Differential frost heave in seasonally frozen soils

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ABSTRACT

The formation of some forms of patterned ground, notably earth hummocks and stone circles, is associated with seasonal freezing and a spatial instability in the resulting frost heave. We analyse the Miller model of frost heave for such spatial instability, by incorporating three-dimensionsal heat and mass transfer, and allowing the frozen soil to deform as a viscous medium. We find that the heaving process is generally (but not always) stable, but that if account is taken of a surface snow cover, then the insulating thermal properties of the snow predict that instability will occur if a dimensionless parameter $\mathcal{N} > 0.02$. The parameter $\mathcal{N}$ is given by $\mathcal{N} = \eta_f v_s/(\rho_f g d^2)$, where η_f is the frozen soil viscosity, v_s is the surface heave rate, ρ_f is the frozen soil density, g is gravity, and d is the depth of the freezing front. This implies that the propensity for differential frost heave depends on the soil heaving characteristics, as well as the rate of frost penetration.

Key words: patterned ground, differential frost heave

INTRODUCTION

In tundra regions subject to a cold climate, many types of regular geometric formations are observed. These formations are referred to as patterned ground, and differential frost heave is thought to be a possible mechanism for their formation. Some forms of isolated patterned ground (e.g. pingos and palsas) can be directly attributed to frost heave. For the more interesting earth hummocks and stone circles (see, for example, Williams and Smith 1989), differential frost heave has been suggested as an organising mechanism (Van Vliet-Lanoë 1991), although other mechanisms have been suggested, for example thermal convection (Krantz et al. 1988) or the 'cryostatic pressure' theory (Van Vliet-Lanoë 1991). In this paper we summarise recent work by Noon (1996) which demonstrates that the Miller model of secondary frost heave, when suitably modified to allow for three-dimensional (differential) frost heave, has an instability mechanism in it which predicts that in certain conditions, differential frost heave will spontaneously occur in the seasonal freezing of soils, and thus lead to the formation of patterned ground.

THE MILLER MODEL

Miller (1972, 1978) developed a model for secondary frost heave which allowed for the existence of a partially frozen fringe between the frozen soil near the surface and the unfrozen soil beneath. In particular, it includes a mechanism for distinct lens formation within the frozen fringe. While it is perhaps the most conceptually complete model available, it is hampered by the fact that its mathematical formulation (O'Neill and Miller 1982, 1985) is dauntingly complex. However, Piper et al. (1988) showed how certain (accurate) approximations allowed the model to be simplified, and more recently Fowler and Krantz (1994) extended this earlier analysis, and showed that the one-dimensional Miller model could be reduced to a single pair of first order ordinary differential equations for the positions of the ground surface and the freezing front, from which all other quantities, such as lens thickness and spacing, can be derived. The nature of this simplification is summarised in the following section.

THE FOWLER-KRANTZ-NOON REDUCTION

Fowler and Noon (1993) and Fowler and Krantz (1994) use four approximations to simplify the Miller model. In turn, these are the assumptions that (i) gravitational effects are small; this is more of a convenience than a necessity, but is accurate except on a regional scale, and certainly in the present case; (ii) the advection of sensible and latent heat is small, and in particular, heat conduction is essentially in equilibrium; (iii) the frozen fringe is thin (relative to the depth of frost penetration): this is an accurate approximation, due to the fact that the generalised Clapeyron equation allows only a small temperature jump across the fringe (relative to a typical seasonal variation). The final approximation which enables a dramatic reduction in the model complexity is based on (iv) the strong dependence of soil permeability on the pore water fraction in the frozen fringe. As a result, the pore water pressure only varies within a boundary layer which lies inside the frozen fringe, and the governing differential equations can be solved in the fringe.

