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Mathematical models of compaction, consolidation and regional groundwater flow

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SUMMARY

A 3-D theory of groundwater flow and consolidation that allows for compressibility of the soil or sediment matrix is that due to Biot (1941). However, this theory is inapplicable to virgin consolidation tests, where the deformation of the soil along the critical state surface is accomplished by elastic–plastic flow. In fact, Biot's model cannot even be applied to overconsolidated soils where the deformation is elastic, as it does not conserve the mass of the solid. In this paper, we provide a modified version of the model, and show how it can be extended to allow for elastic–plastic deformation and large-amplitude deformations in elastic–plastic compaction. A particular example of the potential application of this theory is to the groundwater flow induced by the compaction of sediments by an overriding ice sheet, and we show how the commonly used Dupuit approximation can be employed to simplify the theory.

Key words: compaction, consolidation, groundwater flow.

1 INTRODUCTION

There are a number of problems in the geophysical sciences that involve the compaction or consolidation of porous media. The simplest of these is the consolidation of saturated soils (Lambe & Whitman 1979), in which a superimposed load (for example, a building) will cause slow settlement of the ground through expulsion of groundwater. Exactly the same effect occurs in dry summers as soils dry out. When displacements are small, a satisfactory theory of the process is obtained by supplementing Darcy's law for the flow of pore water with a relationship between the effective pressure p_e and the porosity ϕ of the soil. The resulting *consolidation equation* describes the diffusive relaxation of the pore pressure towards equilibrium. Common applications of this theory occur in geotechnical design problems, as in the construction of dams or embankments. On a larger scale, subsidence caused by groundwater removal has been the cause of flooding in many areas of the world, perhaps the most celebrated being that of Venice. The analysis of such problems involves small strains, and thus can be successfully analysed using linear elastic or elastic–plastic theories (Lewis & Schrefler 1987).

On a larger scale, however, this description becomes less appealing. For example, compaction in sedimentary basins (Audet & Fowler 1992) or in marine ice formed at the base of ice shelves (Jenkins & Bombosch 1995) involves finite strains, and also the application of a simple elastic-type relationship between p_e and ϕ is suspect. Even for soils, consolidation tests imply deformation along a critical state line that forms part of the elastic–plastic yield surface, so that the simple theory may not even strictly apply there.

Other situations where the simple theory is likely to be an inappropriate one are in the groundwater flow induced by compaction under the Quaternary ice sheets (Boulton & Dobbie 1993), and the related problem of permafrost formation in ice-sheet marginal environments. Fig. 1 shows a typical scenario. Our aim in this paper is to provide a theoretical framework wherein these problems may be formulated, and to give explicit criteria that determine when simpler models may be used.

Although we focus on groundwater flow in porous sediments, our theoretical discussion is closely related to the theory of compaction in partially molten crystalline rocks

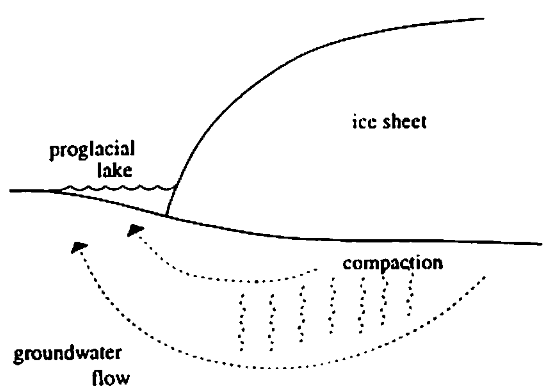

Figure 1. A schematic illustration of the consolidation process that occurs in the sediments underlying an ice sheet. Groundwater flows towards the ice margin, while at the same time, the subglacial sediments are compacted by the overlying ice.

(McKenzie 1984; Scott & Stevenson 1986), with the principal difference lying in the choice of rheological compaction law. In this paper we restrict attention to the formulation of the theory for an elastic or elastic–plastic rheology, while that for compaction in partially molten rocks is based on a viscous compaction law.

The organization of the paper is as follows. In Section 2 we review the classic 1-D consolidation theory given by the Richards equation for saturated, compressible soils, and its generalization to 3-D deformation by Biot (1941). It is a principal purpose of the present paper to illustrate the way in which Biot's theory is conceptually inconsistent, and to show how a correct model can be proposed. In Section 3 we use the assumption of small strains to derive, for an elastic porous medium, a theory that essentially reproduces the model derived by Biot. In Section 4 we derive a small strain theory of consolidation for elastic-plastic materials; in Section 5 this is extended to the case of large strains. Our aim in these sections is limited to showing the structure of the compaction model that is derived, together with an indication of how it might be solved (numerically). By way of application of the theoretical model, we consider in Sections 6 and 7 the derivation of a model for pore pressure evolution in groundwater below ice sheets, and we derive an explicit expression for the rate of formation of massive ground ice. Many problems of geophysical significance have domain geometries with large aspect ratio, and in Section 6 we show how the Dupuit–Forchheimer approximation can be applied in this case. Although the theoretical derivation is complicated, it is necessary for a proper theoretical understanding of the basis on which simpler theories can be properly posed. Section 7 then discusses the application of this reduced theory, and the conclusions follow in Section 8.

2 MATHEMATICAL MODELS

The simplest model of groundwater flow in a (saturated) porous medium is given by the equation of mass conservation,

$$\frac{\partial\phi}{\partial t}+\nabla.\mathbf{u}=0, \tag{2.1}$$

together with Darcy's law,

$$\mathbf{u}=-\frac{k}{\eta}\,[\nabla p+\rho_l g\mathbf{k}], \tag{2.2}$$

where ϕ is porosity, $\mathbf{u}$ is fluid flux, k is permeability, η is liquid viscosity, p is liquid pressure, ρ_l is liquid density, g is gravity and $\mathbf{k}$ is a unit vector pointing vertically upwards. The two equations for the three variables ϕ, $\mathbf{u}$, p must be supplemented by a further relationship, and this is the normal consolidation curve,

$$p_e=P-p=f(\phi), \tag{2.3}$$

where $f(\phi)$ is a monotonically decreasing function. Elimination of $\mathbf{u}$ and p then leads to the non-linear diffusion equation for ϕ in the form

$$\frac{\partial\phi}{\partial t}-\frac{\partial}{\partial z}\left\{\frac{\rho_l g k}{\eta}\right\}=\nabla.\left[\frac{k}{\eta}\,|f'(\phi)|\nabla\phi\right], \tag{2.4}$$

where the consolidation coefficient $k|f'(\phi)|/\eta$ is a diffusivity for the relaxation of ϕ (and hence p) to equilibrium. We refer to (2.4) as the Richards equation, by analogy with the similar equation derived in infiltration theory for unsaturated soils.

Biot (1941) extended this theory to allow for 3-D consolidation in which the isotropic law (2.3) is generalized to allow for more general linear elastic deformations of the soil. Essentially he takes (2.1) to (2.3) as above together with force balance for the medium and an elastic constitutive law relating stress to strain. Here we write such a model in the framework of a two-phase flow (Drew & Wood 1985), since properly speaking, one must conserve mass and momentum for both (solid and liquid) phases separately.

