18 Dunes and Drumlins

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18.1 Introduction

18.1.1 Dunes

Dunes are landforms which occur when a turbulent fluid flow occurs above an erodible substrate. The most obvious example occurs in deserts, where the wind blows sand into a wide variety of different shapes (see Chap. 17 for many illustration of such dunes). Linear dunes, or 'seifs', are ridges which form parallel to the prevailing wind direction, while transverse dunes are ridges perpendicular to the wind. A variety of other shapes can occur, amongst them star dunes and barchan dunes.

Dunes also occur under rivers, for similar reasons. Because the flow in this situation is uni-directional, such exotica as star dunes do not occur. On the other hand, when the flow is rapid enough, *anti-dunes* occur; these are associated with waves at the water surface which are in phase with the underlying bed forms.

Dunes occur due to an instability which arises through a coupling between the bed transport rate and the overlying flow. In rivers and deserts, the bed material is transported (in rivers as *bedload*) through the imposition of a wind or water driven shear stress. If a perturbation in the bed elevation occurs, then the increased roughness alters the bed shear stress, and hence the bed transport rate. It is this feedback which causes the instability. As we shall see, the instability relies crucially on the fact that the perturbed shear stress is out of phase with the perturbed bed form.

18.1.2 Drumlins

Drumlins are small oval hills. They occur in swarms in regions which were formerly covered by ice sheets (in the last ice age). For example, much of the northern part of Ireland is covered by drumlins. They are typically formed of subglacial *till*, which is a dispersion of coarse, angular rock fragments in a matrix of finer grained material. Drumlins have typical dimensions of 100–1000 metres in length, and 10–50 metres elevation. As in the case for dunes, drumlins come in many different forms. In particular, analogues to various different dune types exist. Rogen moraine consists of ridges aligned perpendicular to the former ice flow, while glacial flutes and mega-flutes are lineations parallel to the ice flow. The three-dimensional drumlin forms themselves also have varying styles, such as spindle drumlins and barchanoid drumlins, and it has been suggested that

they occur through the action of massive subglacial floods, which erode the bedforms by analogy with dunes. While this is not inconceivable, it does require subglacial floods on a scale more massive than is commonly thought possible.

In this paper we show how an erosional theory for dune formation can be provided, and we will also show how the analogous theory for ice sheet flow can predict drumlin-forming instabilities, despite the vast disparity in Reynolds number. Notes and references to some of the literature can be found at the end of the paper.

18.2 Dunes

The basic geometry of the system is shown in Fig. 18.1. The water surface is $z = \eta$, where z is a coordinate normal to the mean bed slope, and the bed is $z = s$. For simplicity we consider only two-dimensional motions, so that $s = s(x,t)$, $\eta = \eta(x,t)$. The water depth is thus

$$h = \eta - s\,, \tag{18.1}$$

and the basic equation which describes the evolution of s is the *Exner equation*

$$(1-n)\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0\,. \tag{18.2}$$

Here, n is the porosity of the bed and q is the bedload transport, usually written as a prescribed function of the mean basal shear stress τ; a typical example is the Meyer–Peter and Müller [23] law

$$q = q(\tau) = C(\tau - \tau_c)_+^{3/2}, \tag{18.3}$$

where $[x]_+ = \max(x, 0)$, and τ_c is a yield stress, called the *Shields stress* (after Shields [30]); bedload transport only occurs for stresses above this value.

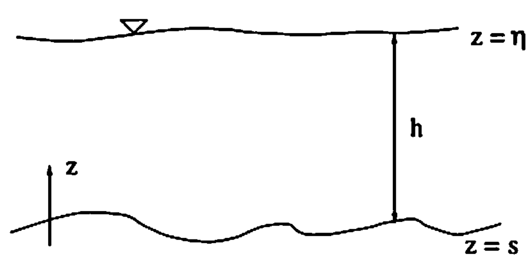

Fig. 18.1. Geometry for dune model. Water of depth h flows over an erodible bed $z = s$

The bedload transport q is an increasing function of τ, which itself depends on the mean flow velocity $\bar{u}$, for example we can take

$$\tau = f\rho_w\bar{u}^2\,, \tag{18.4}$$

where ρ_w is the density of water, and f is a dimensionless friction factor having a typical value of 0.05. It varies somewhat with flow speed and bed roughness, but can be taken as constant in an initial study.

18.2.1 St. Venant Equations

The equations (18.1)–(18.4) provide four equations for the six variables $s, q, \tau, \bar{u}, h, \eta$, and two further equations are necessary to complete the set. These arise from mass and momentum conservation, and an attractive possibility is to use the St. Venant equations to express these.

The St. Venant equations are the classical averaged equations which are used to describe turbulent river flow. They can be derived by taking cross-sectional averages (or, for a wide channel with no cross-stream variation of depth, depth averages) of the Navier–Stokes equations. For a two-dimensional velocity field $(u, 0, w)$ down a slope S, using depth averages, we obtain

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_s^\eta u\,\mathrm{d}z = 0,$$

$$\frac{\partial}{\partial t}\int_s^\eta u\,\mathrm{d}z + \frac{\partial}{\partial x}\int_s^\eta u^2\,\mathrm{d}z = gh(S - \eta_x) - \frac{\mu}{\rho_w}\left.\frac{\partial u}{\partial z}\right|_{z=s}, \tag{18.5}$$

using in addition the assumption of a shallow flow, so that the pressure is approximately hydrostatic,

$$p \approx \rho_w g(\eta - z), \tag{18.6}$$

and neglecting longitudinal stress terms.

The average velocity $\bar{u}$ is defined by

$$\bar{u} = \frac{1}{h}\int_s^\eta u\,\mathrm{d}z, \tag{18.7}$$

and the system is closed by the two additional constitutive assumptions, that

$$\int_s^\eta u^2\mathrm{d}z = h\bar{u}^2, \tag{18.8}$$

and that the basal shear stress is

$$\tau = \mu\left.\frac{\partial u}{\partial z}\right|_{z=s} = f\rho_w\bar{u}^2. \tag{18.9}$$

These conditions, particularly the latter, can be derived providing some suitable assumptions about the form of the turbulent shear flow near the boundary are made. The classical St. Venant equations are then (now writing $\bar{u} = u$)

$$\begin{aligned}&\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0,\\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = g(S - \eta_x) - \frac{fu^2}{h}.\end{aligned} \tag{18.10}$$

Suppose the volume flux (per unit width) of the river is Q_0, and is prescribed. We non-dimensionalise the St. Venant equations using length scales h_0, velocity scale u_0, time scale t_0 and bedload transport scale q_0, chosen so that

$$u_0h_0 = Q_0, \quad gS = \frac{fu_0^2}{h_0}, \quad t_0 = \frac{h_0^2}{q_0(1-n)}, \tag{18.11}$$

where q_0 is chosen so that $q \sim q_0$ when $\tau \sim f\rho_w u_0^2$. The resulting dimensionless Exner–St. Venant model is

$$\begin{aligned}&\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0,\\ &\varepsilon\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0,\\ &F^2\left(\varepsilon\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\eta_x + \delta\left(1 - \frac{u^2}{h}\right),\end{aligned} \tag{18.12}$$

and the parameters are defined by

$$\varepsilon = \frac{(1-n)q_0}{Q_0}, \quad F = \frac{u_0}{\sqrt{gh_0}}, \quad \delta = S. \tag{18.13}$$

Typical values are $\varepsilon \sim 10^{-2}$, $\delta \sim 10^{-3}$, $F < 1$ (if we restrict our attention to dunes), and the simplifying assumptions $\varepsilon \to 0$, $\delta \to 0$ lead to

$$\begin{aligned}&uh = 1,\\ &\tfrac{1}{2}F^2u^2 + \eta = \tfrac{1}{2}F^2 + 1,\end{aligned} \tag{18.14}$$

referring to a basic (scaled) state $u = h = 1$, $s = 0$. These provide the extra two equations to complete the model. Note that the specific assumptions $\varepsilon \ll 1$, $\delta \ll 1$ (both realistic) obviate the necessity to specify (18.8) and (18.9). However (18.14) does require the water flow to be slowly varying (and specifically, $s \ll 1$ or $\partial s/\partial x \ll 1$).