The consequence of the fringe being thin is that its location is effectively specified as a surface $z = z_f(x, y, t)$, where z is a coordinate normal to the (original) ground surface, and x, y

are horizontal. Equally the ground surface is given by a surface $z = z_s(x, y, t)$. It is convenient to express these and other variables in dimensionless form, and to this end we choose a length scale d appropriate to our situation: for example, a typical depth of the active layer. Equally, the temperature scale ΔT represents a typical seasonal freezing temperature (degrees below zero Celsius). In terms of these, we define a (thermal) velocity scale

$$U = \frac{k_f \Delta T}{\rho_w L d}, \tag{1}$$

where k_f is the thermal conductivity of the frozen ground, ρ_w is the density of water, and L its latent heat. For values $k_f \sim 2$ W m^{-1} K^{-1}, $\Delta T \sim 20$ K, $\rho_w \sim 10^3$ kg m^{-3}, $L \sim 3.3 \times 10^5$ J kg^{-1}, $d \sim 5$ m, we have $U \sim 2.4 \times 10^{-8}$ m s^{-1}. From these we have the time scale

$$t_f = \frac{d}{U} \sim 2 \times 10^8 \text{ s}; \tag{2}$$

note that this is longer than the seasonal time scale 3×10^7 s.

If temperatures, distances, lengths and velocities are scaled with the values above, then Noon (1996) showed, following Fowler and Krantz (1994), that the normal velocity of the frozen fringe, denoted V_f, and the normal ice flux at the lowest ice lens within the fringe, denoted V_i, are related by

$$\begin{aligned} V_f(\phi - W_m) &= V_i + G_f - G_l, \\ V_i &= \alpha[(W_l - W_m)V_f + G_l], \end{aligned} \tag{3}$$

where the dimensionless heat fluxes are given by

$$\begin{aligned} G_f &= -(k_u/k_f)\frac{\partial T}{\partial n} \text{ at the freezing front}, \\ G_l &= -\frac{\partial T}{\partial n} \text{ at the lowest ice lens}, \end{aligned} \tag{4}$$

k_u is the thermal conductivity of unfrozen soil, and in the present case we can take the small geothermal heat flux G_f as zero. In the equations (3), ϕ is the unfrozen soil porosity, W_m (which could be zero) is the soil's minimum obtainable water content on freezing, W_l is the pore water volume fraction at the lowest ice lens in the fringe, and α is a dimensionless heaving parameter which depends both on W_l and on the soil characteristics.

W_l itself depends on the dimensionless effective pressure in the unfrozen soil, and it is only insofar as this is taken as independent of z_f that gravity is neglected.

Miller model in one dimension

If we denote the normal velocity of z_s as V_s, then in one spatial dimension (upwards), Miller's rigid ice assumption specified the heave rate by prescribing

$$V_s = V_i. \tag{5}$$

Since, for steady conduction, we can write $G_l \propto (z_s - z_f)^{-1}$, we see that (3) forms a pair of ordinary differential equations for z_s and z_l; solutions have been given by Fowler and Noon (1993).

EXTENSIONS TO THREE DIMENSIONS

The equations (3) apply as much in three dimensions as in one, but we can no longer assume $V_i = V_s$. Instead, V_s must be related to V_i by consideration of the rheology of the frozen soil. In addition, where the rigid ice assumption allows one to take V_i as spatially uniform in one dimension, this is not possible in three dimensions, and we have replaced Miller's assumption of rigidity with the physically based thermal regelation model (based on work by Römkens and Miller (1973) and Gilpin (1979)) which allows the ice flux $\mathbf{u}_i$ *within the fringe* to be proportional to $-\boldsymbol{\nabla} T$. In particular, the normal component of $\mathbf{u}_i$, u_{in}, at the lowest lens is not necessarily equal to V_i. The theory behind the equations (3) is still valid, but the definition of α is different.

We assume that the frozen soil deforms as a viscous medium. The creep behaviour of frozen soil is considerably more complicated (Sayles 1988, Fish 1994), and the assumption of viscous deformation is taken here as a first simple approach. In three dimensions, we have to solve for the temperature T and frozen soil velocity $\mathbf{u}$ in the region $z_f < z < z_s$, where $(3)_1$ determines the location of z_f, and $(3)_2$ gives the normal flux there. We thus have to solve

$$\nabla^2 T = 0 \quad \text{in} \quad z_f < z < z_s, \tag{6}$$

with prescribed temperature on z_s and $T = 0$ on z_f. To examine possible instabilities we first pose the condition

$$T = -1 \quad \text{on} \quad z = z_s, \tag{7}$$

i.e., an isothermal surface.