We consider each phase to be separately incompressible, with densities ρ_s and ρ_w, velocities $\mathbf{u}^s$ and $\mathbf{u}^w$, where the porosity is ϕ, and the solid velocity $\mathbf{u}^s$ is related to the solid strain $\mathbf{U}$ by

$$\mathbf{u}^s=\frac{\partial\mathbf{U}}{\partial t}+(\mathbf{u}^s.\nabla)\mathbf{U}, \tag{2.5}$$

which represents a material time derivative. Conservation of mass is then represented by

$$-\phi_t+\nabla.[(1-\phi)\mathbf{u}^s]=0,$$

$$\phi_t+\nabla.[\phi\mathbf{u}^w]=0, \tag{2.6}$$

where subscripts denote partial derivatives. Momentum conservation for the pore fluid is given by Darcy's law, which is

$$\phi(\mathbf{u}^w-\mathbf{u}^s)=-\frac{k}{\eta}\,[\nabla p+\rho_w g\mathbf{k}], \tag{2.7}$$

and the total momentum equation for the medium is simply a force balance, which can be written in the form

$$\nabla p=\nabla.\boldsymbol{\sigma}^e-[\rho_w\phi+\rho_s(1-\phi)]g\mathbf{k}, \tag{2.8}$$

where $\boldsymbol{\sigma}^e$ is Terzaghi's (1943) effective stress tensor, defined by

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}^e-p\boldsymbol{\delta}, \tag{2.9}$$

where $\boldsymbol{\sigma}$ is the total stress tensor $[=(1-\phi)\boldsymbol{\sigma}^s-\phi p\boldsymbol{\delta}$, where $\boldsymbol{\sigma}^s$ is the phase-averaged solid stress], and $\boldsymbol{\delta}$ is the unit tensor. Skempton (1960) (see also Verruijt 1984) discussed modifications to the definition of $\boldsymbol{\sigma}^e$ that have the effect of replacing p by $(1-a)p$ in (2.9) (and hence 2.8). In this paper we will take $a=0$. The nature of the effective stress is discussed lucidly by Bear & Bachmat (1990).

Biot poses an elastic relation between $\boldsymbol{\sigma}^e$ and the strain tensor $\boldsymbol{\varepsilon}$ given by

$$\boldsymbol{\varepsilon}=\frac{1}{2}\,[\nabla\mathbf{U}+(\nabla\mathbf{U})^{\mathrm{T}}]. \tag{2.10}$$

Biot's values $Q^{-1}=0$, $\alpha=1$ correspond to the model

$$\boldsymbol{\sigma}^e=2\mu\boldsymbol{\varepsilon}+\frac{2\mu\nu}{1-2\nu}\,\Delta\boldsymbol{\delta}, \tag{2.11}$$

where Δ is the dilation,

$$\Delta=\nabla.\mathbf{U}=\varepsilon_{kk}. \tag{2.12}$$

Biot further appends a constitutive relation analogous to (2.3),

$$\phi-\phi_0=\Delta, \tag{2.13}$$

where ϕ_0 is the undeformed porosity. We now wish to show that the assumption of (2.13) is illegal.

Non-dimensionalization

It is convenient at the outset to non-dimensionalize the equations. We choose a length scale d, a velocity scale $[u]=\bar{k}\rho_w g/\eta$ (where $\bar{k}$ is the permeability, or a typical value thereof), a stress scale $[\sigma]=\rho_w g d$, a strain scale $[U]=[\sigma]d/\mu$, and a timescale $[t]=[U]/[u]$. In terms of these scales, the equations can be written in the form

$$\nabla.\mathbf{U}=\Delta,$$

$$\nabla(p+z)=\nabla.\boldsymbol{\sigma}^e-r(1-\phi)\mathbf{k},$$

$$\boldsymbol{\sigma}^e=2\boldsymbol{\varepsilon}+\frac{2\nu}{1-2\nu}\Delta\boldsymbol{\delta}, \tag{2.14}$$

together with

$$\phi(\mathbf{u}^w-\mathbf{u}^s)=-k^*\nabla(p+z),$$

$$\nabla.\mathbf{u}^s=\nabla.[k^*\nabla(p+z)],$$

$$\phi_t=\varepsilon\nabla.[(1-\phi)\mathbf{u}^s],$$

$$\mathbf{u}^s=\frac{\partial\mathbf{U}}{\partial t}+\varepsilon(\mathbf{u}^s.\nabla)\mathbf{U}. \tag{2.15}$$

Here $k^*=k/\bar{k}$ is the dimensionless permeability, and the dimensionless parameters ε and r are given by

$$\varepsilon=\frac{[\sigma]}{\mu},\quad r=\frac{(\rho_s-\rho_l)}{\rho_l}. \tag{2.16}$$

Typically, we have in mind the flow in a self-consolidating layer of depth d. If a surface load larger than $\rho_w g d$ is prescribed, then we would choose $[\sigma]$ to be this value, and rescale the gravitational terms with $\rho_w g d/[\sigma]$; the discussion will hardly be affected.

The parameter ε is a measure of the strain. We expect it to be small in the near-surface environment. For example, if we take $\rho_w=10^3$ kg m^{-3}, $g=10$ m s^{-2}, $d=10^3$ m, $\mu=10^8$ Pa, then $\varepsilon=0.1$. Clearly, small values may be appropriate, although for larger depth scales, finite strains will be important. We call such large deformations *compaction*, and in these, ϕ changes significantly.

Boundary and initial conditions must be supplied to complete the model, but there is nothing unusual about these, and they will be discussed in due course.

3 ELASTIC DEFORMATIONS WITH SMALL STRAINS

We first examine the structure of the model equations (2.14) and (2.15) with ε taken to be small. Also, for simplicity, we take $k^*=1$, although this does not affect the argument. As $\varepsilon\to0$, the 'fluid' equations (2.15) imply $\mathbf{u}^s\approx\partial\mathbf{U}/\partial t$, whence

$$\nabla.\mathbf{u}^s\approx\frac{\partial\Delta}{\partial t}, \tag{3.1}$$

so that $(2.15)_2$ becomes

$$\frac{\partial\Delta}{\partial t}=\nabla.[k^*\nabla(p+z)]. \tag{3.2}$$

We can write (2.14) in the forms

$$\nabla(p+z)=\frac{2(1-\nu)}{1-2\nu}\nabla\Delta-\text{curl curl }\mathbf{U}-r(1-\phi)\mathbf{k},$$

$$=\frac{1}{1-2\nu}\nabla\Delta+\nabla^2\mathbf{U}-r(1-\phi)\mathbf{k}. \tag{3.3}$$

With $k^*=1$, (3.2) is the consolidation equation for Δ,

$$\frac{\partial\Delta}{\partial t}=\frac{2(1-\nu)}{1-2\nu}\nabla^2\Delta+r\frac{\partial\phi}{\partial z}. \tag{3.4}$$

Now mass conservation of solid, $(2.15)_3$, implies $\phi\approx\phi_0(\mathbf{x})$, where ϕ_0 is the initial porosity profile. Therefore, ϕ cannot be prescribed as in Biot's theory: it is determined from solid mass conservation. In fact, from $(2.15)_3$, we have

$$\phi_t\approx\nabla.\left[(1-\phi_0)\frac{\partial\mathbf{U}}{\partial t}\right]=\frac{\partial}{\partial t}\nabla.[(1-\phi_0)\mathbf{U}], \tag{3.5}$$

whence

$$\phi-\phi_0\approx\nabla.[(1-\phi_0)\mathbf{U}] \tag{3.6}$$

if $\mathbf{U}=\mathbf{0}$ at $t=0$. If ϕ_0 is uniform in space, then (3.6) simplifies further to

$$\phi-\phi_0\approx(1-\phi_0)\Delta, \tag{3.7}$$

which is similar, but not identical, to Biot's assumption (2.13). Biot's model can be reclaimed if his values of α and Q are taken as $\alpha=1-\phi_0$, $Q^{-1}=0$; the former corresponds to a choice of effective stress via

$$\boldsymbol{\sigma}^e=\boldsymbol{\sigma}+(1-\phi_0)p\boldsymbol{\delta}, \tag{3.8}$$

which may indeed be more appropriate than (2.9) when intergranular contact area is of a reasonable size.

