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A note on the derivation of the quasi-geostrophic potential vorticity equation

A. C. FOWLER*

Department of Mathematics and Statistics, MACSI, University of Limerick, Limerick, Republic of Ireland

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The derivation of the quasi-geostrophic potential vorticity equation of mathematical meteorology is usually done using fairly sophisticated techniques of perturbation theory, but stops short of deriving self-consistently the stratification parameter of the mean atmospheric state. In this note we suggest how this should be done within the confines of the theory, and as a consequence we raise the possibility that the atmosphere could become globally unstable, with dramatic consequences.

Keywords: Quasi-geostrophic potential vorticity equation; Day after tomorrow

1. Introduction

One of Raymond Hide's key contributions in meteorological science was his introduction of the rotating annulus experiment (Hide 1958, Hide and Mason 1975). In these experiments, he drew a laboratory analogue of the mechanism of baroclinic instability, which provides a vehicle for the formation of atmospheric planetary waves in mid-latitudes (Eady 1949). Hide's intriguing observations of vacillations in the flow led to a whole host of further experimental work (Buzyna *et al*. 1991, Read *et al*. 1992, Castrejón-Pita and Read 2007), as well as to numerous dynamical system studies of simple models which could explain the observations (e.g. Lorenz 1963, Pedlosky 1970, 1971, Pedlosky and Frenzen 1980, Hart 1973, 1981, Gibbon and McGuinness 1981, Moroz and Brindley 1981).

At the heart of this laboratory analogy of the Earth's weather system, and the associated theoretical studies of these experiments, is the quasi-geostrophic potential vorticity (QG) equation, which was the first serious attempt to describe weather in a quantitative way; indeed, it was used as a predictive model in the early days of numerical weather forecasting.

The basis of this approximation is that rotation is large, or equivalently the Rossby number is small, and in addition the troposphere is shallow, in the sense that its depth is

*Email: andrew.fowler@ul.ie

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much less than the synoptic scale length of planetary wave systems. These two facts, together with the fact that the temperature is nearly adiabatic, lead to the assertion that at leading order, the wind velocity is given by the geostrophic approximation, which relates the horizontal wind velocity to the horizontal gradient of the pressure field. The quasi-geostrophic approximation finds a prescription for the horizontal variation of the pressure, and thus the wind velocity, by proceeding to next order in an asymptotic expansion in powers of the Rossby number. This theory is described in most books on meteorology, e.g. those of Gill (1982), Holton (2004), Vallis (2006) and Pedlosky (1987); the last of these gives the most mathematically detailed account, and inspires our discussion here.

Pedlosky's derivation consumes some 20 pages, and it is not our intention to trawl through this again; the bones of the derivation are described in the following section. Further detail will be given in my forthcoming book, *Mathematical Geoscience*, to be published by Springer. The main point of this note is the following. Although Pedlosky starts with a complete set of equations and boundary conditions, he is left towards the end of his analysis with a quantity S, which is called the stratification parameter. This is given in terms of the basic rest state of the atmosphere, and is proportional to the square of the Brunt–Väisälä frequency N, which describes the frequency of small oscillations in a stably-stratified atmosphere. Evidently, S is necessarily positive, but in Pedlosky's derivation, it is prescribed rather than derived. In a self-contained perturbation theory of a complete set of governing equations, this appears anomalous. Our aim here is to show how the stratification parameter can be derived from the model itself.

At the outset, it must be pointed out that in pedagogical texts such as those cited above, the derivation of the QG equation is inevitably based on a model of the Earth's atmosphere which makes numerous simplifying assumptions for the sole purpose of analytic tractability. Most obviously, the detail of radiative transfer is ignored, as is any description of moisture transport. The purpose of the derived equation is thus not that of specific prediction, but rather to provide an understanding of the way in which planetary waves are formed, for example. Our aim here is correspondingly didactic, and not prognostic.