The equations describing slow flow (including gravity) are Stokes's equations, and can be written in the dimensionless form (using $\eta_f k_f \Delta T/\rho_w L d^2$ as the pressure scale, where η_f is the frozen soil viscosity)

$$\boldsymbol{\nabla} p = \nabla^2 \mathbf{u} - \Pi \mathbf{k},$$

$$\boldsymbol{\nabla} . \mathbf{u} = 0, \tag{8}$$

and the gravity parameter is

$$\Pi = \frac{g L d^3}{\eta_f k_f \Delta T}, \tag{9}$$

with typical value, if $\eta_f = 10^{14}$ Pa s (Sayles 1988), of $\Pi \approx 1$.

We apply boundary conditions of no normal or tangential stress at z_s, together with a kinematic condition there, which determines z_s. At z_f, the normal velocity is equal to the normal ice flux V_i given by $(3)_2$, z_f is determined by $(3)_1$, and we finally suppose that the unfrozen soil is undeformable, so that a no slip condition is applied. Other choices are clearly possible.

STABILITY ANALYSIS

The basic solution is that of one-dimensional heave. We analyse its stability by linearising the domain boundaries and the variables about their basic states, thus obtaining a linear set of equations and boundary conditions. The basic solution is time dependent, as $z_s - z_f$ increases with time as the freezing front penetrates downwards. We adopt a quasi-static approach (Robinson 1976) in which it is assumed that instabilities occur much faster than frost penetration, so that solutions of the perturbation equations proportional to $e^{\sigma t}$ can be sought. We restrict attention to two spatial dimensions x, z; solutions are then proportional to $\exp(\sigma t + ikx)$, where σ is the growth rate and k is the wavenumber, and in the usual way we derive a dispersion relation in the form $\sigma = \sigma(k)$ from the analysis.

In keeping with the two time derivatives ($\partial z_s/\partial t$ and $\partial z_f/\partial t$) which occur in the problem, we find that there are two modes, and these can in fact be characterised as being due to the gravitational relaxation of z_s (we call this the gravity mode), and the relaxation of z_f in the presence of heave: this is called the thermal mode.

In figure 1 we show typical examples of the dependence of σ on k for the gravity mode and the thermal mode. The formulae describing these functions are excessively complicated and the details of the analysis will appear elsewhere. Typically, the growth rates are negative, indicating stability. However, it is possible to obtain unstable modes, as indicated in figure 2. We see that instability can occur through a degeneracy in the solutions of the dispersion relation $a(k)\sigma^2 + b(k)\sigma + c(k) = 0$, when a goes through zero. The infinite growth rate at finite wavelength is reminiscent of a resonance phenomenon, and while it is unusual, is by no means unknown; see Murray (1989, chapter 17.4) for examples in biology. The formulae are so complicated that it is difficult to be dogmatic, but essentially this instability is predicted to occur for large enough values of α, corresponding to higher permeabilities and thus coarser soils. If this instability does occur, it seems likely to be rather violent (due to the infinite growth rate!).