To solve the model, we first solve for Δ from (3.4), given $\phi\approx\phi_0(\mathbf{x})$. Given this Δ, we then see that (3.3) together with (2.12) is a pair of Navier–Stokes-type equations that can be solved (in principle) to find $\mathbf{U}$ and p. The other variables can then be determined immediately. An application of this theory is given in Section 6 below.

4 ELASTIC–PLASTIC FLOW WITH SMALL STRAINS

In this section, we relax the assumption that the deformation is elastic. When soils are consolidated in isotropic stress, they deform along a path in (ϕ, p_e) space called the normal consolidation line (NCL). In triaxial tests, where an effective pressure p_e and a shear stress τ can be determined, the NCL forms part of a *yield surface* in (ϕ, p_e, τ) space (Atkinson & Bransby 1978; Schofield & Wroth 1968; Chen & Mizuno 1990). When the soil compacts on this surface, the deformation is elastic–plastic, and the generalization of the elastic relationship in (2.14) is determined as follows. First we can invert the elastic constitutive law by writing the trace as

$$\sigma^e_{kk}=\frac{2(1+\nu)\Delta}{1-2\nu}, \tag{4.1}$$

and hence

$$\boldsymbol{\varepsilon}=\frac{1}{2}\boldsymbol{\sigma}^e-\frac{\nu}{2(1+\nu)}\sigma^e_{kk}\boldsymbol{\delta}. \tag{4.2}$$

In plastic flow, the strains are irreversible, and the increments of the strains (from t to $t+dt$) are considered to be partly elastic as in (4.2):

$$d\varepsilon^{e}=\frac{1}{2}\,d\sigma^{e}-\frac{\nu}{2(1+\nu)}\,\sigma^{e}_{kk}\boldsymbol{\delta}\,, \tag{4.3}$$

where $d\varepsilon^{e}$ is the *elastic* strain increment (but $d\sigma^{e}$ is the *effective* stress increment); and partly plastic, so that the total strain increment is

$$d\varepsilon=d\varepsilon^{e}+d\varepsilon^{p}\,. \tag{4.4}$$

The determination of the plastic strain is a matter of measurement, but a common assumption is that of *normality*, for which the plastic strains are taken as perpendicular (in stress space) to the yield surface. If this is given by

$$\phi=g(\sigma^{e}_{ij})\,, \tag{4.5}$$

then this implies that

$$d\varepsilon^{p}_{ij}=\frac{\partial g}{\partial\sigma^{e}_{ij}}\,d\chi\,, \tag{4.6}$$

where the indeterminate scalar field χ plays a similar role to the pressure in the Navier–Stokes equations, and acts as a Lagrange multiplier, allowing (4.5) to be satisfied. (In the Navier–Stokes equation, the pressure allows the incompressibility condition $\nabla.\mathbf{u}=0$ to be satisfied.) We thus have the elastic–plastic strain increment equation in the form

$$d\varepsilon=\frac{1}{2}\,d\sigma^{e}-\frac{\nu}{2(1+\nu)}\,d\sigma^{e}_{kk}\boldsymbol{\delta}+\mathbf{G}d\chi\,, \tag{4.7}$$

where $G_{ij}=\partial g/\partial\sigma^{e}_{ij}$.

In order to illustrate the structure of the theory, we firstly assume $\mathbf{G}$ is constant. This is unrealistic for soils, but will be appropriate in small deformations from a uniform state; we also assume that strains are small, as in the preceding section. We can then invert (4.7) (and indeed integrate it in time) to obtain

$$\sigma^{e}=2\varepsilon+\frac{2\nu}{1-2\nu}\,\Delta\boldsymbol{\delta}-2\left[\mathbf{G}+\frac{\nu}{1-2\nu}\,G_{kk}\boldsymbol{\delta}\right]\chi\,. \tag{4.8}$$

We write $\gamma_{ij}=G_{ij}+\{\nu/(1-2\nu)\}G_{kk}\delta_{ij}$, and then from (2.14) we have

$$\nabla(p+z)=\frac{2(1-\nu)}{1-2\nu}\,\nabla\Delta-\text{curl curl }\mathbf{U}-r(1-\phi)\mathbf{k}-2\gamma.\nabla\chi\,, \tag{4.9}$$

whence the elastic–plastic consolidation equation is (with $k^{*}=1$)

$$\frac{\partial\Delta}{\partial t}=\frac{2(1-\nu)}{(1-2\nu)}\,\nabla^{2}\Delta+r\,\frac{\partial\phi}{\partial z}-2\gamma_{ij}\,\frac{\partial^{2}\chi}{\partial x_i\partial x_j}\,. \tag{4.10}$$

Suppose the scalar function χ is known. As before $\phi\approx\phi_0(\mathbf{x})$, and then (4.10) determines Δ. With χ given, (4.9) together with $\nabla.\mathbf{U}=\Delta$ will serve to determine p and $\mathbf{U}$ and hence also σ^{e}; finally χ is determined (in principle) from the extra condition (4.5). We see, however, that the problem of computing the solution is much less straightforward than in the elastic case.

If γ is not constant, then the incremental form of (4.8) must be used. This is

$$d\sigma^{e}=2d\varepsilon+\frac{2\nu}{1-2\nu}\,d\Delta\boldsymbol{\delta}-2\gamma d\chi\,, \tag{4.11}$$

and for small strains we have

$$\dot{\sigma}^{e}=\partial\sigma^{e}/\partial t=2\dot{\varepsilon}+\frac{2\nu}{1-2\nu}\,\dot{\Delta}\boldsymbol{\delta}-2\gamma\dot{\chi}\,, \tag{4.12}$$

where $\dot{\varepsilon}=\frac{1}{2}[\nabla\mathbf{u}^{s}+(\nabla\mathbf{u}^{s})^{T}]$, $\dot{\Delta}=\nabla.\mathbf{u}^{s}(=\partial\Delta/\partial t)$. We write $\xi=\nabla.\mathbf{u}^{s}=\dot{\Delta}$, then

$$\nabla.\dot{\sigma}^{e}=\frac{2(1-\nu)}{(1-2\nu)}\,\nabla\xi-\text{curl curl }\mathbf{u}^{s}-2\nabla.[\gamma\dot{\chi}]\,, \tag{4.13}$$

so that $(2.15)_2$ implies (with $k^{*}=1$ and $\partial\phi/\partial t\approx0$)

$$\frac{\partial\xi}{\partial t}=\frac{2(1-\nu)}{(1-2\nu)}\,\nabla^{2}\xi-2\,\frac{\partial^{2}}{\partial x_i\partial x_j}\,(\dot{\chi}\gamma_{ij})\,. \tag{4.14}$$

The complete set of equations governing the motion can then be written in the form

$$\frac{\partial\xi}{\partial t}=\frac{2(1-\nu)}{1-2\nu}\,\nabla^{2}\xi-2\,\frac{\partial^{2}}{\partial x_i\partial x_j}\,(\gamma_{ij}V)\,,$$

$$\frac{\partial\sigma^{e}_{ij}}{\partial t}=\frac{\partial u^{s}_{i}}{\partial x_j}+\frac{\partial u^{s}_{j}}{\partial x_i}+\frac{2\nu}{1-2\nu}\,\xi\delta-2V\gamma_{ij}\,,$$

$$\nabla\Pi=\frac{2(1-\nu)}{1-2\nu}\,\nabla\xi-\text{curl curl }\mathbf{u}^{s}-2\gamma.\nabla V\,,$$

$$\nabla.\mathbf{u}^{s}=\xi\,,$$

$$g(\sigma^{e}_{ij})=\phi\,, \tag{4.15}$$

where $V=\dot{\chi}$, $\Pi=\dot{p}$. Suppose V and γ_{ij} are known. Then $(4.15)_1$ determines ξ; then $(4.15)_{3,4}$ are Navier–Stokes-type equations to determine Π and $\mathbf{u}^{s}$ (whence p and $\mathbf{U}$); then σ^{e}_{ij} is determined from $(4.15)_2$, while $(4.15)_5$ determines V. The system of equations is fully coupled, but the above discussion indicates that the model is properly posed, and indicates an iterative scheme whereby the solution may be computed.