Elimination of h and η yields $s = s(u)$,

$$s = 1 - \frac{1}{u} + \tfrac{1}{2}F^2(1 - u^2), \tag{18.15}$$

and since also $q = q(\tau) = q(u)$, then $q = q(s)$, and the Exner equation is simply

$$\frac{\partial s}{\partial t} + q'(s)\frac{\partial s}{\partial x} = 0, \tag{18.16}$$

or equivalently

$$\frac{\partial q}{\partial t} + \frac{1}{s'(q)}\frac{\partial q}{\partial x} = 0. \tag{18.17}$$

For example, the (scaled) choice $q = \tau^{3/2} = u^3$ leads to

$$v(q) = \frac{1}{s'(q)} = \frac{3q^{4/3}}{1 - F^2q}. \tag{18.18}$$

When $F < 1$, the wave speed $v(q)$ is an increasing function of q (at $q = 1$), so that shocks form and propagate downstream. Dunes are indeed shocks (and do propagate downstream); also

$$\frac{\mathrm{d}s}{\mathrm{d}\eta} = \frac{F^2 - h^3}{F^2}, \tag{18.19}$$

so that for $F < 1$, $\mathrm{d}s/\mathrm{d}\eta < 0$ at $h = 1$, and the water surface is out of phase with the bed, as is the case for dunes.

18.2.2 Instability

Of course, (18.16) does not predict instability, it simply evolves prescribed perturbations into dunes. The key to instability lies in the observation that the presence of a perturbation of the bed causes a disturbance to the flow, and this is not manifested in the St. Venant model. However, it is not in fact the shallow water assumptions (18.14) that are at fault, but rather the prescription of the shear stress in (18.9). From (18.15) $u = u(s)$, and $\mathrm{d}u/\mathrm{d}s = u^2/(1 - F^2u^3)$; at $u = 1$, $\mathrm{d}u/\mathrm{d}s = 1/(1 - F^2) > 0$ for $F < 1$: u is exactly in phase with s. Hence also τ is exactly in phase with s.

This is not realistic, because the presence of the bump perturbs the flow, and intuitively we expect that the shear stress will be greatest on the upstream face of a bump. One simple way to represent this is to replace the dimensionless version of (18.9), $\tau = u^2$, by

$$\tau = u^2|_{x+l}, \tag{18.20}$$

so that the shear stress leads the velocity (and thus the bed). This idea was introduced by Kennedy [19], and leads to instability, since with

$$q = q[s(x + l, t)], \tag{18.21}$$

linearisation of the Exner equation via

$$q = 1 + \bar{q}e^{ikx+\sigma t}, \quad s = \bar{s}e^{ikx+\sigma t}, \tag{18.22}$$

yields

$$\bar{q} = q'(0)e^{ikl}\bar{s}, \quad \sigma\bar{s} + ik\bar{q} = 0, \tag{18.23}$$

and thus

$$\sigma = kq'(0)(\sin kl - i\cos kl), \tag{18.24}$$

and instability for $l > 0$. The model (18.21) is not actually very good ($\sigma \sim k$ at $k \to \infty$ indicating ill-posedness), but it does point out the modification which needs to be made.

18.2.3 The Orr–Sommerfeld Model

In order to compute the effect of the bed on the shear stress, we must consider the structure of the shear flow above the bed. The simplest model for the turbulent flow is one in which there is an eddy viscosity, which we take to be μ_T, and independent of position (for simplicity). The governing equations for the two-dimensional time-averaged velocity field $(u, 0, w)$ (note u is not the depth-averaged mean, but reverts to its original meaning) are

$$\begin{aligned} u_t + uu_x + wu_z &= -\frac{1}{\rho_w}\frac{\partial p}{\partial x} + \nu_T\nabla^2 u + gS, \\ w_t + uw_x + ww_z &= -\frac{1}{\rho_w}\frac{\partial p}{\partial z} + \nu_T\nabla^2 w - g, \\ u_x + w_z &= 0; \end{aligned} \tag{18.25}$$

here $\nu_T = \mu_T/\rho_w$ is the kinematic eddy viscosity. If the downstream slope is $S = \sin\alpha$, then g in the second equation is an approximation for $g\cos\alpha$, but the difference is slight.

Boundary conditions are those of zero stress at the top surface, and no slip at the base (modifications may be necessary for more realistic eddy viscosity models). The object is to calculate the basal shear stress

$$\tau \approx \mu_T \left.\frac{\partial u}{\partial z}\right|_{z=s} \tag{18.26}$$

for s small but non-zero (whence the approximation in (18.26) is valid). We take as a reference point the supposition that $\tau = f\rho_w\bar{u}^2$ when $s = 0$; this will provide us with a consistent (flow-dependent) definition of μ_T.

In the uniform state where $s = 0$, we find

$$u = \frac{gS}{\nu_T}\left(hz - \frac{1}{2}z^2\right), \tag{18.27}$$

so that

$$\bar{u} = \frac{gS}{3\nu_T}h^2, \tag{18.28}$$

and these are consistent with $\tau = f\rho_w\bar{u}^2$ providing we have

$$\nu_T = \varepsilon_T\bar{u}h, \tag{18.29}$$

and we choose

$$\varepsilon_T = f/3. \tag{18.30}$$

Thus, we suppose ν_T is defined by (18.29) and (18.30), and this reproduces (18.9) for a uniform flow.

Now we want to modify the solution to find an expression for τ when $s \neq 0$. First we scale the model as before, and scale $p - \rho g(h_0 - z)$ with $\rho_w u_0^2$. Neglecting the small time derivatives as before ($\varepsilon \to 0$), we have

$$u u_x + w u_z = -p_x + \frac{1}{R}\nabla^2 u + \frac{S}{F^2},$$
$$u w_x + w w_z = -p_z + \frac{1}{R}\nabla^2 w,$$
$$u_x + w_z = 0, \tag{18.31}$$

and the new parameter is the turbulent Reynolds number

$$R = \frac{u_0 h_0}{\nu_T} = \frac{1}{\varepsilon_T}. \tag{18.32}$$

Although in general, $\bar{u}$ and h may differ (when $s \neq 0$) from u_0 and h_0, we note that $\bar{u}h = u_0 h_0$, and thus it is consistent to define the eddy viscosity $\nu_T = \varepsilon_T Q_0$, so that for a given discharge it is constant, whether s is constant or not.

We specifically choose u_0 to be the mean steady velocity (even if $s \neq 0$), so that the dimensionless uniform velocity profile (when $s = 0$) is

$$U(z) = 3\left(z - \frac{1}{2}z^2\right). \tag{18.33}$$

Now we write

$$(u, w) = (U(z) + \psi_z, -\psi_x), \tag{18.34}$$

where ψ is small (as s is), and thus the model is linearised: we find

$$U\nabla^2\psi_x - U''\psi_x = \frac{1}{R}\nabla^4\psi. \tag{18.35}$$

The condition of zero normal stress at the surface becomes, approximately,

$$\eta \approx 1 - F^2 p|_{z=1}, \tag{18.36}$$

and if we conveniently suppose $F^2 \ll 1$, then we can take $\eta \equiv 1$ in the perturbed flow. The linearised boundary conditions for the flow are then

$$\psi = \psi_{zz} = 0 \quad \text{at} \quad z = 1,$$
$$\psi = 0, \quad \psi_z = -U_0' s \quad \text{at} \quad z = 0, \tag{18.37}$$

where $U_0' = U'(0) = 3$.