A number of authors have considered the issue of how to determine the stratification of the troposphere. One can of course simply measure it. To provide a theoretical description, Stone (1972, 1973) and Stone and Carlson (1979) consider a radiative-convective model, which is based on an approximate form of the energy equation involving large-scale eddy transport, together with a number of *ad hoc* assumptions. Thuburn and Craig (1997) and Barry *et al.* (2000) consider a more elaborate mechanism, based on the concept of baroclinic adjustment: the atmosphere adjusts its state (through baroclinic eddies) so that its mean profile is neutrally stable to further baroclinic disturbance. The last two studies are based on numerical simulations, as is the more recent study of Schneider and O'Gorman (2008). Our purpose in this article is similarly motivated, but is couched in the formal framework of matched asymptotic expansions of the governing equations.

2. The quasi-geostrophic potential vorticity equation

Pedlosky (1987) begins with the equations of mass, momentum and energy for a compressible fluid on the surface of a sphere. There are eddy viscous and conductive (or radiative) terms, but these are small (though important), and the equations are in essence hyperbolic. To deal with mid-latitude motions, we use spherical coordinates r (radius), ϕ (azimuthal longitude) and latitude $\lambda = (\pi/2) - \theta$ (θ being the polar angle); then, because we are concerned with synoptic scale motions with horizontal length scales $l \sim 1000\,\text{km} \ll r_0 \approx 6370\,\text{km}$, where r_0 is the Earth's radius, we take local (near a particular latitude $\lambda = \lambda_0$) orthogonal coordinates (x, y, z), which are almost Cartesian, and in which x is eastwards, y is northwards and z is radially outwards from the Earth's geopotential surface. We omit detail of the mass and momentum equations, but write the energy equation explicitly as

$$\rho c_{\mathrm{p}} \frac{\mathrm{D}T}{\mathrm{D}t} - \frac{\mathrm{D}p}{\mathrm{D}t} = \nabla \cdot (\bar{k}\, \nabla\, T) + \rho LC, \tag{1}$$

where T is the absolute temperature, ρ is the air density, c_{p} is the specific heat at constant pressure, p is the pressure, $\bar{k}$ is an effective thermal conductivity, L is the latent heat and C is the condensation rate. One could also include a term representing absorption of short wave radiation, but this is small and is neglected here.

We conceive of the atmospheric motion largely occurring in the troposphere of typical depth $h \approx 10\,\text{km}$, and that the horizontal motions occur on the synoptic scale of length $l \approx 1000\,\text{km}$. We then define two geometric parameters,

$$\delta = \frac{h}{l}, \quad \Sigma = \frac{l}{r_0}. \tag{2}$$

The parameter δ is small, of order 0.01, and represents the fact that the flow is shallow, while the parameter $\Sigma \approx 0.16$ is also relatively small, and represents the degree of sphericity. Where it appears, it is multiplied either by the aspect ratio δ or by the (small) Rossby number, and is thus genuinely negligible.

The horizontal velocities are scaled with a typical value U, and then the vertical velocity is scaled with δU; the time scale is the advective time scale l/U, and the density, pressure and temperature are scaled with typical values ρ_0, p_0 and T_0, related by

$$p_0 = \frac{\rho_0 R T_0}{M_{\mathrm{a}}} = \rho_0 g h. \tag{3}$$

The extra scale can be taken to be the temperature scale T_0 determined by radiation balance. The surface pressure p_0 is determined by the mass of the atmosphere, so that (3) determines the scale height h.

It is usual in scaling equations that all the undetermined scales can be chosen by suitable internal balances. In the present case, the undetermined scales are U and l, and we determine these below. Although unrealistic for the Earth, it is common pedagogical practice (e.g. Houghton 2002) to assume a radiatively grey atmosphere, and if, in addition, we take the optical density to be large, the non-local radiative transfer equation can be approximately solved to determine an effective radiative thermal conductivity

$$k_R = \frac{16\sigma T^3}{3\kappa\rho}, \tag{4}$$

where σ is Stefan's constant, T is the absolute temperature and κ is the absorption coefficient. In addition, we can define a (vertical) eddy thermal conductivity k_T as

$$k_T = \rho c_{\mathrm{p}} \varepsilon_V, \tag{5}$$

where we may estimate $\varepsilon_V \sim 0.1\delta Uh$ (Pedlosky 1987). Estimates give $k_R \sim k_T \sim 10^5\,\mathrm{W\,m^{-1}\,K^{-1}}$, and so we suppose that the effective conductivity $\bar{k}$ in (1) is of order $k_0 \sim 2\times 10^5\,\mathrm{W\,m^{-1}\,K^{-1}}$, and then we put

$$\bar{k} = k_0 k^*. \tag{6}$$

The consequent definition of the (reduced) Péclet number is

$$Pe = \frac{Uh^2}{\kappa_0 l}, \tag{7}$$

where

$$\kappa_0 = \frac{k_0}{\rho_0 c_\mathrm{p}} \tag{8}$$

is the effective thermal diffusivity scale.