5 COMPACTION: ELASTIC–PLASTIC FLOW WITH LARGE STRAINS

Finally, we consider the situation where $\varepsilon=O(1)$, and we can choose d so that $\varepsilon=1$. This is the situation normally described by the term compaction, for example in sedimentary basins (Audet & Fowler 1992). Here we use the term compaction to distinguish large deformations from consolidation, where we suppose the strains to be small. Finite strain plasticity is described by Chen & Mizuno (1990) and by Khan & Huang (1995), for example. The governing equations can be written, from (2.14) and (2.15), as

$$\nabla.\mathbf{U}=\Delta\,,$$

$$\nabla(p+z)=\nabla.\sigma^{e}-r(1-\phi)\mathbf{k}\,,$$

$$\phi(\mathbf{u}^{w}-\mathbf{u}^{s})=-k^{*}\nabla(p+z)\,,$$

$$\nabla.\mathbf{u}^{s}=\nabla.[k^{*}\nabla(p+z)]\,,$$

$$\phi_t=\nabla.[(1-\phi)\mathbf{u}^{s}]\,,$$

$$\mathbf{u}^{s}=\frac{\partial\mathbf{U}}{\partial t}+(\mathbf{u}^{s}.\nabla)\mathbf{U}\,, \tag{5.1}$$

and must be supplemented by a suitable constitutive law. This is of the incremental form (4.11), but the difficulty here is that for large strains the differentials in (4.11) are essentially time derivatives following the solid; however, the tensor derivative $d\sigma_{ij}/dt_s = \partial\sigma_{ij}/\partial t + \mathbf{u}^s.\nabla\sigma_{ij}$ is not *objective* (i.e. frame-indifferent) and the stress increments must be defined differently. Two possibilities (there are others) are the *Jaumann derivative* and the *Truesdell derivative*. Of these, the Jaumann derivative preserves tensor invariants, and also seems more appropriate to the 'updated Lagrangian' description (Chen & Mizuno 1990), which refers the increments to the Lagrangian configuration at the preceding time instant. This is appropriate for soils or uncemented sediments, where the solid structure retains no long-range memory of its previous state (i.e. soil grains touch but are not tied). In this model, the (non-dimensional) elastic–plastic incremental stress is given by

$$d\sigma^{e,J} = 2d\varepsilon + \frac{2\nu}{1-2\nu}\,d\Delta\delta - 2\gamma d\chi\,, \tag{5.2}$$

where the Jaumann differential is given by

$$d\sigma^{e,J}_{ij} = d\sigma^e_{ij} + \sigma^e_{kj}d\Omega_{ik} - \sigma^e_{ik}d\Omega_{kj}\,, \tag{5.3}$$

and the spin tensor Ω_{ij} is given by

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial U_j}{\partial x_i} - \frac{\partial U_i}{\partial x_j}\right). \tag{5.4}$$

($d\sigma^e_{ij}$ refers to a material increment following the solid.) The strain derivatives refer to infinitesimal displacements (i.e. the linearized Eulerian strain rates), since they refer to displacements relative to the configuration at the preceding time instant.

The analogue of (4.12) is thus

$$\dot{\sigma}^{e,J} = 2\dot{\varepsilon} + \frac{2\nu}{1-2\nu}\,\xi\delta - 2\gamma V\,, \tag{5.5}$$

where $V = d\chi/dt_s$ $(=\partial\chi/\partial t + \mathbf{u}^s.\nabla\chi)$, $\dot{\varepsilon}$ is the usual Eulerian strain rate tensor, and $\xi = \nabla.\mathbf{u}^s$; the Jaumann derivative can be written

$$\dot{\sigma}^{e,J}_{ij} = \frac{\partial\sigma^e_{ij}}{\partial t} + \mathbf{u}^s.\nabla\sigma^e_{ij} + \omega_{ik}\sigma_{kj} - \sigma_{ik}\omega_{kj}\,, \tag{5.6}$$

where ω is the vorticity tensor,

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u^s_j}{\partial x_i} - \frac{\partial u^s_i}{\partial x_j}\right). \tag{5.7}$$

It is possible to derive analogues to (4.15) in the case $k^* = 1$; these are

$$\frac{d\xi}{dt} = \frac{2(1-\nu)}{(1-2\nu)}\nabla^2\xi - 2\nabla.\{\nabla.(V\gamma)\} - \nabla.\{\nabla.(\omega:\sigma^e - \sigma^e:\omega)\}$$
$$- \frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k} - \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_j}\frac{\partial\sigma^e_{ij}}{\partial x_k}\right) + \frac{\partial}{\partial z}[(1-\phi)\xi]\,, \tag{5.8}$$

$$\nabla\frac{dp}{dt} - \mathbf{e}_i\frac{\partial u_k}{\partial x_i}\frac{\partial p}{\partial x_k} = \frac{2(1-\nu)}{1-2\nu}\nabla\xi - \text{curl curl }\mathbf{u}^s - 2\nabla.(V\gamma)$$
$$- \nabla.[\omega:\sigma^e - \sigma^e:\omega] - \frac{\partial u_k}{\partial x_j}\frac{\partial\sigma^e_{ij}}{\partial x_k} + (1-\phi)\xi\mathbf{k}\,, \tag{5.9}$$

$$\frac{d\sigma^e}{dt} + [\omega:\sigma^e - \sigma^e:\omega] = 2\dot{\varepsilon} + \frac{2\nu}{1-2\nu}\,\xi\delta - 2\gamma V\,, \tag{5.10}$$

together with

$$\nabla.\mathbf{u}^s = \xi\,,$$
$$g(\sigma^e_{ij}) = \phi\,,$$
$$\frac{d\phi}{dt} = (1-\phi)\xi\,. \tag{5.11}$$

Here d/dt is the material derivative $\partial/\partial t + \mathbf{u}^s.\nabla$, following the solid. As before, (5.8) is a diffusive-type equation for ξ, (5.9) with $(5.11)_1$ should determine $\mathbf{u}^s$ and p, while σ^e is determined from (5.10). The remaining relations $(5.11)_{2,3}$ then serve to determine V and ϕ, respectively. We see, however, that the extra parts of the Jaumann derivative cause all the equations to become more strongly coupled.

6 APPLICATIONS: THE DUPUIT APPROXIMATION

We are most interested in *regional groundwater flow problems*, for which we expect the consolidation/compaction to be essentially 1-D, due to the naturally large horizontal scale of the system. That is, the vertical length scale may be of the order of 100–1000 m, while the horizontal length scale may be 10–1000 km, giving an aspect ratio between the two of the order of 10^{-1}–10^{-4}. We denote the aspect ratio by δ, and assume in what follows that δ is very small.

Two particular problems attract our attention. These are the growth of permafrost in Arctic regions (Fowler & Noon 1997), and the groundwater flow beneath ice sheets (Boulton & Dobbie 1993). In both of these it is necessary to determine the effective normal stress at the upper surface of the flow region.