If we can solve this problem, then the dimensional basal shear stress is

$$\tau = \rho_w \varepsilon_T u_0^2 U_0' \left(1 + s\frac{U_0''}{U_0'} + \frac{1}{U_0'}\psi_{zz}|_{z=0}\right), \tag{18.38}$$

and since $f = 3\varepsilon_T = \varepsilon_T U_0'$, $u_0 = \bar{u}$, this is

$$\tau = f\rho_w \bar{u}^2 \left(1 + \frac{sU_0''}{U_0'} + \frac{1}{U_0'}\psi_{zz}|_0\right). \tag{18.39}$$

We write (time dependence is implicit)

$$s = \int_{-\infty}^{\infty} \hat{s}(k)e^{ikx}\,dk,$$
$$\psi = -U_0' \int_{-\infty}^{\infty} \hat{s}e^{ikx}\Psi(z,k)\,dk, \tag{18.40}$$

so that Ψ satisfies

$$U(\Psi'' - k^2\Psi) - U''\Psi = \frac{1}{ikR}(\Psi^{iv} - 2k^2\Psi'' + k^4\Psi), \tag{18.41}$$

with

$$\Psi = \Psi'' = 0 \qquad \text{on } z = 1,$$
$$\Psi = 0, \quad \Psi' = 1 \qquad \text{on } z = 0. \tag{18.42}$$

Engelund [8] and Smith [31] solved this problem numerically, incorporating it into a linearised model of the Exner equation, and finding instability to occur, essentially because of the phase shift of the shear stress. However, since the parameter $1/R = f/3$ is relatively small (e.g. $1/R = 0.02$ if $f = 0.06$), an alternative approach is to derive an asymptotic solution based on the limit $R \gg 1$.

This can be done using an analysis pioneered by Bill Reid, and expounded in the book by Drazin and Reid [6]. The neglect of the terms of $O(1/R)$ in (18.41) gives a second order equation which can be solved by Frobenius's method to give power series solutions in z. Near $z = 0$, there is a boundary layer of (complex) thickness $(ikRU_0')^{-1/3}$, in which the approximating equations can be solved in terms of a class of generalised Airy functions introduced by Reid [25], defined explicitly by contour integral representations, whose asymptotic form far from $z = 0$ can be explicitly computed. Matching of the two expansions can be carried out, and it is found that

$$\Psi''(0) \sim -3(ikRU_0')^{1/3}\mathrm{Ai}(0) + O(1) \tag{18.43}$$

for $k > 0$, where $i^{1/3} = e^{i\pi/6}$ (and $\Psi''(0,-k) = \overline{\Psi''(0,k)}$, where the overbar denotes the complex conjugate).

Now the basal stress is, from (18.39),

$$\tau = f\rho_w\bar{u}^2\left[1 - s - \int_{-\infty}^{\infty} e^{ikx}\hat{s}(k)\Psi''(0,k)\,dk\right], \tag{18.44}$$

and by use of the convolution theorem, this is

$$\tau = f\rho_w\bar{u}^2\left[1 - s + \int_{-\infty}^{\infty} K(x-\xi)\frac{\partial s}{\partial \xi}\,d\xi\right], \tag{18.45}$$

where the kernel $K(x)$ is

$$K(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Psi''(0,k)}{ik}e^{ikx}\,dk. \tag{18.46}$$

Writing $c = 3(RU_0')^{1/3}\mathrm{Ai}(0)$ in (18.43), we have

$$K(x) = \frac{c}{\pi}\int_0^\infty \cos(kx - \tfrac{\pi}{3})\frac{\mathrm{d}k}{k^{2/3}}\,, \tag{18.47}$$

and this can be evaluated to give

$$\begin{aligned} K(x) &= \frac{\mu}{x^{1/3}}\,, \quad x > 0\,, \\ &= 0\,, \quad x < 0\,, \end{aligned} \tag{18.48}$$

where

$$\mu = \frac{3^{2/3}R^{1/3}}{\Gamma(\frac{2}{3})^2} \approx 1.98R^{1/3}\,. \tag{18.49}$$

Thus (18.45) is

$$\tau = f\rho_w\bar{u}^2\left[1 - s + \mu\int_0^\infty \xi^{-1/3}\frac{\partial s}{\partial x}(x-\xi,t)\,\mathrm{d}\xi\right]: \tag{18.50}$$

the shear stress is corrected by a weighting which increases its value upstream of maxima of s, consistent with our previous heuristic expectations. For stability purposes, note that with

$$K(x) = \int_{-\infty}^{\infty}\hat{K}(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k, \tag{18.51}$$

then

$$\hat{K} = -\frac{\Psi''(0,k)}{2\pi\mathrm{i}k} = \frac{c}{2\pi k^{2/3}}\mathrm{e}^{-\mathrm{i}\pi/3}\,,\quad k > 0\,. \tag{18.52}$$

18.2.4 Orr–Sommerfeld–Exner–St. Venant Model

As explained previously, as long as s is small, the shallow water approximation (18.14) applies, and thus $s \approx s(u)$. In fact, when F is small, then $\eta \approx 1$, $h \approx 1 - s$ and thus

$$u \approx \frac{1}{1-s}\,. \tag{18.53}$$

The dimensionless Exner model thus becomes

$$\begin{aligned} &\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0\,, \\ &\tau \approx \frac{1}{1-s} + \frac{1}{(1-s)^2}\int_{-\infty}^{\infty} K(x-\xi)\frac{\partial s}{\partial \xi}(\xi,t)\,\mathrm{d}\xi\,, \end{aligned} \tag{18.54}$$

where $q = q(\tau)$. We write $s = \hat{s}\mathrm{e}^{\mathrm{i}kx+\sigma t}$, $\tau = 1 + \hat{\tau}\mathrm{e}^{\mathrm{i}kx+\sigma t}$, and then a linearisation of (18.54) yields

$$\begin{aligned} &\sigma\hat{s} + \mathrm{i}kq'(1)\hat{\tau} = 0\,, \\ &\hat{\tau} = \hat{s} + 2\pi\mathrm{i}k\hat{K}\hat{s}\,, \end{aligned} \tag{18.55}$$

whence (for $k > 0$)

$$\begin{aligned} \sigma &= 2\pi k^2 q'(1)\hat{K} - \mathrm{i}kq'(1) \\ &= cq'(1)k^{4/3}\mathrm{e}^{-\mathrm{i}\pi/3} - \mathrm{i}kq'(1)\,. \end{aligned} \tag{18.56}$$

The growth rate is thus

$$\mathrm{Re}\,\sigma = \tfrac{1}{2}cq'(1)k^{4/3}\,, \tag{18.57}$$

which is positive, denoting instability; the wave speed is

$$-\frac{\mathrm{i}\sigma}{k} = q'(1)\left(1 + \frac{\sqrt{3}}{2}ck^{1/3}\right)\,, \tag{18.58}$$

and waves propagate downstream.