To estimate the condensation term C, we assume that the atmosphere is saturated, so that the vapour pressure is the saturation vapour pressure, given by the Clapeyron equation. Approximately, $C \approx -\mathrm{D}m/\mathrm{D}t$, where $m=\rho_\mathrm{v}/\rho$ is the mixing ratio (ρ_v is the vapour density), and using the perfect gas law and the Clapeyron equation, m can be obtained as a function of temperature and pressure. When written in terms of the dimensionless variables, we arrive, after some algebra, at the dimensionless energy equation in the form

$$\frac{p}{\theta}\left[1+\frac{\nu\Lambda aM}{T^2}\right]\frac{\mathrm{D}\theta}{\mathrm{D}t} = -\frac{\nu\Lambda M(\alpha a - T)}{T^2}\frac{\mathrm{D}p}{\mathrm{D}t} + \frac{1}{Pe}\left[\frac{\partial}{\partial z}\left(k^*\frac{\partial T}{\partial z}\right)\right], \tag{9}$$

where potential temperature θ and moisture M are defined by

$$\theta = \frac{T}{p^\alpha}, \quad M = \frac{1}{p}\exp\left[a\left(1-\frac{1}{T}\right)\right], \tag{10}$$

and

$$\nu = \frac{M_\mathrm{v} p_\mathrm{SV}^0}{M_\mathrm{a} p_0}, \quad \Lambda = \frac{L}{c_\mathrm{p} T_0}, \quad a = \frac{M_\mathrm{v} L}{RT_0}, \quad \alpha = \frac{R}{M_\mathrm{a} c_\mathrm{p}}; \tag{11}$$

M_v and M_a are molecular weights of vapour and air, respectively, p_SV^0 is a reference saturation vapour pressure (at temperature T_0) and R is the gas constant. The terms on the right-hand side of the energy equation (9) give explicit form to the general internal heating term Q used by Pedlosky (1987). Typical values of the parameters are $\nu \approx 0.01$, $\Lambda \approx 8.7$, $a \approx 18.8$ and $\alpha \approx 0.29$.

If we anticipate that $U \sim 20\,\mathrm{m\,s^{-1}}$, $l \sim 10^3\,\mathrm{km}$, then we find that $Pe \sim 10$, and the conductive term is small, and in fact comparable to the Rossby number. If we neglect the conductive term, then we can define a basic state from (9), given by solving

$$\frac{\mathrm{d}p_\mathrm{w}}{\mathrm{d}z} = -\frac{p_\mathrm{w}^{1-\alpha}}{\theta_\mathrm{w}}, \tag{12a}$$

$$\frac{\mathrm{d}\theta_\mathrm{w}}{\mathrm{d}z} = \frac{\nu\Lambda(a\alpha - \theta_\mathrm{w} p_\mathrm{w}^\alpha)M}{[\theta_\mathrm{w}^2 p_\mathrm{w}^{2\alpha} + \nu\Lambda aM]p_\mathrm{w}^\alpha}, \tag{12b}$$

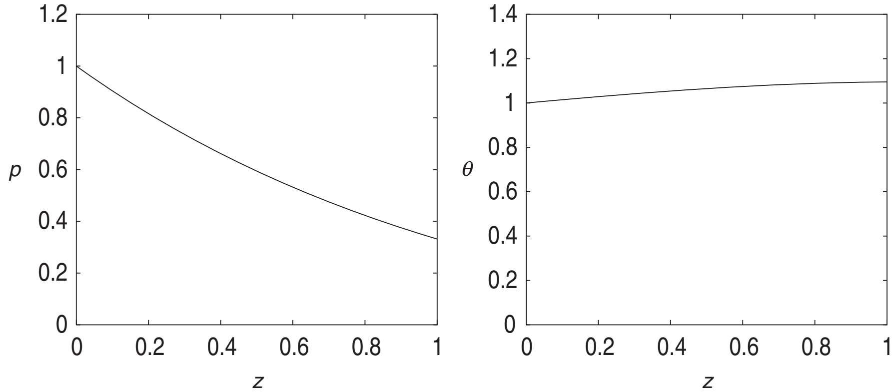

Figure 1. Basic dimensionless pressure and potential temperature profiles obtained by solving (12a) and (12b).

