Asymptotic analysis of the O'Neill/Miller model for secondary frost heave

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1. Introduction

Frost heave occurs during the freezing of soils when water is sucked towards the freezing front by capillary forces associated with the ice/water interface. As this water freezes, it forces the soil surface upwards. The resultant 'frost heave' causes structural damage to roads and buildings, and an understanding of the phenomenon is of significant commercial interest. A detailed physical model of the process has been developed over many years by Miller and co-workers, and a numerical computation of this model was implemented by O'Neill and Miller (1985). The numerical solution is, however, extremely difficult to obtain, and an easier method of solution would be of much use. In this report, we summarise recent work by Fowler (1989) and Fowler and Krantz (1990), which enables an enormous simplification of the O'Neill/Miller model to be made on the basis of realistic asymptotic approximations. We then comment briefly on the extension of these methods to situations not considered by Miller's model: compressible, unsaturated, three-dimensional, saline soils.

2. The O'Neill/Miller model

Fundamental to the concept of the O'Neill/Miller model is the existence of a 'frozen fringe' within which ice and water coexist in the soil's pore space. In essence, this is quite analogous to the existence of mushy zones in solidifying alloys, and in Miller's model is due to the dependence of freezing temperature on pressure. Through this fringe, the pore-water volume fraction decreases as freezing progresses, and the permeability to water flow decreases significantly. In turn the capillary pressure difference between ice and water phases increases as freezing progresses, analogously to drying/wetting curves for unsaturated (unfrozen) soils.

More specifically, the generalised Clapeyron equation for the freezing temperature is

$$T = T_0[1 + \frac{p_w}{L}(\frac{1}{\rho_w} - \frac{1}{\rho_i}) - \frac{1}{\rho_i L}(p_i - p_w)] , \qquad (1)$$

in which the pressure difference is given by

$$p_i - p_w = f(W) , \qquad (2)$$

and represents the Gibbs-Thomson effect. Typically, $f(W)$ is monotone decreasing to zero at $W = \phi$ (the porosity). These relations are supplemented by equations of conservation of mass of both water, ice (and in principle) soil phases, although the latter is neglected in Miller's version, since the soil is assumed incompressible; the water flow is given by Darcy's law, and temperature is described by the energy equation, although within the fringe, temperature is given by the generalised Clapeyron relation, and energy conservation determines the freezing rate, which acts as a source term in the mass conservation equations. For reasons of space, we do not present the complete model here.

Lens formation

Crucial to the heaving process is a criterion for the formation of separate ice '<u>lenses</u>' in the soil. During freezing, a series of discrete ice lenses is observed to form, and it is this process which really lies behind the heaving which occurs: expansion on freezing contributes, but is incidental. Miller essentially associates lens formation with a simple fracture criterion for the soil, which arises through the partitioning of pressure between soil and pore constituents. The pressure transmitted through the soil skeleton is Terzaghi's <u>effective pressure</u> p_e, and is related to the confining pressure P by

$$P = p_e + \chi p_w + (1-\chi)p_i \quad , \tag{3}$$

where the stress partition factor $\chi(W)$ indicates the relative importance of water and ice pressures within the pore. For a coherent soil, we require $p_e > 0$, and Miller's criterion for lens formation is that $p_e = 0$ within the fringe, when a (non-cohesive) soil can spontaneously 'fracture' (since there is nothing to keep the soil particles together). In O'Neill and Miller's numerical computations, the energy and mass flow equations are solved pointwise, with a restart procedure after the formation of each lens.

Regelation

It is important to the Miller concept that the fringe ice forms a continuous mass with the ice in the lowest lens. Heaving occurs as water freezes on to the lowest lens, thus forcing the lens upwards. Miller's concept of 'rigid ice' allows the necessary resultant fringe ice motion to occur via regelation: ice moves relatively to the soil down temperature gradients. However, O'Neill and Miller actually take the ice velocity $\tilde{v}_i$ as a function only of time $\tilde{v}_i = \tilde{v}_i(t)$. In fact this is inconsistent with any three-dimensional model, and a more realistic model might be

$$\underset{\sim}{v}_i - \underset{\sim}{v}_s = -\lambda\underline{\nabla}T , \tag{4}$$

where $\underline{v}_s$ is the soil velocity (cf. Gilpin 1979). This is consistent with ice being 'rigid', since the total <u>average</u> ice flux contains a 'non-rigid' part due to regelation.

3. Approximations

We make four key approximations in analysing the model.