18.2.5 Well-posedness

This model is also ill-posed because of the rapid growth of short wavelength disturbances. The remedy here is to account for the local slope of the bed on the mobility. For a particle of grain diameter D_s, the buoyancy-induced stress τ_p acting in the x direction on the particle is approximately $\tau_p = -\Delta\rho g D_s \partial s/\partial x$, where $\Delta\rho = \rho_s - \rho_w$, ρ_s is sediment density, for small s. Thus the effective driving stress for bedload transport is $\tau + \tau_p$, and we should take q as a function of $\tau + \tau_p$. Equivalently we add τ_p to the definition of τ, and when this is non-dimensionalised, we replace $(18.54)_2$ (the notation $(a)_b$ indicates the b-th equation of the equation set (a)) by

$$\tau = \frac{1}{1-s} + \frac{\mu}{(1-s)^2}\int_0^\infty \xi^{-1/3}\frac{\partial s}{\partial x}(x-\xi,t)\,\mathrm{d}\xi - \beta\frac{\partial s}{\partial x}\,, \tag{18.59}$$

where

$$\beta = \frac{\Delta\rho g D_s}{f\rho_w u_0^2} = \frac{\Delta\rho D_s}{\rho_w S h_0}\,. \tag{18.60}$$

Typical values of $D_s = 1\,\mathrm{mm}$, $h_0 \gtrsim 1\,\mathrm{m}$, $S \gtrsim 10^{-3}$, give values $\beta \sim O(1)$. The extra term is diffusive, and the growth rate (18.57) is modified as

$$\mathrm{Re}\,\sigma = q'(1)\left(\tfrac{1}{2}ck^{4/3} - \beta k^2\right)\,, \tag{18.61}$$

and high wave number disturbances decay.

18.2.6 The Canonical Dune Equation

The linear integral correction to the stress is computed on the basis that s is small compared to the depth, and also uses the fact that R is large. On the face of it, this implies that only the linearisation of (18.59) gives a self-consistent approximation. However, it is possible to argue that some nonlinear terms in (18.59) may be included, at least formally, in certain circumstances. Suppose the amplitude of variations in s is of order $s_0 \ll 1$ and varies on a length scale $L \gg 1$. Then if $s_0 \sim \mu/L \sim \beta/L$, a self consistent approximation correct to $O(s_0^2)$ is

$$\tau \approx 1 + s + s^2 + \mu \int_0^\infty \xi^{-1/3} \frac{\partial s}{\partial x}(x-\xi,t)\,d\xi - \beta \frac{\partial s}{\partial x}\,, \tag{18.62}$$

and if we write

$$q(\tau) \approx q(1) + q'(1)(\tau-1) + \frac{q''(1)}{2}(\tau-1)^2 + O(\tau-1)^3\,, \tag{18.63}$$

and define the moving spatial coordinate

$$X = x - q'(1)t\,, \tag{18.64}$$

then we find that, correct to terms of $O(s^3)$,

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial X}\left[\{q'(1) + \tfrac{1}{2}q''(1)\}s^2 + \mu \int_0^\infty \xi^{-1/3}\frac{\partial s}{\partial X}(X-\xi,t)\,d\xi - \beta\frac{\partial s}{\partial X}\right] = 0\,, \tag{18.65}$$

and by a suitable rescaling of s, t and X we obtain the canonical equation

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial X}\left[\tfrac{1}{2}s^2 + \int_0^\infty \xi^{-1/3}\frac{\partial s}{\partial X}(X-\xi,t)\,d\xi - \frac{\partial s}{\partial X}\right] = 0\,. \tag{18.66}$$

We propose this equation as a first canonical equation for the study of nonlinear dune formation. It bears comparison to the Kuramoto–Sivaskinsky equation, and we may hope that the properties of its solutions may be dune-like.

18.2.7 Caveats

The principal feature of real dunes which we have neglected, but which it would be essential to include in future models, is that of boundary layer separation at the dune crest. Essentially, we may expect growth of unstable perturbations to lead to shock formation (smoothed by the diffusion term), but in fact when the bed slope reaches the angle of repose, spontaneous slip occurs, and it is a familiar feature of desert dunes that there is separation at the resulting slope discontinuity.

It may in fact still be possible to model the separated flow in the same way, except that in the lee of the dune a constant pressure (or vorticity) cavity exists, whose extent is unknown *a priori*, but this is a more difficult problem to address.

18.3 Drumlins

The basic geometry of the model for drumlin formation is shown in Fig. 18.2; it is similar to the fluvial dune geometry of Fig. 18.1. We suppose the ice flows over a layer of till lying above an impermeable basement. Typical ice sheet

thicknesses are on the order of kilometres, and it is commonly the case that the basal ice reaches the melting point, due to geothermal heat input, together with the insulating effect of the ice cover. In this situation, basal meltwater is produced and the underlying till will become deformable if its pore pressure is high enough (within about a bar of the overburden pressure). The resulting ice motion may then be almost entirely due to deformation of the till, which can be thought of as acting like a power-law viscous fluid, with an effective viscosity which decreases as the pore water pressure increases. This is the situation we study. The vertical coordinate is z, the ice-till interface is $z = s$, and the top surface is denoted by $z = \eta$, thus the ice depth is $h = \eta - s$.

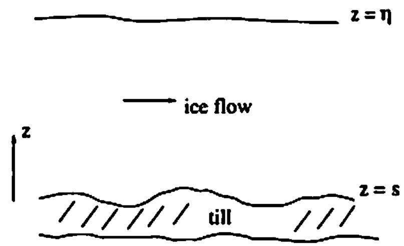

Fig. 18.2. Geometry for the drumlin model. Ice flows over a layer of deformable till

In the uniform state, h and s are constant, and a uniform shear flow exists in the ice. There are two features (other than scale) which distinguish the ice-till flow problem from the water-sediment flow problem. The first is that the flow of till is thought to depend both on basal stress (as for bedload) and on the effective pressure N, defined as the difference between overburden pressure and the interstitial pore water pressure within the sediments. Without this dependence, the instability does not occur. The second is that the Reynolds number for ice flow is essentially zero. Despite these differences, the problems are similar in structure. In particular, it is essential to the instability to take account of the effect of a perturbed bed on the basal stress.

18.3.1 The Hindmarsh Model

The basic model is due to Hindmarsh [16,17], and the present formulation is due to Fowler [13]. For a slow, two-dimensional flow in $s < z < \eta$, we define a stream function ψ and the reduced pressure Π via

$$p = p_a + \rho_i g(\eta - z) + \Pi\,, \tag{18.67}$$

where p_a is atmospheric pressure and ρ_i is ice density. Then Stokes's equations are

$$\begin{aligned} \rho_i g \eta_x + \Pi_x &= \mu \nabla^2 \psi_z\,, \\ \Pi_z &= -\mu \nabla^2 \psi_x\,, \end{aligned} \tag{18.68}$$

where μ is the viscosity of ice, which we take to be constant, and we require conditions of stress continuity at $z = \eta$, and stress and velocity continuity at $z = s$. (Velocity continuity implies no flow of ice into the till, and also implies that the ice does not slip over the till. This is reasonable, though it may be inaccurate, but there is little information on which to base a description of such slip.) The normal and tangential stresses at $z = s$ are, respectively,

$$\begin{aligned}\tau_{nn} &= -\frac{2\mu}{1+s_x^2}[(1-s_x^2)\psi_{xz} + s_x(\psi_{zz} - \psi_{xx})] \,, \\ \tau &= \frac{\mu}{1+s_x^2}[(1-s_x^2)(\psi_{zz} - \psi_{xx}) - 4s_x\psi_{xz}] \,,\end{aligned} \tag{18.69}$$

and similar expressions apply at $z = \eta$ (simply replace s by η).