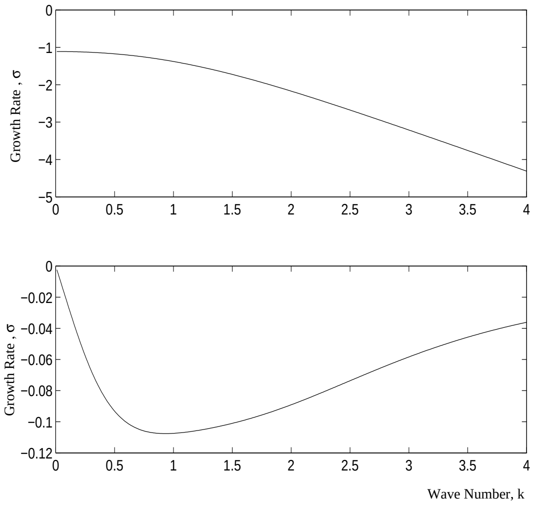

Figure 1: *A typical example of the dependence of thermal mode σ (upper) and gravity mode σ (lower) versus k for $\alpha = 0.1$, $\Pi = 1$. Note the difference in scale. Essentially $\sigma \sim \Pi$ for the gravity mode, while $\sigma \sim \alpha$ for the thermal mode.*

THE EFFECT OF SNOW COVER

One of the mechanisms commonly cited as a reason for the growth of hummocks is the variability in snow covering overlying the hummocks (Williams and Smith 1989). During the winter months troughs of hummocks fill with snow and hence the snow covering is greater in

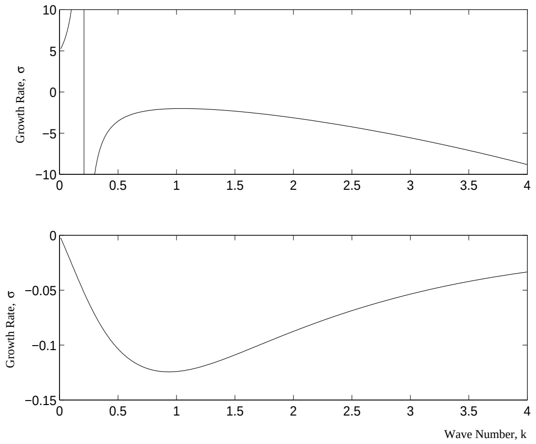

Figure 2: *The thermal mode has become violently unstable for wavenumber $k < k_c \approx 0.2$. This would suggest rapid formation of patterned ground, presumably at a wavelength close to $2\pi/k_c$ (in this example). The parameters are still $\alpha = 0.1$, $\Pi = 1$, but another parameter C in the dispersion relation has been changed from 1 to 1.2; this can be effected by changing γ, for example.*

the troughs than on the crests. Because snow acts as an insulator, the crests are subject to greater cooling and thus more heave, which allows for a further possible instability mechanism.

To study this, we consider a layer of snow of (dimensionless) thickness h overlying an initially flat ground surface. We suppose that the snow surface remains flat during the evolution of z_s, and we replace the condition $T = -1$ at $z = 0$ (the undisturbed ground surface) by solving $\nabla^2 T = 0$ in $z_s < z < h$, with $T = -1$ at $z = h$, T continuous at $z = z_s$, and

$$\delta \left.\frac{\partial T}{\partial n}\right|_{z_s+} = \left.\frac{\partial T}{\partial n}\right|_{z_s-}, \qquad (10)$$

where $\delta = k_s/k_f$ is the ratio of the thermal conductivities of snow and frozen soil.

As before, we derive a quadratic dispersion relation for $\sigma(k)$. For simplicity, we neglect the effects of the perturbation in load on the heave parameter α. (In the absence of snow cover, the same assumption always gives stability, so that the instability (in the thermal mode) mentioned in section 5 for high α relies on its variation with load.) Figure 3 shows a typical plot of the gravity and thermal modes. For these unstable modes, the freezing front and surface perturbations are out of phase, as observed (Tarnocai and Zoltai 1978), Williams and Smith 1989, Van Vliet-Lanoë 1991).