with $p_{\rm w}=\theta_{\rm w}=1$ at $z=0$. Figure 1 shows numerically computed profiles for the pressure and potential temperature thus defined. Note that the potential temperature varies by about 0.1 over the troposphere, so that the temperature profile is close to adiabatic (but the variation is significant, see (25) below, in that the Richardson number, defined by $Ri=(1/F^2\theta)\partial\theta/\partial z$ is large, of order $1/\varepsilon^2$, cf. (13) (Charney 1963)).

In scaling the momentum equation, we also find dimensionless Froude and Rossby numbers, defined respectively by

$$F=\frac{U}{\sqrt{gh}},\quad \varepsilon=\frac{U}{fl}, \tag{13}$$

where f is the Coriolis parameter

$$f=2\Omega\sin\lambda_0. \tag{14}$$

The whole basis of the quasi-geostrophic approximation is the notion that the Rossby number and heating terms are small, and the consequent balance of scales can be used to choose velocity and length scales. Specifically, we choose

$$\frac{F^2}{\varepsilon}=\frac{\alpha}{Pe}=\varepsilon^2, \tag{15}$$

and this leads to

$$U=\left(\frac{\alpha\kappa_0 g}{fh}\right)^{1/2},\quad l=U\left(\frac{h^2}{\alpha\kappa_0 f^2}\right)^{1/3}, \tag{16}$$

and calculation of these using values given previously leads to $U\approx 26\,\mathrm{m\,s^{-1}}$, $l\approx 1290\,\mathrm{km}$.

The quasi-geostrophic approximation is based on the formal limit $\varepsilon\ll 1$, and in particular, we assume the distinguished limits

$$\nu\Lambda\sim\Sigma\sim\varepsilon,\quad \delta\sim\varepsilon^2. \tag{17}$$

The mass and momentum equations then lead, as described by Pedlosky (1987) to the geostrophic wind at leading order, while the energy equation is approximately

$$\frac{p}{\theta}\left[1+\frac{\nu\Lambda a M}{T^2}\right]\frac{\mathrm{D}\theta}{\mathrm{D}t}=-\frac{\varepsilon s M(\alpha a-T)}{T^2}\frac{\mathrm{D}p}{\mathrm{D}t}+\varepsilon^2\left[\frac{\partial}{\partial z}\left(\frac{k^*}{\alpha}\frac{\partial T}{\partial z}\right)\right], \tag{18}$$

where we have written

$$\nu\Lambda=\varepsilon s \tag{19}$$

to delineate the smallness of $\nu\Lambda$ (but noting that $\nu\Lambda a\approx 1.64$ is O(1)).

Following Pedlosky (1987), and as suggested by (12), we define the perturbed potential temperature Θ by

$$\theta=\bar{\theta}(z)+\varepsilon^2\Theta; \tag{20}$$

evidently $\bar{\theta}(z)$ is the time and space-horizontal average of θ correct to $\mathrm{O}(\varepsilon^2)$, and we can in fact define it to be the exact such average of θ, without loss of generality. In addition, the vertical velocity w is scaled as

$$w=\varepsilon W; \tag{21}$$

the energy equation then takes the form

$$\frac{\mathcal{D}\Theta}{\mathcal{D}t}=H-WS, \tag{22}$$

where $\mathcal{D}/\mathcal{D}t$ is the horizontal material derivative

$$\frac{\mathcal{D}}{\mathcal{D}t}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\approx\frac{\partial}{\partial t}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}, \tag{23}$$

in which ψ is the geostrophic stream function. The heating term is

$$H=\frac{\partial}{\partial z}\left(\frac{k^*}{\alpha}\frac{\partial\bar{T}}{\partial z}\right)\bigg/\left\{\frac{\bar{p}}{\bar{\theta}}\left[1+\frac{\nu\Lambda a M(\bar{T},\bar{p})}{\bar{T}^2}\right]\right\} \tag{24}$$

(note that $H=H(z)$), and we define the stratification function $S(z)$ by

$$S(z)=\frac{1}{\varepsilon}\left[\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}z}-\frac{\mathrm{d}\theta_{\mathrm{w}}}{\mathrm{d}z}\right], \tag{25}$$

and note that by observation (and assumption) it is positive and O(1).