The flow in an ice sheet is driven by the surface slope η_x; η varies on a length scale of order 1000 km, so that $\eta_x \sim 10^{-3}$. On the more relevant drumlin length scale of $\lesssim 1000$ m, it is convenient to take η_x as small and constant, but also to take η as approximately constant. We thus define

$$-\eta_x = \delta \sim 10^{-3} \tag{18.70}$$

to be constant in (18.68), but we solve the resulting model equations assuming the top surface η is constant.

These equations then have a uniform solution in which $\Pi = 0$, $s = 0$, $\eta = \bar{h}$, and the velocity and shear stress at the ice-till interface are $\bar{u}$ and $\bar{\tau}$, respectively. This solution is (with zero shear stress at the surface)

$$\psi = \bar{u}z + \frac{\bar{\tau}}{2\mu}\left(z^2 - \frac{z^3}{3\bar{h}}\right) \,, \tag{18.71}$$

and the basal shear stress is

$$\bar{\tau} = \rho_i g \delta \bar{h} \,. \tag{18.72}$$

In allowing uniform η, we are in fact letting $\delta \to 0$ while keeping $\bar{\tau}$ finite, an approximation of Boussinesq type.

Next, consider the zero normal stress condition at the surface. Under a perturbation of the flow, the perturbation to the surface $\Delta\eta$ is given from (18.67) by

$$\Delta\eta = -\frac{(\Pi - \tau_{nn})}{\rho_i g} \,, \tag{18.73}$$

and we can suppose $\Pi - \tau_{nn} \lesssim \mu u/l$, where u is ice velocity and l is the drumlin length scale. Then

$$\Delta\eta \lesssim \frac{\mu u \delta \bar{h}}{l\bar{\tau}} \,, \tag{18.74}$$

and with $\bar{\tau} \sim \mu u/\bar{h}$, then $\Delta\eta/\bar{h} \sim \delta\bar{h}/l \sim \delta$. Thus we can consistently neglect variations of η under flow perturbations (just as we did for dunes when the Froude number was small).[1]

[1] This observation is due to Christian Schoof.

Now we perturb the flow, supposing s to be small relative to the depth, but not zero, by writing

$$\psi = \bar{u}z + \frac{\bar{\tau}}{2\mu}\left(z^2 - \frac{z^3}{3\bar{h}}\right) + \Psi \,, \tag{18.75}$$

so that Ψ satisfies

$$\begin{aligned}\Pi_x &= \mu\nabla^2\Psi_z \,, \\ \Pi_z &= -\mu\nabla^2\Psi_x \,,\end{aligned} \tag{18.76}$$

with

$$\Psi_x = 0 \,, \quad \Psi_{zz} = 0 \quad \text{on} \quad z = \bar{h} \,, \tag{18.77}$$

and, correct to terms of $O(s)$, (18.69) gives

$$\begin{aligned}\tau_{nn} &\approx -2\mu\Psi_{xz} - 2\bar{\tau}s_x \,, \\ \tau &\approx \bar{\tau}(1 - s/\bar{h}) + \mu(\Psi_{zz} - \Psi_{xx}) \,,\end{aligned} \tag{18.78}$$

which can be taken to be evaluated on $z = 0$ rather than $z = s$.

If the horizontal till velocity at $z = s$ is u, then the velocity continuity conditions are

$$-\psi_x = s_t + us_x \,, \quad \psi_z = u \quad \text{at} \quad z = s \,, \tag{18.79}$$

and, under linearisation about $z = 0$, these imply

$$\begin{aligned}u &= \bar{u} + \frac{\bar{\tau}s}{2\mu} + \Psi_z \,, \\ -\Psi_x &= s_t + \bar{u}s_x \,,\end{aligned} \tag{18.80}$$

at $z = 0$.

The ice flow problem thus reduces to the solution of the biharmonic equation

$$\nabla^4\Psi = 0 \,, \tag{18.81}$$

together with the two conditions in (18.77), and the four conditions in (18.78) and (18.80). Only two of these latter four are necessary, and so the solution gives two extra relations, which can be taken to be those in (18.78), i.e. we obtain τ_{nn} and τ as linear functionals of u and s. The model is now completed by relating u to τ, τ_{nn}, and s through the dynamics of the deforming till. When this is done, we can eliminate u to find τ and τ_{nn} in terms of s, and hence the till flux q can be written in terms of s, and an Exner equation will provide a model for s.

We also need to specify boundary conditions as $x \to \pm\infty$, and if we suppose $s \to 0$ there, then we would have $\Psi \to 0$. This implies that we can replace (18.77) with the conserved flux condition, thus

$$\Psi = 0 \,, \quad \Psi_{zz} = 0 \quad \text{on} \quad z = \bar{h} \,. \tag{18.82}$$

The solution of the problem is now completed by consideration of the till flux.

18.3.2 Till Rheology and Flow

This has been considered in some detail by Fowler [13]. The till is shallow, and the velocity and flux can be determined explicitly. The principal feature of the till flow is that it should increase with applied shear stress and also with pore water pressure. A simple (empirical) choice is of the form

$$\frac{\partial v}{\partial z} = A\exp(\alpha\tau/p_e)\,, \tag{18.83}$$

where v is horizontal velocity and p_e is the effective pressure within the till. This has some basis in experimental measurements on deformation in clays and in till, and is consistent with observed plastic-like behaviour when α is large. The effective pressure increases with depth from the ice-till interface due to hydrostatic effects:

$$p_e = N + (1-n)\Delta\rho_{sw}g(s-z)\,, \tag{18.84}$$

where N is the value at $z = s$, n is the till porosity, $\Delta\rho_{sw} = \rho_s - \rho_w$, ρ_s is sediment density, ρ_w is water density, and g is gravity. The shear stress is taken as constant, while the interfacial effective pressure itself is given by

$$N = \bar{N} + \Delta\rho_{wi}gs + \Pi - \tau_{nn}\,, \tag{18.85}$$

evaluated on $z = s$, or (since linearised) on $z = 0$. Here $\bar{N}$ is a reference value which is supposed fixed, and corresponds to the effective pressure which we suppose is determined by the subglacial drainage characteristics.

The upshot of these assumptions is that the till flux is (approximately)

$$q = A^*\zeta^{*2}[1-(1+X)e^{-X}]\,, \tag{18.86}$$

while the ice-till interface velocity is

$$u = A^*\zeta^*(1-e^{-X})\,, \tag{18.87}$$

and the parameters are

$$X = s_0/\zeta^*\,,\quad \zeta^* = \frac{N^2}{\alpha r\tau}\,,\quad A^* = A\exp(\alpha\tau/N)\,, \tag{18.88}$$

where s_0 is the till thickness, and we can take $s_0 = \bar{s}+s$, where $\bar{s}$ is the (uniform) till depth in the undisturbed flow; also the parameter r is defined by

$$r = \Delta\rho_{sw}g(1-n) \approx 0.1\ \text{bar}\,\text{m}^{-1}\,. \tag{18.89}$$