In the case of permafrost growth, the migration of the frost line downwards is related to the formation of ice lenses at the surface, which also depends on the effective load N. Although in laboratory or small-scale field experiments this can be prescribed, on a regional scale it must be determined through a solution of the groundwater flow problem—which itself depends on the location of the frost line.

The groundwater flow beneath ice sheets is coupled to the ice sheet above through the fact that the effective pressure at the ice-sheet base depends on the subglacial drainage, which itself depends on the ice-sheet flow.

In both cases it is therefore of interest to determine a consistent model for the groundwater flow, and we show how to do this here by developing a version of the Dupuit–Forchheimer approximation based on $\delta \ll 1$, appropriate to consolidating soils.

We will consider only the simplest case, that of elastic small strains, which should be appropriate to depths $\lesssim 1$ km and for pre-consolidated sediments. For simplicity we consider a constant permeability. The basic equations are given in Sections 2 and 3, and may be written in the form (for $\varepsilon \to 0$ and $k^* = 1$)

$$\nabla.\mathbf{U} = \Delta\,,$$
$$\nabla(p+z) = \nabla.\sigma^e - r(1-\phi)\mathbf{k}\,,$$
$$\sigma^e = 2\varepsilon + \frac{2\nu}{1-2\nu}\Delta\delta\,,$$
$$\frac{\partial\Delta}{\partial t} = \frac{2(1-\nu)}{1-2\nu}\nabla^2\Delta + r\frac{\partial\phi}{\partial z}\,. \tag{6.1}$$

For these equations, we require suitable boundary conditions. We consider a domain such as that shown in Fig. 2. We suppose that the level $z=0$ is taken at the height of the (far-field) water table, and that groundwater flows in $z_b < z < z_f$, with impermeable basement below $z=z_b$ and a horizon at $z=z_f$. Depending on the application we have in mind, the region $z_s > z > z_f$ could represent an ice sheet, permafrost, or even simply the unsaturated zone, or it could represent another impermeable horizon.

At the sides of the domain (denoted by ∂D), we prescribe zero normal and tangential effective stress and hydrostatic pressure. At the base $z=z_b$, we have no solid strain, and no normal water flow. At the upper surface $z=z_f$, we prescribe the normal load due to the substrate above z_f, the normal effective pressure and zero tangential strain. The resultant forms of the boundary conditions are then the following, in dimensionless form:

$$\mathbf{U}\times\mathbf{n}=0,$$

$$-\sigma_3'+p=P_s,$$

$$-\sigma_3'=N,$$

$$\phi(u_3^w-u_3^s)=U_f, \quad \text{on } z=z_f;$$

$$\mathbf{U}=0, \quad \mathbf{n}.\nabla(p+z)=0 \quad \text{on } z=z_f;$$

$$p+z=0, \quad \boldsymbol{\sigma}.\mathbf{n}=0 \quad \text{on } \partial D. \tag{6.2}$$

Here, $\mathbf{n}$ is a unit normal, $\sigma_3'=\sigma_{33}^e$ is the normal effective stress, and

$$P_s=\rho_f(z_s-z_f)/\rho_w \tag{6.3}$$

is the dimensionless surface load, where ρ_f is the density of the region above z_f. The effective load N and water flux U_f cannot both be prescribed. Either one or the other, or a relationship between the two, can be given, and this will be determined by the process dynamics at z_f.

Now we rescale the equations using the small aspect ratio. The coordinates are x, y horizontal, and z vertical. The corresponding components of the strain are taken as (U_1, U_2, U_3), while the effective stress tensor is written

$$\boldsymbol{\sigma}^e=\begin{pmatrix}\sigma_1' & \tau_{12} & \tau_{13}\\ \tau_{12} & \sigma_2' & \tau_{23}\\ \tau_{13} & \tau_{23} & \sigma_3'\end{pmatrix}; \tag{6.4}$$

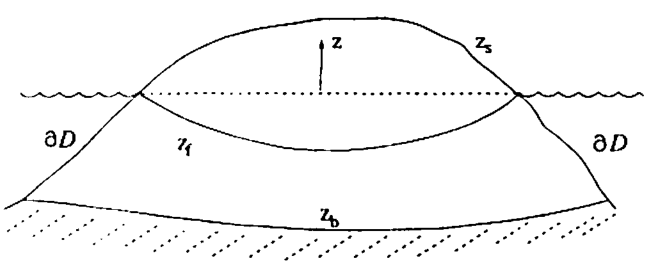

Figure 2. Regional groundwater flow geometry. The flow is between an impermeable basement $z=z_b$ and an upper surface $z=z_f$. The side boundaries are denoted ∂D.

the component-wise forms of eqs (6.1) are then

$$\frac{\partial p}{\partial x}=\frac{\partial\sigma_1'}{\partial x}+\frac{\partial\tau_{12}}{\partial y}+\frac{\partial\tau_{13}}{\partial z},$$

$$\frac{\partial p}{\partial y}=\frac{\partial\tau_{12}}{\partial x}+\frac{\partial\sigma_2'}{\partial y}+\frac{\partial\tau_{23}}{\partial z},$$

$$\frac{\partial p}{\partial z}+1=\frac{\partial\tau_{13}}{\partial x}+\frac{\partial\tau_{23}}{\partial y}+\frac{\partial\sigma_3'}{\partial z}-r(1-\phi),$$

$$\sigma_1'=2\frac{\partial U_1}{\partial x}+\frac{2\nu}{1-2\nu}\Delta,$$

$$\sigma_2'=2\frac{\partial U_2}{\partial y}+\frac{2\nu}{1-2\nu}\Delta,$$

$$\sigma_3'=2\frac{\partial U_3}{\partial z}+\frac{2\nu}{1-2\nu}\Delta,$$

$$\tau_{12}=\frac{\partial U_1}{\partial y}+\frac{\partial U_2}{\partial x},$$

$$\tau_{13}=\frac{\partial U_1}{\partial z}+\frac{\partial U_3}{\partial x},$$

$$\tau_{23}=\frac{\partial U_2}{\partial z}+\frac{\partial U_3}{\partial y},$$

$$\Delta=\frac{\partial U_1}{\partial x}+\frac{\partial U_2}{\partial y}+\frac{\partial U_3}{\partial z},$$

$$\frac{\partial\Delta}{\partial t}=\frac{2(1-\nu)}{1-2\nu}\left[\frac{\partial^2\Delta}{\partial z^2}+\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial^2\Delta}{\partial y^2}\right]+r\frac{\partial\phi}{\partial z}. \tag{6.5}$$

We rescale the variables as follows:

$$x,\, y\sim 1/\delta,$$

$$U_1,\, U_2\sim\delta,$$

$$\tau_{13},\, \tau_{23}\sim\delta,$$

$$\tau_{12}\sim\delta^2. \tag{6.6}$$

With the variables rescaled in this way, (6.5) becomes

$$\frac{\partial p}{\partial x}=\frac{\partial\sigma_1'}{\partial x}+\frac{\partial\tau_{13}}{\partial z}+\delta^2\frac{\partial\tau_{12}}{\partial y},$$

$$\frac{\partial p}{\partial y}=\frac{\partial\sigma_2'}{\partial y}+\frac{\partial\tau_{23}}{\partial z}+\delta^2\frac{\partial\tau_{12}}{\partial x},$$

$$\frac{\partial p}{\partial z}+1=\frac{\partial\sigma_3'}{\partial z}-r(1-\phi)+\delta^2\left(\frac{\partial\tau_{13}}{\partial x}+\frac{\partial\tau_{23}}{\partial y}\right),$$

$$\sigma_1'=\frac{2\nu}{1-2\nu}\Delta+2\delta^2\frac{\partial U_1}{\partial x},$$

$$\sigma_2'=\frac{2\nu}{1-2\nu}\Delta+2\delta^2\frac{\partial U_2}{\partial y},$$

$$\sigma_3'=\frac{2\nu}{1-2\nu}\Delta+2\frac{\partial U_3}{\partial z},$$

$$\tau_{12}=\frac{\partial U_1}{\partial y}+\frac{\partial U_2}{\partial x},$$

$$\tau_{13}=\frac{\partial U_1}{\partial z}+\frac{\partial U_3}{\partial x},$$

$$\tau_{23}=\frac{\partial U_2}{\partial z}+\frac{\partial U_3}{\partial y},$$

$$\Delta=\frac{\partial U_3}{\partial z}+\delta^2\left(\frac{\partial U_1}{\partial x}+\frac{\partial U_2}{\partial y}\right),$$

$$\frac{\partial \Delta}{\partial t}=\frac{2(1-\nu)}{(1-2\nu)}\left[\frac{\partial^2\Delta}{\partial z^2}+\delta^2\left(\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial^2\Delta}{\partial y^2}\right)\right]+r\,\frac{\partial \phi}{\partial z}. \tag{6.7}$$