Thus far the presentation exactly parallels that of Pedlosky (1987), with the exception that the assumption of a grey, opaque, saturated atmosphere allows us to specify the internal heating term, and by construction the horizontal average of Θ is zero.

Expansion of the hydrostatic component of the momentum equation shows that

$$\Theta=\frac{\partial\psi}{\partial z}, \tag{26}$$

and manipulation of the mass and momentum equations yields the vorticity equation

$$\frac{\mathcal{D}}{\mathcal{D}t}[\zeta+\beta y]=\frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho}W)}{\partial z}, \tag{27}$$

where

$$\zeta = \nabla^2 \psi \tag{28}$$

is the vorticity, $\bar{\rho}$ is the density profile corresponding to a hydrostatic atmosphere with potential temperature $\bar{\theta}(z)$ and the term in β arises from the variation of sin λ with latitude; β is defined by

$$\beta = \frac{\Sigma \cot \lambda_0}{\varepsilon}, \tag{29}$$

and we take $\beta = O(1)$.

In summary, we have the two separate equations for ψ and $\Theta = \partial\psi/\partial z$ (27) and (22), from which W and $S(z)$ must also be determined, the latter by averaging the equations, so that $\bar{\Theta} = 0$ (an overbar denoting the horizontal space and time average).

By an application of Green's theorem in the plane, we have that

$$\iint_A \frac{\mathcal{D}\Gamma}{\mathcal{D}t}\, \mathrm{d}S = \frac{\partial}{\partial t}\iint_A \Gamma\, \mathrm{d}S - \oint_{\partial A} \Gamma\, \mathrm{d}\psi, \tag{30}$$

where A is any horizontal area at fixed z and Γ is any continuously differentiable scalar field. In particular, if A is a closed region on the boundaries of which ψ is constant in space, i.e. there is no flow through ∂A, then the boundary integral is zero.† With an overbar denoting a time and horizontal space average over A (assuming solutions are stationary, i.e., with long term mean of zero), we apply this to (22) and (27) to find

$$H = \overline{W}S, \quad \bar{\rho}\overline{W} = \overline{W}_0, \tag{31}$$

where $\overline{W}_0$ is the surface value of $\overline{W}$ at $z = 0$ (since $\bar{\rho}(0) = 1$).

Near the Earth's surface, the value of $\overline{W}_0$ is determined by an analysis of the Ekman boundary layer, which occurs due to the presence of the small eddy diffusive friction terms (ignored thus far). When this is done, we find that

$$\overline{W}_0 = E^* \bar{\zeta}_0, \tag{32}$$

where $\bar{\zeta}_0$ is the space averaged vorticity at the surface (assuming flat topography), and

$$E^* = \sqrt{\frac{E}{2\varepsilon^2}}, \tag{33}$$

with the Ekman number being defined as

$$E = \frac{\varepsilon_V}{fh^2}. \tag{34}$$

Practical estimates are $\varepsilon \approx 0.2$, $E \approx 10^{-2}$, and thus $E^* \approx 0.35$.

The two equations in (31) define S and $\overline{W}$, and in particular we find that

$$\frac{\bar{\rho}}{S} = \frac{E^* \bar{\zeta}_0}{H}. \tag{35}$$

†We have in mind that A is the region of zonal mid-latitude flow, bounded to the north by the polar front, and to the south by the tropical front. We can allow A to be a periodic strip on the sphere also.

This equation thus defines the stratification function $S(z)$ for a stationary (but not necessarily steady) atmosphere. Evidently, the wet adiabatic profile ($S = 0$) is obtained (in stationary conditions) only if the heating rate H is zero.

