l = 2400$ km. The effect of heat sources depends on the value of q. If we choose $q = 6 \times 10^{-12}$ W kg^{-1} [*Turcotte and Schubert*, 1982], $\rho = 4$ kg m^{-3}, $K = 4$ W m^{-1}K^{-1}, then $\rho q / 2KT_a \sim 1.5 \times 10^{-6}$ km^2, and this term becomes significant for $\bar{d} \sim$ 1000 km (which would in fact be the maximum possible depth). In fact, solving (65) gives $\bar{d} =$ 660 km, and if any concentration of q occurs, this value would be lower. In the Archaean, a value of q twice as high gives a value around 490 km.

In summary, we wish to draw two possible conclusions and offer one suggestion. The first conclusion is that if allowance is made for a yield stress in numerical models of variable viscosity convection, then high Nusselt number rigid lid convection is possible if η_a is low enough (so that the lid is thin, specifically

$\epsilon Ra^{1/5} > 1$), and the lid will yield if η_a is high enough ($\eta_a > \eta_c$ given by (53)). This should be clearly demonstrable by adequate numerical computations.

As far as Earth is concerned, a straightforward application of these results to a proto-Earth having no subduction suggests that rigid lid convection will yield and cause subduction if $\eta_a > 10^{21}$ Pa s. That is to say, subduction would be marginally feasible. However, there are other features of the real Earth, notably internal heat production, which may alter this numerical value. In particular, we expect internal heat to cause a decrease in the lid thickness, although the effect of this on the lid stresses is unknown.

The suggestion is this. Active plate tectonics occurs as a consequence of subduction. The resulting volcanism depletes the mantle, leading to a concentration of radioactive elements in the continental crust. Higher heat production leads to thinner lids, and it is tempting to suppose that while oceanic lithosphere is the surface expression of a mobile thermal boundary layer, so continental lithosphere is the surface expression of a rigid lid convection cell, and that the two types of convective system coexist on Earth. These and other issues require further scrutiny.

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