The boundary conditions can be written in the form

$$\mathbf{U}_{\rm H}=\mathbf{0},$$
$$-\sigma_3'+p=P_s,$$
$$-\sigma_3'=N,$$
$$-\left(\frac{\partial p}{\partial z}+1\right)=U_f,\quad \text{on } z=z_f;$$
$$\mathbf{U}=\mathbf{0},\qquad \frac{\partial p}{\partial z}+1=\delta^2\nabla_{\rm H}z_b.\nabla_{\rm H}p,\quad \text{on } z=z_b;$$
$$p+z=0,$$
$$\tau_{13}=\sigma_1'\,\frac{\partial z_s}{\partial x}+\delta^2\tau_{12}\,\frac{\partial z_s}{\partial y},$$
$$\tau_{23}=\sigma_2'\,\frac{\partial z_s}{\partial y}+\delta^2\tau_{12}\,\frac{\partial z_s}{\partial x},$$
$$\sigma_3'=\delta^2\left[\tau_{13}\,\frac{\partial z_s}{\partial x}+\tau_{23}\,\frac{\partial z_s}{\partial y}\right],\quad \text{on } z=z_s. \tag{6.8}$$

Here $\mathbf{U}_{\rm H}=(U_1,U_2)$, $\nabla_{\rm H}=(\partial/\partial x,\partial/\partial y)$, and z_s denotes the side ∂D: it is the part of z_s in Fig. 2 that lies in $z<0$.

To leading order as $\delta\to 0$, we have the following approximate system:

$$\frac{\partial p}{\partial z}+1=\frac{\partial \sigma_3'}{\partial z}-r(1-\phi),$$
$$\sigma_1'=\frac{2\nu}{1-2\nu}\Delta=\sigma_2',$$
$$\sigma_3'=\frac{2\nu}{1-2\nu}\Delta+2\frac{\partial U_3}{\partial z},$$
$$\Delta=\frac{\partial U_3}{\partial z},$$
$$\frac{\partial \Delta}{\partial t}=\frac{2(1-\nu)}{(1-2\nu)}\frac{\partial^2\Delta}{\partial z^2}+r\,\frac{\partial\phi}{\partial z}, \tag{6.9}$$

while to leading order, the relevant boundary conditions are

$$U_3=0,\quad \frac{\partial p}{\partial z}+1=0\quad \text{on } z=z_b;$$
$$-\sigma_3'+p=P_s,$$
$$-\sigma_3'=N,$$
$$-\left(\frac{\partial p}{\partial z}+1\right)=U_f\quad \text{on } z=z_f. \tag{6.10}$$

We omit the boundary conditions on z_s for the moment.

From (6.9) we have

$$\frac{\partial p}{\partial z}+1=\frac{\partial \sigma_3'}{\partial z}-r(1-\phi),$$
$$\sigma_3'=\frac{2(1-\nu)}{(1-2\nu)}\Delta,$$
$$\frac{\partial U_3}{\partial z}=\Delta. \tag{6.11}$$

Hence the equation for Δ can be written, to leading order, as

$$\frac{(1-2\nu)}{2(1-\nu)}\frac{\partial \sigma_3'}{\partial t}=\frac{\partial}{\partial z}\left[\frac{\partial\sigma_3'}{\partial z}-r(1-\phi)\right], \tag{6.12}$$

where σ_3' satisfies

$$\sigma_3'=-N\text{ on } z=z_f,$$
$$\frac{\partial\sigma_3'}{\partial z}=r(1-\phi)\text{ on } z=z_b. \tag{6.13}$$

After an initial transient, it follows that the solution is of approximate hydrostatic and lithostatic equilibrium: (6.12) (with $\partial/\partial t=0$) and (6.13) imply

$$\frac{\partial\sigma_3'}{\partial z}\approx r(1-\phi) \tag{6.14}$$

everywhere, while $(6.11)_1$ and $(6.10)_1$ imply

$$\frac{\partial p}{\partial z}+1\approx 0; \tag{6.15}$$

hence

$$\sigma_3'\approx-\left[N+r\int_z^{z_f}(1-\phi)\,dz\right],$$
$$p\approx P_s-N+(z_f-z). \tag{6.16}$$

Now in general z_f and N will vary with x and t, and these variations drive small lateral flows. To account for these, we use (6.16) to seek improved values for σ_3' and p. The following procedure is essentially a generalization of the classic Dupuit–Forchheimer approximation, applied here to elastically consolidating porous media flows.

We define

$$\mathbf{U}_{\rm H}=(U_1,U_2),\quad \boldsymbol{\tau}_{\rm H}=(\tau_{13},\tau_{23}), \tag{6.17}$$

so that from (6.7) we have, using $(6.11)_2$,

$$\frac{\partial\boldsymbol{\tau}_{\rm H}}{\partial z}=\nabla_{\rm H}\left[p-\frac{\nu\sigma_3'}{1-\nu}\right]+O(\delta^2), \tag{6.18}$$

and also

$$\frac{\partial\mathbf{U}_{\rm H}}{\partial z}=\boldsymbol{\tau}_{\rm H}-\nabla_{\rm H}U_3,$$
$$\frac{\partial U_3}{\partial z}=\frac{(1-2\nu)}{2(1-\nu)}\sigma_3'+O(\delta^2), \tag{6.19}$$

whence

$$\frac{\partial^2\mathbf{U}_{\rm H}}{\partial z^2}=\nabla_{\rm H}\left[p-\frac{\nu}{1-\nu}\sigma_3'\right]-\frac{(1-2\nu)}{2(1-\nu)}\nabla_{\rm H}\sigma_3'+O(\delta^2), \tag{6.20}$$

with boundary conditions $\mathbf{U}_{\rm H}=0$ on z_f and z_b. We use the approximations for p and σ_3' in (6.16) to compute $\mathbf{U}_{\rm H}$ and U_3 (with $U_3=0$ on z_b) to leading order, and then

$$\nabla_{\rm H}.\boldsymbol{\tau}_{\rm H}=\frac{\partial}{\partial z}\nabla_{\rm H}.\mathbf{U}_{\rm H}+\nabla_{\rm H}^2U_3, \tag{6.21}$$

which is the correction in the vertical force balance,

$$\frac{\partial p}{\partial z}+1=\frac{\partial\sigma_3'}{\partial z}-r(1-\phi)+\delta^2\nabla_{\rm H}.\boldsymbol{\tau}_{\rm H}. \tag{6.22}$$

The consolidation equation can be written as

$$\delta^2\left[\frac{\partial\Delta}{\partial\tau}-\frac{2(1-\nu)}{(1-2\nu)}\nabla_{\rm H}^2\Delta\right]=\frac{2(1-\nu)}{(1-2\nu)}\frac{\partial^2\Delta}{\partial z^2}+r\,\frac{\partial\phi}{\partial z}, \tag{6.23}$$

where we put

$$t=\tau/\delta^2, \tag{6.24}$$

in anticipation of the controlling lateral-flow timescale. We also have

$$\frac{2(1-\nu)}{(1-2\nu)}\Delta=\sigma_3'+2\delta^2\nabla_H\cdot\mathbf{U}_H. \tag{6.25}$$

With $\boldsymbol{\tau}_H$ and $\mathbf{U}_H$ calculated from the leading-order expressions, (6.22), (6.23) and (6.25) are three equations for the variables Δ, σ_3', p. [Note that the correction to $\partial\phi/\partial z$ is $O(\varepsilon\delta^2)\ll O(\delta^2)$, since $U^* = \delta^2\partial\mathbf{U}/\partial\tau\sim O(\delta^2)$.]