(i) Gravitational effects are small

If the characteristic function f(W) is of typical magnitude σ (e.g. 1 bar), and if freezing penetrates a distance of order d, then gravity is small if $\rho gd/\sigma << 1$, where ρ is the soil density, g is gravity. If $\rho \sim 2 \times 10^3$ kg m^{-3}, g $\sim$ 10 m s^{-2}, d $\sim$ 1 m, $\sigma \sim 10^5$ N m^{-2}, then $\rho gd/\sigma \sim 0.2$. For laboratory scale tests, this is a good approximation, but may be less good in the field.

(ii) Advective heat transport is small

This is measured by the Peclet number Pe $= Ud/\kappa$, where U is a typical velocity, and κ is the thermal difusivity. With U $= 10^{-6}$ cm s^{-1} (a significant heave rate), d = 1 m, $\kappa = 10^{-2}$ cm^2 s^{-1}, then Pe $\sim 10^{-2}$.

(iii) The fringe is thin

The temperature variation across the fringe is, from (1), of order $[T] = \sigma T_0/\rho_i L$. With $\rho_i L \sim 3 \times 10^3$ bar, $T_0 \sim 300$ K, $\sigma \sim 1$ bar, we get $[T] \sim 10^{-1}$ K. The relative thickness of the fringe is measured by the parameter $\epsilon = [T]/\Delta T$, where ΔT is a typical temperature deficit, e.g. the degree of sub-cooling at the surface. If, for example, $\Delta T = 10$ K, then $\epsilon \sim 10^{-2}$, and if $d = 1$ m, the fringe has a thickness of order 1 cm.

The net result of these approximations, which are well known, the last being due to Piper et al. (1988), is that the fringe is essentially a (free) boundary dividing regions of frozen and unfrozen soil, in each of which the temperature T satisfies the heat equation. In fact, since the Stefan number $St = L/c_p\Delta T$ (where c_p is specific heat) is ~ 15 for $L \sim 3 \times 10^5$ J kg^{-1}, $\Delta T \sim 10$ K, $c_p \sim 2 \times 10^3$ J kg^{-1} K^{-1}, the temperature effectively satisfies

$$\nabla^2 T = 0 \tag{5}$$

outside the fringe. At the fringe $z = z_f$ (where z is the vertically upward coordinate), T is given by the Clapeyron equation, which is approximately

$$T = T_0 \quad \text{at} \quad z = z_f \tag{6}$$

(since $[T] \ll \Delta T$). At the top surface $z = z_s$, we prescribe the sub-cooling, thus

$$T = T_s \quad \text{at} \quad z = z_s \ . \tag{7}$$

In addition, we suppose that at the base $z = z_b$, temperature is prescribed:

$$T = T_b \quad \text{at} \quad z = z_b \ .$$

The model must thus be completed by two further boundary conditions at z_f and z_s.

(iv) Large permeability exponent

It is in prescribing these two extra relations that the difficulty lies. The Stefan condition at z_f, for example, takes the form

$$\rho_W L[U_n - WV_{fn}]_-^+ = k[\frac{\partial T}{\partial n}]_-^+ \ , \tag{9}$$

where U_n is the normal water flux at z_f, and V_{fn} is the normal velocity of z_f. This relation, however, contains several further unknowns, for example the water flux U_n, which must be computed from solutions of the fringe variables. The water flux is given by Darcy's law, where the permeability is modelled by a function of the form

$$k_p = \bar{k}W^\gamma \ , \tag{10}$$

where the exponent γ is quite high, for example 7 or 9. We can exploit the limit $\gamma >> 1$ using the ideas of high activation energy asymptotics. Essentially, the water pressure varies over a thin boundary layer (of thickness ~ $\epsilon d/\gamma$), and is quasi-static. This approximation enables us to solve the equations in the fringe analytically, and collapses the second boundary condition on z_f to nine algebraic relations for nine unknowns, including V_{fn} (the normal front velocity) and v_{in}^+, the normal ice lens velocity. Elimination leads to dimensionless expressions of the form

$$V_{fn} = [c \frac{\partial T}{\partial n}]_{-}^{+} \,, \tag{11}$$

where c_+, c_- are (unequal) complicated functions of the applied load P and soil type, which affects f(W) as well as the permeability, and

$$v^{+}_{in} = \alpha[\frac{\partial T}{\partial n}\bigg|_{z_{f^+}} - W_1 V_{fn}] \,. \tag{12}$$

Where $W_1 = O(1)$ depends on load, and α is a parameter which is a measure of the heaving rate of the soil (if it heaves). The magnitude of α is determined by soil type; coarse soils have $\alpha = O(1)$, fine soils have $\alpha << 1$. However, for coarse soils (and large loads), α may be negative, which implies that heaving does not occur.