These expressions give $u = u(s,\tau,N)$, $q = q(s,\tau,N)$, as shown in Fig. 18.3. From (18.78), (18.80) and (18.85), we can write

$$u = L(s,\tau-\tilde{\tau})\,,\quad N - \bar{N} = M(s,\tau-\tilde{\tau})\,, \tag{18.90}$$

where L and M are bilinear (integral) operators. Thus the ice-till velocity is

$$u[s,\tau,\bar{N} + M\{s,\tau-\tilde{\tau}\}] = L\{s,\tau-\tilde{\tau}\}\,, \tag{18.91}$$

whence $\tau = P[s]$ and thus $q = q[s]$. We then evolve s via the Exner-type equation

$$\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0\,. \tag{18.92}$$

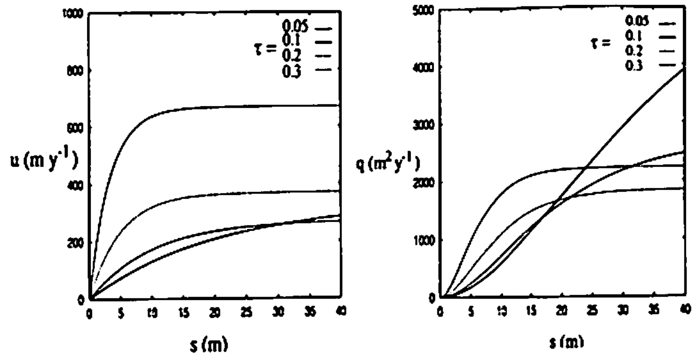

Fig. 18.3. Profiles of $u(s,\tau)$ and $q(s,\tau)$ calculated using (18.87) and (18.86). Graphs are shown for $\tau = 0.05$, 0.1, 0.2, and 0.3 bars. The constants are taken as $\alpha = 10$, $\Delta\rho_{sw}g(1-n) = 0.1\,\text{bar}\,\text{m}^{-1}$, $N = 1$ bar, $A = 10\ \text{y}^{-1}$

18.3.3 Fourier Integral Solution

We can write a solution of (18.76), (18.78), (18.80) and (18.82) in terms of Fourier integrals. It is convenient to suppose that $\bar{h} = \infty$, corresponding to a limit in which $\bar{h}/l \gg 1$. This is in fact marginal, since we expect $l \sim 100$–1000 m, and $\bar{h} \sim 1000$ m, but simplifies the algebra without (apparently) seriously compromising the physics. We can then write the solution in the form

$$\begin{aligned}
\Psi &= \int_{-\infty}^{\infty}(a+bz)e^{-|k|z}e^{ikx}dk,\\
\Pi &= -\int_{-\infty}^{\infty}2\mu ikbe^{-|k|z}e^{ikx}dk\,,\\
u-\bar{u} &= \int_{-\infty}^{\infty}\tilde{u}e^{ikx}dk\,,\\
\tau-\bar{\tau} &= \int_{\infty}^{\infty}\tilde{\tau}e^{ikx}dk,\\
s &= \int_{-\infty}^{\infty}\tilde{s}e^{ikx}dk,\\
N &= \bar{N} + \int_{\infty}^{\infty}\tilde{N}e^{ikx}dk\,,
\end{aligned} \tag{18.93}$$

and the boundary conditions imply

$$\begin{aligned}
b-|k|a &= \tilde{u} - \frac{\bar{\tau}\tilde{s}}{\mu}\,,\\
-ika &= \tilde{s}_t + ik\bar{u}\tilde{s},\\
\tilde{\tau} &= -2|k|u(b-|k|a)\,,\\
\tilde{N} &= -2\mu ik|k|a + 2ik\bar{\tau}\tilde{s} + \Delta\rho_{wi}g\tilde{s}\,,
\end{aligned} \tag{18.94}$$

from which we derive

$$\tilde{\tau} = -2|k|\mu\tilde{u} + 2|k|\bar{\tau}\tilde{s}\,,$$
$$\tilde{N} = 2\mu|k|(\tilde{s}_t + ik\bar{u}\tilde{s}) + (\Delta\rho_{wi}g + 2ik\bar{\tau})\tilde{s}\,. \tag{18.95}$$

Suppose also that $\tilde{s}$, $\tilde{N}$, and $\tilde{\tau}$ are small; then linearisation of the ice-till velocity $u(s,\tau,N)$ gives the relation

$$\tilde{u} = u_s\tilde{s} + u_\tau\tilde{\tau} + u_N\tilde{N}\,, \tag{18.96}$$

where $u_s = \partial u/\partial s$ evaluated at $s = 0$, $\tau = \bar{\tau}$, $N = \bar{N}$, etc. Fowler [13] omitted to include the second term in $(18.95)_1$; he also argued that the last term in $(18.95)_2$ was negligible, on the basis that, typically, $\Delta\rho_{wi}g, 2k\bar{\tau} \ll 2\mu k^2\bar{u}$, for values of interest when $k^{-1} \sim 100\,\mathrm{m}$, and we do the same here.

From $(18.95)_1$ and (18.96), we have

$$\tilde{\tau} = -\frac{2|k|\mu}{1+2|k|\mu u_\tau}\left[\left(u_s - \frac{\bar{\tau}}{\mu}\right)\tilde{s} + u_N\tilde{N}\right]\,. \tag{18.97}$$

The $\bar{\tau}/\mu$ term (which arises from the second term in $(18.95)_1$) is negligible if $\bar{\tau} \ll \mu\partial u/\partial s$. Consulting Fig. 18.3, we see that a typical range of $\partial u/\partial s$ is 10–100 y^{-1} for $s < 10\,\mathrm{m}$. Then if $\mu = 6\,\mathrm{bar\,y}$, $\mu\partial u/\partial s \gtrsim 60\,\mathrm{bar}$, and it is safe to neglect this term. Thus we derive the approximations

$$\tilde{N} = 2\mu|k|(\tilde{s}_t + ik\bar{u}\tilde{s})\,,$$
$$\tilde{\tau} = -\frac{2|k|\mu}{1+2|k|\mu u_\tau}(u_s\tilde{s} + u_N\tilde{N})\,,$$
$$\tilde{u} = \frac{u_s\tilde{s} + u_N\tilde{N}}{1+2|k|\mu u_\tau}\,. \tag{18.98}$$

18.3.4 Linear Stability

If we now linearise the till-flux expression (18.86) and the Exner equation (18.92), we obtain two further relations

$$\tilde{q} = q_s\tilde{s} + q_\tau\tilde{\tau} + q_N\tilde{N},$$
$$\tilde{s}_t + ik\tilde{q} = 0\,, \tag{18.99}$$

and from these we can derive the growth rate in the form

$$\tilde{s}_t/\tilde{s} = \rho - ikc\,, \tag{18.100}$$

where ρ is the growth rate and c is the wave speed. Explicit expressions are given by Fowler [13], for example the growth rate is

$$\rho = \frac{2\mu k^2|k|\Delta_1\Delta_2}{(1+2\mu|k|u_\tau)^2 + 4\mu^2k^4\Delta_2^2}\,, \tag{18.101}$$

where

$$\Delta_1 = (1+2\mu|k|u_\tau)(\bar{u} - q_s) + 2\mu|k|q_\tau u_s\,,$$
$$\Delta_2 = q_N + 2\mu|k|[u_\tau q_N - q_\tau u_N]\,. \tag{18.102}$$

It is clear from these formulae that it is essential for instability that the flow law for till depend on N, for otherwise $\Delta_2 = 0$ and thus stability is neutral.