We can now use the identity

$$\frac{\partial}{\partial z}\left[K(z)\frac{\mathcal{D}\Theta}{\mathcal{D}t}\right] = \frac{\mathcal{D}}{\mathcal{D}t}\left[\frac{\partial}{\partial z}\left(K(z)\frac{\partial\psi}{\partial z}\right)\right] \tag{36}$$

(where we use the approximating definition of $\mathcal{D}/\mathcal{D}t$ in (23) together with the thermal wind equation (26)) to show, using (22), that

$$\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}[\bar{\rho}W] = \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left[\frac{\bar{\rho}H}{S}\right] - \frac{\mathcal{D}}{\mathcal{D}t}\left[\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\frac{\bar{\rho}}{S}\frac{\partial\psi}{\partial z}\right)\right], \tag{37}$$

and therefore (27) can be written

$$\frac{\mathcal{D}}{\mathcal{D}t}\left[\nabla^2\psi + \beta y + \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\frac{\bar{\rho}}{S}\frac{\partial\psi}{\partial z}\right)\right] = \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left[\frac{\bar{\rho}H}{S}\right]. \tag{38}$$

This is one form of the *quasi-geostrophic potential vorticity equation* for the geostrophic stream function ψ (cf. Pedlosky 1987, equation (6.5.18)).

3. Discussion

Our derivation here exactly mirrors that of Pedlosky, except that we prescribe the heating term H in (24), and we determine the stratification parameter via (35). This indicates that the right-hand side of (38) is zero even when heating is non-zero, and the equation can be written in the form

$$\left\{\frac{\partial}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right\}\left[\nabla^2\psi + \frac{E^*\overline{(\nabla^2\psi)}_0}{\bar{\rho}}\frac{\partial}{\partial z}\left(\frac{1}{H}\frac{\partial\psi}{\partial z}\right)\right] + \beta\frac{\partial\psi}{\partial x} = 0. \tag{39}$$

This must be supplemented with suitable boundary conditions on the surface $z = 0$ and (for example) the tropopause $z = z_T \sim O(1)$, which are not discussed further here.

Our main point is that the stratification parameter S is self-consistently determined by the quasi-geostrophic approximation itself, via (35). There is no intrinsic necessity for the time and space averaged vorticity $\overline{\nabla^2\psi_0}$ to be positive. In fact, neither is there an intrinsic necessity for H to be positive. Therefore it seems perfectly plausible that in solving (39), we may find that the solution develops a negative value of S.

Apart from an initial condition, natural boundary conditions for (39) are to prescribe boundary value expressions in terms of ψ on all the boundaries, which carries with it the implication that S should be non-negative in order that the model should be well-posed. Discussions of static stability (Pedlosky 1987, Holton 2004) generally suggest that should S become negative, then the atmosphere becomes top heavy and thus unstable. Holton (2004, p. 53) states that "on the synoptic scale the atmosphere is always stably stratified because any unstable regions that develop are stabilized quickly by convective overturning", and indeed such "rearrangements" are routinely employed in numerical codes for weather forecasting.

It is fairly obvious that, indeed, negative S would lead to rapid overturning. What does this mean for solutions of the QG equation? The derivation of the QG equation involves a number of assumptions, for example that w is small and the mean potential temperature gradient is close to that of the wet adiabat. Suppose that negative stratification leads to local small scale convection; it seems reasonable that although vertical velocities are then large, a locally spatially averaged velocity will have small vertical component. With this proviso, one can still derive the QG equation, with the modification that where S defined by (35) becomes negative, we should replace the mean vorticity $\bar{\zeta}_0 = \overline{(\nabla^2 \psi)_0}$ by, for example,

$$\zeta^* = \max(\bar{\zeta}_0, 0). \tag{40}$$

In our simple model, there is no distinction between continent and ocean, and the stratification is independent of horizontal location. If we are to interpret Earth's weather in terms of the QG equation, we would suppose that $S > 0$. But it is not inevitable that this should be the case. We have only to look at the other planets in the solar system to realise that planetary weather can work in different ways. Because (we suppose) weather on Earth has always been the way it is, we suppose that it will remain so. But another possibility is that as planetary conditions change, whether through changing continental configurations or, more recently, through anthropogenic carbon production, the mean vorticity predicted by (39), or the heating term, may become negative. In this case, the assumption of slowly spatially varying winds must break down, and the Earth's weather systems would undergo a régime change in places where $\bar{\zeta}_0 < 0$, perhaps to a climate in which the atmosphere boils relentlessly, and storms fill the sky. Such regions might occur patchily in space, or in extremes might occupy the entire planet.