4. Simple solutions: one dimension

In one dimension, the heaving rate $\dot{z}_s$ is determined by mass conservation from the lowest ice lens velocity: $z_s^+ = v_{in}$. The solution for the temperature field can be written down, and after some algebra, the equations for $\dot{z}_f = V_{fn}$, $\dot{z}_s = v_{in}^+$, can be written

$$\begin{aligned} \dot{z}_f &= - \frac{A}{(z_s - z_f)} + \frac{B}{(z_f - z_b)} \,, \\ \dot{z}_s &= \alpha[\frac{1}{z_s - z_f} + W_1 \dot{z}_f] \,, \end{aligned} \tag{13}$$

which we can in fact solve exactly. For the interesting case $\alpha \ll 1$, we can also rapidly reproduce various qualitative features of O'Neill and Miller's computational results, for example that $\dot{z}_s \sim t^{\frac{1}{2}}$ at small t, but $z_s \to$ constant at larger times. We are hoping to obtain more specific quantitative comparison in the near future.

5. Generalisations

The simplicity afforded by this asymptotic reduction suggests that the approach is a useful one by which to incorporate more complications of the process. In this section, we consider some of these in turn.

(a) Three-dimensionality

As soon as the flow is three dimensional, it can no longer be true that the heaving rate is equal to the lowest ice lens velocity. At the freezing front z_f, the normal stress is P, the stress is P, the normal strain rate is v_{in}^{+}, and we can suppose the shear stress is zero. At the top surface z_s, the normal and shear stresses are zero, and the heaving rate is v_{sn}. If a suitable rheology for frozen soil is given, then only four conditions (two at each end) are needed, and hence P and v_{sn} can be determined in terms of v_{in}^{+}. For example, if the frozen soil behaves elastically, we can write $\partial P/\partial t$ and v_{sn} as linear functionals of v_{in}^{+}. If slow (secondary) creep is appropriate (Sayles 1988), then we would have P and v_{sn} as linear functionals of v_{in}^{+}. For slowly (horizontally) varying heaving rates, one should be able to obtain simple expressions using the ideas of lubrication theory.

(b) Compressibility

Soils are compressible. In its simplest form, this means that the effective pressure p_e is a function of the porosity ϕ. Thus the unfrozen soil below z_f will deform, and the deformation problem for the frozen soil will be coupled to the compression below. More realistic relations between p_e and ϕ involve the ideas of consolidation theory, and in particular the fact that loading and unloading histories are different. However, for small heaving deformations, such niceties may be practically irrelevant. It is unlikely that compressibility will affect the solution in the fringe, either.

(c) Saturation

The O'Neill/Miller model assumes saturation of the soil. Miller (1980) gives some discussion as to how the model might be extended to cover unsaturated soils containing air and water. As the soil begins to freeze, there may be three pore constituents: ice, air and water. However, Miller shows that three such co-existing pore phases will be dynamically unstable, except over a very narrow temperature range. Thus as the soil freezes, we should find a relatively sudden jump from unfrozen, unsaturated soil to frozen (saturated) soil. There are two ways this information can be reconciled with the idea of a partially frozen fringe. The first is that the fringe is saturated as before, with the base of the fringe corresponding to the transition temperature ($\approx T^*$, say), and below the fringe the unsaturated region exists. The other, possibly more likely, is that there is an unsaturated frozen fringe in which ice, water and air coexist, which thus requires that $T \approx T^*$ in the fringe. Then T^* depends on p_i and p_w as before and will be a generalisation to unsaturated soils of the

Clapeyron relation (1). We can expect the variation of T* to be small, so that the previous analysis should apply, with some modifications to allow for the air phase in the fringe.

(d) Salinity

Salinity will affect the permeability of the soil by osmotic effects, since clay particles may act as semi-permeable barriers. In addition, salt depresses the freezing point of water. Thus, the Clapeyron relation needs to be extended to include solute concentrations. This is usually done in terms of the osmotic pressure Π, which modifies (1) to be

$$T = T_0 [1 + \frac{(p_w - \Pi)}{\rho_w L} - \frac{p_i}{\rho_i L}] . \tag{14}$$

It is more common in solidification models to use the salt concentration, which for dilute solutions is related to the molar concentration c by the Van't Hoff equation (Glasstone and Lewis 1960)

$$\Pi = RTc , \tag{15}$$

where R is the gas constant and T is absolute temperature. A one molar solution (56 g litre^{-1}) of sodium chloride corresponds to an osmotic pressure of about 23 bars. Therefore, for moderately saline soils (> 5 g litre^{-1}), the freezing temperature may be significantly determined by the salt concentration. However, the temperature change through the fringe is $O(\Pi T_0/\rho_w L)$, and only 1°C for $\Pi = 10$ bars, so, except for very saline soils, the fringe will still be thin, and the present method of analysis should work; however, in computing the normal front velocity V_{fn}, it will be necessary to compute the distribution of salt concentration in the unfrozen soil. Partition on freezing causes a significant variation to occur.

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