Further simplification is possible, using the anticipated fact that $k^{-1} \sim 100\,\mathrm{m}$, whence typically $2\mu|k|u_\tau \gg 1$. In fact, if we define (cf. (18.88))

$$X = \xi Y\,,\quad \xi = \frac{r\bar{s}}{N}\,,\quad Y = \frac{\alpha\tau}{N}\,,\quad K = \frac{2\mu\alpha A|k|}{r}\,, \tag{18.103}$$

then we find that $K \gg 1$, and thence (for $X, Y = O(1)$)

$$\Delta_1 \approx \frac{A^*\zeta^*W(X)Ke^Y[Y - F(X)]}{Y^2}\,,$$
$$\Delta_2 \approx \frac{qKe^YJ(X)}{NY^2}\,, \tag{18.104}$$

where the functions W, J and F are positive, and

$$F(X) = \frac{1 - 2Xe^{-X} - e^{-2X}}{1 - (1+X)e^{-X}}\,; \tag{18.105}$$

F increases monotonely from 0 at $X = 0$ to 1 as $X \to \infty$. Roughly, $F \approx 1 - e^{0.7X}$. We suppose typical values $\bar{s} \sim 10\,\mathrm{m}$, $N \sim 1\,\mathrm{bar}$, $\alpha = 10$, $\tau = 0.1\,\mathrm{bar}$, so that $X, Y = O(1)$, and instability occurs (since $\Delta_2 > 0$) if

$$Y > F(X) = F(\xi Y)\,. \tag{18.106}$$

(We also require $X > O(1/\sqrt{K})$.) A delineation of the instability region is shown in Fig. 18.4. We see that instability generally occurs for $\xi, Y = O(1)$, and thus in general for low effective pressures.

With the same approximation, that $K \gg 1$, we find that the growth rate can be written in the approximate form

$$\rho \approx \left(\frac{AN}{2\mu}\right)^{1/2}\left[\frac{D|k/k^*|^3}{B^2 + C^2(k/k^*)^4}\right]\,, \tag{18.107}$$

where

$$k^* = \frac{r}{(2\mu AN)^{1/2}}\,, \tag{18.108}$$

and B, C, D are $O(1)$ functions of X and Y. This indicates that the preferred wavelength of growth is $O(k^{*-1})$, and the growth time scale is $(2\mu/AN)^{1/2}$. We find unstable wavelengths in the range 100–1000 m, and the growth time is of order 1 year. Of course, the model is only two-dimensional, but we might expect three-dimensional instability also, perhaps arising as a secondary instability on the primary (Rogen moraine) ridge forms.

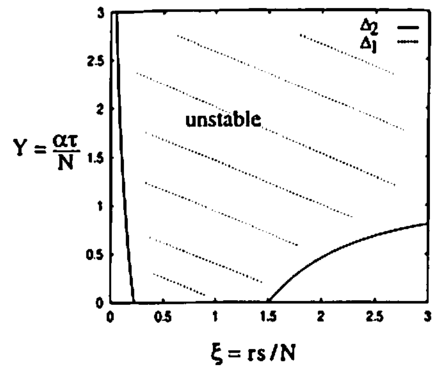

Fig. 18.4. Instability region in Y, ξ space, bounded by the curves $\Delta_1 = 0$, $\Delta_2 = 0$ (see (18.102)). A value of $K = 120$ (see (18.103)) has been used

18.3.5 A Nonlinear Model

With the approximation $2|k|\mu u_\tau \gg 1$, (18.98) may be written in the approximate form

$$\begin{aligned} \tilde{N} &= 2\mu|k|(\tilde{s}_t + ik\bar{u}\tilde{s}) , \\ \tilde{\tau} &\approx -\frac{(u_s\tilde{s} + u_N\tilde{N})}{u_\tau} , \\ \tilde{u} &\approx \frac{(u_s\tilde{s} + u_N\tilde{N})}{2|k|\mu u_\tau} , \end{aligned} \tag{18.109}$$

and $\tilde{N}$ and $\tilde{\tau}$ can be explicitly inverted to obtain

$$N - \bar{N} = -\frac{2\mu}{\pi} \fint_{-\infty}^{\infty} \frac{\partial a}{\partial \xi} \frac{d\xi}{\xi - x} , \tag{18.110}$$

where the barred integral indicates that the principal value is taken, a is given by

$$\frac{\partial s}{\partial t} + \bar{u}\frac{\partial s}{\partial x} = a , \tag{18.111}$$

and

$$\tau - \bar{\tau} = -\frac{(u_s s + u_N N)}{u_\tau} . \tag{18.112}$$

The principal restriction used to obtain these formulae is that $s \ll k^{-1}$ (and $s \ll \bar{h}$), while the assumption that perturbations in τ, s and N are small is only used in linearising $u(s, \tau, N)$. However, the fact that the resulting perturbation in u is approximately zero suggests that the restriction to small perturbations is not essential. This suggests that the formulae (18.110) and (18.112) can be used in the Exner model, but retaining the full nonlinear prescription for $q(s, \tau, N)$; in this way we obtain a nonlinear evolution equation for s.

It is appealing to seek the weakly nonlinear form of this by expanding q for small s out to quadratic terms in s, just as we did for the dune model. We retain only linear terms in $\tau - \bar{\tau}$ and $N - \bar{N}$; the result of this is

$$\frac{\partial s}{\partial t} + \left(q_s - \frac{q_\tau u_s}{u_\tau}\right)\frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\left[\tfrac{1}{2}q_{ss}s^2 - \frac{2\mu(q_N u_\tau - q_\tau u_N)}{u_\tau}H(a_x)\right] = 0 , \tag{18.113}$$

where $H(g)$ is the Hilbert transform

$$H(g) = \frac{1}{\pi}\fint_{-\infty}^{\infty} \frac{g(\xi)\,d\xi}{\xi - x} \tag{18.114}$$

(whose Fourier transform $\widehat{H(g)} = -\pi\hat{g}\ \mathrm{sgn}k$), and a is given by (18.111).

Define the parameters

$$\begin{aligned} f &= \bar{u} + \frac{q_\tau u_s}{u_\tau} - q_s , \\ G &= 2\mu(u_\tau q_N - u_N q_\tau)/u_\tau . \end{aligned} \tag{18.115}$$

With $2|k|\mu u_\tau \gg 1$, then (18.102) implies

$$\Delta_1 \approx 2\mu|k|u_\tau f , \quad \Delta_2 \approx |k|u_\tau G . \tag{18.116}$$

We define

$$Z = x - \bar{u}t ; \tag{18.117}$$

then in the Z frame moving with the interfacial velocity, $a = \partial s/\partial t$, and the equation (18.113) for s can be written in the form

$$\frac{\partial s}{\partial t} - f\frac{\partial s}{\partial Z} + \frac{\partial}{\partial Z}\left[\tfrac{1}{2}q_{ss}s^2 - GH\left(\frac{\partial^2 s}{\partial Z \partial t}\right)\right] = 0 , \tag{18.118}$$

and this is our candidate canonical nonlinear evolution equation for drumlins (or more properly, their two-dimensional version – Rogen moraine).