We use (6.25) to write (6.23) as

$$\frac{\partial}{\partial z}\left[\frac{\partial\sigma_3'}{\partial z}-r(1-\phi)\right]=\delta^2\left[\frac{(1-2\nu)}{2(1-\nu)}\frac{\partial\sigma_3'}{\partial\tau}-\nabla_H^2\sigma_3'-2\frac{\partial^2}{\partial z^2}(\nabla_H\cdot\mathbf{U}_H)\right]. \tag{6.26}$$

The boundary conditions are then

$$\frac{\partial p}{\partial z}+1=-\delta^2V_f \quad \text{on } z=z_f, \tag{6.27}$$

where we write $U_f=\delta^2V_f$; from (6.22) this implies

$$\frac{\partial\sigma_3'}{\partial z}-r(1-\phi)=\delta^2[-V_f-\nabla_H\cdot\boldsymbol{\tau}_H] \quad \text{on } z=z_f. \tag{6.28}$$

In addition we have from (6.8) (using 6.22 and 6.16)

$$\frac{\partial\sigma_3'}{\partial z}-r(1-\phi)=\delta^2[-\nabla_H\cdot\boldsymbol{\tau}_H+\nabla_Hz_b\cdot\nabla_H(P_s-N+z_f)] \quad \text{on } z=z_b. \tag{6.29}$$

Integration of (6.26) between z_b and z_f finally yields the evolution equation

$$\int_{z_b}^{z_f}\left[\frac{(1-2\nu)}{2(1-\nu)}\frac{\partial\sigma_3'}{\partial\tau}-\nabla_H^2\sigma_3'-2\frac{\partial^2}{\partial z^2}(\nabla_H\cdot\mathbf{U}_H)\right]dz$$
$$=\{-V_f-\nabla_H\cdot\boldsymbol{\tau}_H\}|_{z_f}$$
$$-\{-\nabla_H\cdot\boldsymbol{\tau}_H+\nabla_Hz_b\cdot\nabla_H(P_s-N+z_f)\}|_{z_b}, \tag{6.30}$$

in which σ_3' is given by $(6.16)_1$, $\mathbf{U}_H$ by solving (6.20), U_3 by solving (6.19), and $\nabla_H\cdot\boldsymbol{\tau}_H$ from (6.21). Using (6.21) and (6.19), (6.30) can be simplified to

$$\int_{z_b}^{z_f}\left[\frac{(1-2\nu)}{2(1-\nu)}\frac{\partial\sigma_3'}{\partial\tau}-\nabla_H^2p\right]dz=-V_f-\nabla_Hz_b\cdot\nabla_H(P_s-N+z_f)|_{z_b}. \tag{6.31}$$

7 SUBGLACIAL GROUNDWATER FLOW

We now apply this theory to the consideration of conditions under large ice sheets in the last ice age. Boulton & Dobbie (1993) studied groundwater flow under the Fennoscandian ice sheet and deduced that subglacial meltwater could realistically be evacuated through the underlying sediments. Partly this conclusion rested on the assumption that both vertical and horizontal pressure gradients were uniform (i.e. in equilibrium).

In order to examine the basis for this conclusion, we focus on the basic governing equation (6.31). In general, it is of differential type for the effective pressure N at the ground surface [assuming the vertical groundwater flux V_f is given, or a prescribed function of N; if N is given, then (6.31) determines V_f]. The boundary conditions follow from the requirement that σ_3' is prescribed at the sides. In particular $-\sigma_3'=N=0$ at the intersection of z_s and z_f, which will be a curve ∂A bounding the area A in which the flow takes place. We therefore prescribe

$$N=0 \quad \text{on } \partial A. \tag{7.1}$$

A general procedure to solve (6.31) will be numerical, since in general z_f, ϕ, z_b may all be functions of space and time. The simplest case to consider is when z_f, z_b, P_s and ϕ are constants. Then, from (6.16),

$$\sigma_3'=-[N+r(1-\phi)(z_f-z)],$$
$$p=P_s-N+(z_f-z). \tag{7.2}$$

It follows that (6.31) can be written in the form

$$\int_{z_b}^{z_f}\left[-\frac{(1-2\nu)}{2(1-\nu)}\frac{\partial N}{\partial\tau}+\nabla_H^2N\right]dz=-V_f,$$

and thus (if $z_f-z_b=1$)

$$\frac{(1-2\nu)}{2(1-\nu)}\frac{\partial N}{\partial\tau}=\nabla_H^2N+V_f, \tag{7.3}$$

an equation of diffusive type. This equation and the earlier (6.12) form the basis of our analysis. The solution N of (7.3) relaxes over a time τ of $O(1)$ to an equilibrium drainage state. Similarly, the solution of (6.12) relaxes to a state of hydrostatic equilibrium on the shorter timescale $t=O(1)$. It is convenient to write (7.3) in dimensional form, using the scales introduced before (2.14):

$$\frac{(1-2\nu)}{2(1-\nu)}\frac{\partial N^*}{\partial t^*}=\frac{\mu k}{\eta}\nabla_H^{*2}N^*+\frac{\mu U_f^*}{d}, \tag{7.4}$$

where the starred variables denote dimensional quantities, thus $N^*=[\sigma]N$, etc. We see that the *consolidation coefficient* c is defined by

$$c=\frac{2(1-\nu)}{(1-2\nu)}\frac{\mu k}{\eta}, \tag{7.5}$$

which is the same as that given by Biot. This equation defines a *lateral consolidation timescale* for regional groundwater flow,

$$t_{gf}=\frac{\eta l^2}{\mu k}, \tag{7.6}$$

where l is the lateral extent of the region. In a similar way, we can define the hydrostatic equilibrium time t_{hyd} for (6.12) as

$$t_{hyd}=\delta^2 l^2=\frac{\eta d^2}{\mu k}, \tag{7.7}$$

where d is the vertical scale. If we take values $\mu=10^8$ Pa, $l=1000$ km, $\eta=10^{-3}$ Pa s, $\bar{k}\sim10^{-10}$–10^{-16} m^2 (sand → clay), then we have

$$t_{gf}\sim10^4-10^{10}\ \text{yr} \quad (\text{sand}\rightarrow\text{clay}), \tag{7.8}$$

with the vertical consolidation timescale being a factor of 10^6 smaller (for a vertical scale of 1 km).