4. The day after tomorrow

The popular film *The day after tomorrow* portrayed an apocalyptic climate catastrophe, in which ice shelf collapse induced ocean circulation changes, which produced global storms of enormous severity. The film is commonly denigrated, partly on the basis that the time scale of the events is entirely unrealistic.

But the film might have a hidden grain of truth. We now know that ice shelves do indeed collapse suddenly (Rott *et al.* 1996), and indeed both the Antarctic and Greenland ice sheets are currently undergoing rapid change in a way that could not have been imagined 20 years ago (Joughin *et al.* 2004, Wingham *et al.* 2009). In addition, the retrieval of ice cores from Antarctica and Greenland has shown that during the last ice age, there were many sudden shifts of climate (Dansgaard-Oeschger events) (Johnsen *et al.* 1992), whose origin is thought to lie in the alteration of North Atlantic circulation following injection of fresh water fluxes (Ganopolski and Rahmstorf 2001).

So we have learned that ice sheets can behave more rapidly than their convective time scale of thousands of years would suggest, and we have also learned that ocean circulation can possibly change rapidly, on time scales of decades. What we have not contemplated, however, is the possibility that the atmosphere could also alter its fundamental style of behaviour. Ice sheets and oceans have shown their capacity for rapid change; could the atmosphere behave similarly?

There is a potentially interesting analogy to be drawn with two-phase flows of, for example, air and water in a pipe. Two-phase flows occur in a number of régimes (e.g. Brennen 2005), but it is not known what determines the transition between these. One suggestion for the bubbly to slug transition is that bubbly flow becomes unstable to kinematic waves as the bubble volume (void) fraction increases (Matuszkiewicz *et al.* 1987). A common feature of two-phase flow models is that they can be ill-posed† in certain circumstances (e.g. Drew and Passman 1999), and one can show that in a bubbly flow, the onset of instability is a harbinger for ill-posedness, but instability occurs before ill-posedness (Prosperetti and Satrape 1990, Robinson *et al.* 2008). It is as if the system organises its régime in order to avoid ill-posedness, and it does this by engineering a transition by means of instability. One can equivalently show that in the primitive equations of atmosphere dynamics, negative values of S lead to an ill-posedness: the atmosphere is unstable at arbitrarily small wavelengths; further, a purely zonal flow (e.g. with uniform vertical shear) becomes baroclinically unstable for sufficiently small positive (prescribed) S, thus producing the cyclonic depressions whose positive vorticity may enable S to remain positive. But, perhaps the atmosphere is susceptible to the same kind of régime transition which one sees in two-phase flows. What might we expect to see if this was the case? Increasing fluctuations and storminess, presumably; for example, an increase in the frequency of tropical cyclones (Emanuel 2005).

While the atmosphere in the present day appears to behave in a classical quasi-geostrophic manner, we suggest here that, for one particular realisable set of assumptions (a grey, saturated, opaque atmosphere), the stability of the (model) atmosphere is determined by its own dynamics, and that in different climatic circumstances, it is at least possible that the atmosphere could become globally unstable, and as a consequence undergo a transition to a state in which stability is maintained by widespread convective overturning. We cannot say whether such a thing will occur for the QG equation (39) without further study; but if it would, then the prospect of similar behaviour for the Earth's atmosphere could not be discounted, and it would be prudent to be aware of this possibility. If the atmosphere were to undergo such a régime transition, we can be assured that, as in the film, it would be rapid. The governing convective time scale of synoptic scale motions is a bare 14 h.

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†By ill-posed, we mean (for linear models) that normal modes of the typical form $\exp(\sigma t + i\mathbf{k}\cdot\mathbf{x})$ have unbounded growth rates at high wave number; a typical form for negative diffusion is $\sigma \sim k^2$. More generally, nearby solutions can diverge arbitrarily rapidly.

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