Finally, it is convenient to write the model in non-dimensional form. We use the length and time scales suggested by the stability results, namely

$$l = \frac{(2\mu A\bar{N})^{1/2}}{r} , \quad [t] = \left(\frac{2\mu}{A\bar{N}}\right)^{1/2} , \tag{18.119}$$

and thus the velocity scale is

$$[u] = \frac{l}{[t]} = \frac{A\bar{N}}{r} . \tag{18.120}$$

The scale for s is taken as $\bar{s}$. Following [13], we find

$$f = [u]v(X, Y) , \quad G = l^2\gamma(X, Y) , \tag{18.121}$$

where the $O(1)$ functions v and γ are defined by

$$v = \frac{e^Y[Y - F(X)]}{Y[(Y-1)U + XU']}\,,\quad \gamma = \frac{L(X)e^Y}{Y^2[(Y-1)U + XU']}\,, \tag{18.122}$$

the subsidiary functions being

$$\begin{aligned} U &= 1 - e^{-X}\,,\\ W &= 1 - (1+X)e^{-X}\,,\\ L &= UW + X(U'W - UW')\,. \end{aligned} \tag{18.123}$$

Recall that, from (18.103), $X = \alpha r\bar{s}\bar{\tau}/\bar{N}^2$, $Y = \alpha\bar{\tau}/\bar{N}$. In terms of the scaled variables, the dimensionless version of (18.118) is

$$\frac{\partial s}{\partial t} - v\frac{\partial s}{\partial Z} + \frac{\partial}{\partial Z}\left[\tfrac{1}{2}\beta s^2 - \gamma H\left(\frac{\partial^2 s}{\partial Z\partial t}\right)\right] = 0\,, \tag{18.124}$$

where the parameter β is defined by

$$\beta = \frac{\bar{s}q_{ss}}{[u]}\,. \tag{18.125}$$

Using the expression (18.86) for q we find

$$q_{ss} = \frac{q}{s^2}\left[\left(\frac{XW'}{W}\right)^2 - \left(\frac{XW'}{W}\right) + X\left(\frac{XW'}{W}\right)'\right]\,, \tag{18.126}$$

and thus

$$\beta = \frac{We^Y}{XY}\left[\left(\frac{XW'}{W}\right)^2 - \left(\frac{XW'}{W}\right) + X\left(\frac{XW'}{W}\right)'\right]\,, \tag{18.127}$$

and is indeed $O(1)$. A Fourier transform of the linearisation of (18.124) gives a growth rate of

$$\sigma = \frac{-ikv + k^2|k|v\gamma}{1 + k^4\gamma^2}\,, \tag{18.128}$$

which, since $v \propto \Delta_1$ and $\gamma \propto \Delta_2$, reproduces (18.102). Instability occurs if $v\gamma > 0$; and with $\gamma > 0$, this is essentially $v > 0$: the resulting waveforms move forwards relative to the ice flow. (18.124) closely resembles the integrable Benjamin–Ono equation $s_t + ss_x = H(s_{xx})$ [24].

18.4 Discussion

The canonical equation for both dune and drumlin evolution follows from the Exner equation

$$\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0\,, \tag{18.129}$$

together with a suitable prescription for q. In the case of fluvial dunes, $q = q(\tau)$, and the basal shear stress depends on the mean flow speed u and the bed elevation s. When s is small, the stress can be calculated by linearisation of a suitable turbulent flow model over a flat bed. When the turbulent Reynolds number is reasonably large (which is typically the case), then an explicit asymptotic approximation for τ can be determined, in the form of a Fourier convolution of a kernel-function with the bed slope. In this way we derive (18.54), which is a nonlinear model for bed evolution. The essence of this model is captured by the reduced form (18.65), or (18.66).

Ideally, one wants an extension of the model to three dimensions, although fluvial dunes are essentially two-dimensional features. Such a generalisation is the Exner equation

$$\frac{\partial s}{\partial t} + \nabla.\mathbf{q} = 0\,, \tag{18.130}$$

together with $\mathbf{q} = \mathbf{q}(\boldsymbol{\tau})$, where the basal stress vector must now be computed from the Orr–Sommerfeld equation. It seems this should be relatively straightforward to do, simply involving a double Fourier transform.

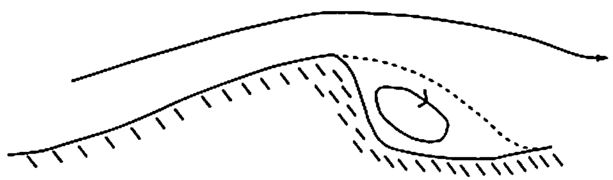

Fig. 18.5. Separation in the lee of a dune

A more important extension is to allow for boundary layer separation in the lee of dunes. Within the present framework, this could be done by supposing, for example, that the recirculating wake is a region of constant pressure or vorticity (see Fig. 18.5). The analysis is the same, but now s is unknown in the wake, whereas the pressure (or vorticity) is prescribed there. Presumably we can write the pressure as a linear functional of the bed, so that we would gain an extra equation to solve for s in the wake. It remains to be seen whether this strategy is feasible.

For drumlins, the extension to three dimensions is more essential, but seems equally straightforward. In addition, inclusion of a finite depth is straightforward, though perhaps messy. There is also a nice analogy with the formation of wakes, because the generation of bedform is likely to cause cavities to occur, and we can imagine that sediment deposition in such cavities may be one way in which layered stratigraphy is formed in drumlins, through successive fluvial deposition events. Cavity formation also complicates the prescription of the shear stress, although in this case the problem becomes one of Hilbert type, and it is possible to solve this [11].

Apart from such developments in the mathematical model, the equations (18.66):

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial X}\left[\frac{1}{2}s^2 + \int_0^\infty \xi^{-1/3}\frac{\partial s}{\partial X}(X-\xi,t)\,d\xi - \frac{\partial s}{\partial X}\right] = 0\,, \qquad (18.131)$$

and (18.124):

$$\frac{\partial s}{\partial t} - \frac{\partial s}{\partial Z} + \frac{\partial}{\partial Z}\left[\frac{1}{2}s^2 - H\left(\frac{\partial^2 s}{\partial Z \partial t}\right)\right] = 0 \qquad (18.132)$$

(where we can put $\beta = \gamma = v = 1$ by appropriate rescaling of s, Z and t), are interesting nonlinear evolution equations, whose study in the context of dynamical systems is of interest in its own right. In particular (18.131) bears comparison with the Kuramoto–Sivashinsky equation, while (18.132) similarly resembles the Benjamin–Ono equation. While we may expect the dune equation (18.131) to provide a coherent model with the stabilising diffusion term, it is less clear that the drumlin model will be. In this context, it may be worth noting that there is a further stabilising term which could be included, since till will also creep down pressure gradients.

Notes and References

Dunes. Principles of sediment transport are described in the book by Allen [1]. Theories of dune formation are given by Kennedy [19], Reynolds [26], Engelund [8] and Smith [31]. A review of this and other work is by Engelund and Fredsøe [9]. Subsequently, theoretical development has been hindered by the necessity of solving the Orr–Sommerfeld equation numerically, and this has precluded the development of nonlinear theories other than through direct numerical computation.

A description of desert dunes can be found in [14] and in Chap. 17. Recently, theoretical models similar to the nonlinear models proposed here have been advanced by Herrmann and co-workers to explain the form of barchan dunes [22]; they use an integral correction for the bed shear stress proposed by Jackson and Hunt [18]. The correction is similar to that used in (18.131), but the convolution kernel is of Cauchy type. The mechanics of the resulting instability is very similar though.

River flow. The hydraulics and processes of river flow are described in the books by Chow [4], Richards [28] and Knighton [21]. An account which is aimed at applied mathematicians is in the book by Fowler [12], and there are also the classic accounts of Stoker [32] and Whitham [36].

Drumlins. The literature on drumlins is substantial; their formation has been debated for well over one hundred years. However, the debate has been largely geological, and the dynamical concept of an instability is virtually absent. Amongst early authors, the papers of Davis [5], Upham [35] and Tarr [34] may be mentioned, as well the seminal paper of Kinahan and Close [20] – the last not easily accessible, but a copy may be obtained from The Royal Irish Academy, Dawson Street, Dublin. Useful early reviews are by Charlesworth [3] and Gravenor [15], and later developments can be followed in [7] and [33]. The erosional and flood theories are expounded by Boulton [2] and Shaw [29], for example. The instability theory derives from Hindmarsh [16]. A voluminous bibliography is that of Everett [10].

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