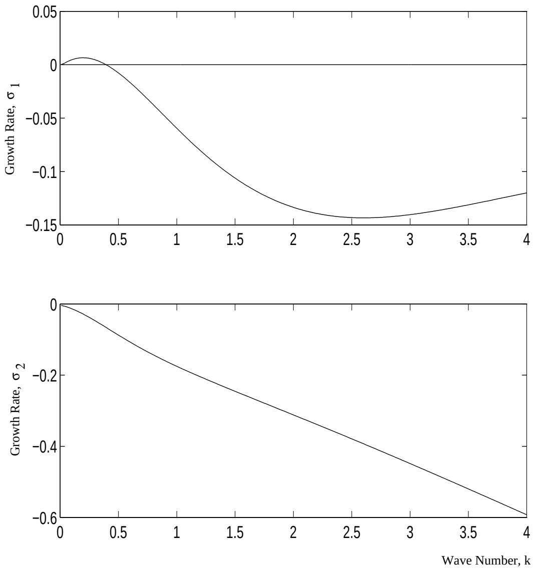

Figure 3: *Growth modes in the presence of snow cover. We take $\alpha = 0.1$, $\Pi = 1$, $\delta = 0.05$, $h = 1$. The upper (gravity) mode indicates long wavelength instability for $k \lesssim 0.5$.*

If we assume that instability arises via an exchange of stability (i.e. $\sigma = 0$), then one can calculate analytically the critical condition for instability to occur. The relevant bifurcation parameter is $\mathcal{N}$, defined by

$$\mathcal{N} = \frac{(\phi - W_l)\alpha}{\Pi\{\phi - W_m - \alpha(W_l - W_m)\}}, \qquad (11)$$

or more simply, $\mathcal{N} = V_i/\Pi$. Since in one dimension, V_i is the heave rate, we can write $\mathcal{N}$ as the dimensionless group

$$\mathcal{N} = \frac{\eta_f v_s}{\rho_f g d_f^2}, \qquad (12)$$

where v_s is the dimensional heave rate, ρ_f is the frozen soil density, and in this analysis the length scale d $(= d_f)$ has been specifically taken to be the (slowly evolving) depth of the frozen front.

We then find that instability occurs at wavenumber k if

$$\mathcal{N} > \frac{(\sinh 2k - 2k)}{4k^3(1-\delta)}(\tanh k \tanh kh + \delta), \qquad (13)$$

and this defines a positive increasing critical value of $\mathcal{N}$ as a fraction of k. The minimum value occurs at $k = 0$, and defines a critical value of $\mathcal{N}$,

$$\mathcal{N}_c = \frac{\delta}{3(1-\delta)}. \quad (14)$$

With typical values $k_s = 0.1$ W m^{-1} K^{-1}, $k_f = 2$ W m^{-1} K^{-1}, we have $\delta \approx 0.05$, thus $\mathcal{N}_c \approx 0.02$. We conclude that differential frost heave is predicted by the Miller model provided the parameter $\mathcal{N}$ given by (11) is greater than this critical value.

DISCUSSION

The Miller model can be extended to describe three-dimensional frost heave, but the rigid ice assumption must be replaced. Gilpin's model of thermal regelation provides a useful alternative. It can then be systematically reduced to give two messy but explicit expressions for the ice flux to the fringe, and the frost penetration rate.

Together with Laplace's equation for the temperature, the frozen soil rheology must be prescribed. We have chosen to model the frozen soil as a viscous medium overlying a rigid substratum, though this choice is easily modified. We have then found two possible instability mechanisms. In the first, instability relies on the variation of α with load (excess load suppresses heave), and is only practical for relatively high values of α, corresponding, for example, to silts or sands. This mechanism could then serve as an explanation for stone circles. Alternatively, while this mechanism is not viable for finer soils, the effect of snow cover can cause instability by enhancing heave at the crests. Here we find instability if the frost depth d_f is less than a critical value,

$$d_f < \left[\frac{\eta_f v_s}{\rho_f g \mathcal{N}_c}\right]^{1/2}. \quad (15)$$

For a silt with $v_s = 10^{-9}$ m s^{-1}, this gives $d_f < 14$ m, and suggests that this mechanism of instability is feasible, although it must also be pointed out that in order for the effect to be viable in practice over a seasonal time scale, the growth rates need to be higher than indicated in figure 3, and this is likely to provide a more severe constraint on d_f. Also, v_s decreases as d_f increases, and of course the whole analysis has been based on a quasi-static approach. Further investigations will warrant a full two or three-dimensional numerical simulation.

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