In considering the growth and decay of ice sheets, the relevant timescale is

$$t_{is}\sim10^3-10^5\ \text{yr}. \tag{7.9}$$

so that a generally reasonable assumption would be that

$$t_{hyd} \ll t_{is} \ll t_{gf} . \tag{7.10}$$

This implies that in the groundwater flow beneath ice sheets, the pore pressure (except, perhaps, in clays) rapidly relaxes towards hydrostatic equilibrium, but that equilibrium of the lateral flow will *not* be approached, in contradiction to Boulton & Dobbie's (1993) assumption. In fact, for U_f^* varying on a timescale $t_{is} \ll t_g$, the solution of (7.4) will be the secular solution

$$N^* \approx \frac{2\mu(1-\nu)}{(1-2\nu)(1-\phi)} \int^{t^*} (U_f^*/d)\, dt^* \tag{7.11}$$

away from the margins, where lateral boundary layers of width

$$l_{bl} \sim l(t_{is}/t_{gf})^{1/2} \tag{7.12}$$

will exist, in which N^* decreases to zero. Only for the most permeable sands can the lateral gradients of effective pressure approach equilibrium. In particular, it is generally inadmissible to assume that subglacial meltwater can be evacuated through groundwater flow. Instead, the sediments expand to accommodate the meltwater.

In fact, from (7.11) we see that $N^* < 0$ if $U_f^* < 0$; that is, any meltwater flux from an ice sheet causes negative effective normal stress. This is unphysical, and implies that failure at z_f and a channelized drainage flow must exist at the bed z_f (Walder & Fowler 1994). In this case N^* is determined by the hydraulic drainage dynamics, and then (7.4) simply determines U_f^*.

Massive ice formation

When an ice sheet advances, permafrost develops in front of the ice; similarly, when an ice sheet thins, permafrost develops in the ground below. In this situation, we can expect frost heave to occur, and there is a suction of groundwater upwards towards the freezing front, i.e. U_f^* is positive. It has been suggested (Noon 1996; Fowler & Noon 1997) that such subglacial frost heave is responsible for the formation of massive ground ice (Mackay 1971; Mackay & Dallimore 1992). In addition, the explanation of Heinrich events through repeated surges of the Laurentide ice sheet (MacAyeal 1993) requires repeated freezing on of subglacial sediments (Alley & MacAyeal 1994), and this can be accomplished through formation of a massive ice lens below the frozen sediment surface (Fowler & Noon 1997).

A typical frost-heave velocity scale is $U_{fh} = q/\rho_w L$, where q is the heat flux into the ice sheet (or from the ground), and L is latent heat. A geothermal heat flux of 60 mW m^{-2} gives a value of $U_{fh} \sim 2\times 10^{-10}$ m s^{-1}. From eq. (13) of Fowler & Noon (1997), we find that the heave rate (and thus the value of U_f^*) is

$$U_f^* = \alpha_{fh}(N^*)U_{fh} , \tag{7.13}$$

where α_{fh} is a decreasing function of N^*. Although it is a very complicated function, an adequate representation is

$$\alpha_{fh} = \alpha_0 \exp[-\gamma N^*/N_{fh}] ; \tag{7.14}$$

here α_0 is a heave parameter depending primarily on the permeability of the soil; it takes typical values of 1 for clay, 10^3 for silt and 10^8 for sand (Fowler & Krantz 1994), while reasonable values of γ and N_{fh} (the former from permeability variation, the latter from the suction characteristic) are 8 and 10^5 Pa. If we define

$$N^* = \frac{N_{fh}\lambda}{\gamma} , \quad t^* = \frac{(1-2\nu)(1-\phi)}{2(1-\nu)} \left(\frac{N_{fh} d}{\gamma\mu\alpha_0 U_{fh}}\right)\xi , \tag{7.15}$$

then $d\lambda/d\xi = \exp(-\lambda)$, and if $\lambda = 0$ when $\xi = 0$, then

$$\lambda = \ln[1+\xi] . \tag{7.16}$$

This indicates that effective pressure grows indefinitely (but slowly), and that heaving decreases, becoming small when λ is large. The relevant timescale over which heaving is significant is then $t_{heave} = N_{fh} d/\mu\gamma\alpha_0 U_{fh}$, which is small. For example, if $N_{fh}/\mu \sim 10^{-3}$, $\gamma = 8$, $\alpha \sim 1$, $d \sim 1000$ m, $U_{fh} \sim 2\times 10^{-10}$ m s^{-1}, then $t_{heave} \sim 20/\alpha_0$ yr. For values of $t > t_{heave}$, $\xi \gg 1$, $\lambda \simeq \ln\xi$, the heave rate is

$$U_f^* \approx \frac{(1-2\nu)(1-\phi)}{2(1-\nu)} \left(\frac{N_{fh}}{\gamma\mu}\right) \frac{d}{(t_{heave} + t^*)} , \tag{7.17}$$

and the massive ice thickness is approximately

$$d_f^* \approx \frac{(1-2\nu)(1-\phi)}{2(1-\nu)} \left(\frac{N_{fh} d}{\gamma\mu}\right) \ln(t^*/t_{heave}) . \tag{7.18}$$

Over glacial timescales of 10^5 yr, thicknesses of the order of metres can be obtained in this way, and such thicknesses are observed.

8 CONCLUSIONS

Our principal motivation in this paper has been to address the problem of determining the effective pressure and hence the groundwater flow in consolidating sediments on a regional scale. The results of our analyses are two-fold. First, the obvious route of generalizing Biot's (1941) classical theory of poroelasticity to large strains or an elastic–plastic rheology cannot be followed immediately, since Biot's theory is in fact conceptually inconsistent, although in its context (small strains, elastic rheology, uniform medium) it is actually effectively correct. By posing the poro-compactive model in terms of two-phase flow, we have indicated the way in which appropriate (but complicated) models can be derived.

Our second achievement has been to show how a Dupuit-type approximation can be used to obtain a diffusion equation to describe the lateral transport of groundwater beneath ice sheets or permafrost, and although this procedure has been carried out here for the limited case of small strains and an elastic rheology, it can presumably be applied in more complicated cases as well. Moreover, although (7.3) is very similar to (3.4), there is in fact no obvious way in which the connection can be made.

Study of the reduced equation for effective pressure indicates that when subglacial meltwater is generated, the effective pressure will remain near zero at the bed, and a basal drainage system will always develop. *Below the ice the pore pressure is* locally hydrostatic, but is not equilibrated horizontally. When freezing occurs, basal freeze-on of sediments can then occur, and thicknesses of massive ice lenses of the order of metres can be formed through the mechanism of frost heave.

Finally, we compare the relevance of the ideas presented here to models of viscous compaction, as are appropriate in the description of magma transport in the Earth's asthenosphere

(McKenzie 1984; Scott & Stevenson 1986), or in sedimentary basins (Birchwood & Turcotte 1994). In these situations, the models are simpler, but pressure solution (in sedimentary rocks) or dislocation creep (in partially molten mantle rocks) both lead to constitutive relations that relate the effective pressure p_e to the dilatation $\nabla.\mathbf{u}^s$. This is essentially because the effective pressure is related to $d[1-\phi]/dt_s$, the derivative of solid volume following the solid, which from (2.6) is just $-(1-\phi)\nabla.\mathbf{u}^s$. It is also reasonable to suppose that if the compactive process is viscous, then a viscous rheology is appropriate; thus the basic viscous compactive model is (2.6)–(2.8), together with a viscous equivalent of (2.11), and the compactive relation between p_e and $\nabla.\mathbf{u}^s$ then takes the place of an equation of state. Dupuit-type approximations (now termed lubrication approximations) may be suitable, but there is no conceptual difficulty in describing such poro-viscous models. It is, however, important to realize that the description of the medium rheology and the compaction relation are conceptually distinct, and this distinction has tended to be blurred in the literature (McKenzie 1984). It is, for example, conceivable to have an elastic medium with a viscous compaction law.

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