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Multilevel Monte Carlo For Exponential Lévy Models

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Abstract We apply the multilevel Monte Carlo method for option pricing problems using exponential Lévy models with a uniform timestep discretisation. For lookback and barrier options, we derive estimates of the convergence rate of the error introduced by the discrete monitoring of the running supremum of a broad class of Lévy processes. We then use these to obtain upper bounds on the multilevel Monte Carlo variance convergence rate for the Variance Gamma, NIG and α-stable processes. We also provide analysis of a trapezoidal approximation for Asian options. Our method is illustrated by numerical experiments.

Keywords multilevel Monte Carlo · exponential Lévy models · Asian options · lookback options · barrier options

Mathematics Subject Classification (2000) MSC 65C05 · MSC 91G60

1 Introduction

Exponential Lévy models are based on the assumption that asset returns follow a Lévy process [25,10]. The asset price follows

$$S_t = S_0 \exp(X_t) \tag{1.1}$$

where X_t is an (m,σ,ν)-Lévy process

$$X_t = mt + \sigma B_t + \int_0^t \int_{|z|\geq 1} z\, J(\mathrm{d}z,\mathrm{d}s) + \int_0^t \int_{|z|<1} z\,(J(\mathrm{d}z,\mathrm{d}s) - \nu(\mathrm{d}z)\mathrm{d}s)$$

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where m is a constant, B_t is a Brownian Motion, J is the jump measure and ν is the Lévy measure [24].

Models with jumps give an intuitive explanation of implied volatility skew and smile in the index option market and foreign exchange market [10]. The jump fear is mainly on the downside in the equity market which produces a premium for low-strike options; the jump risk is symmetric in the foreign exchange market so the implied volatility has a smile shape. [10] shows that models building on pure jump processes can reproduce the stylized facts of asset returns, like heavy tails and the asymmetric distribution of increments. Since pure jump processes of finite activity without a diffusion component cannot generate a realistic path, it is natural to allow the jump activity to be infinite. In this work we deal with infinite-activity pure jump exponential Lévy models, in particular models driven by Variance Gamma (VG), Normal Inverse Gaussian (NIG) and α-stable processes which allow direct simulation of increments.

We are interested in estimating the expected payoff value $\mathbb{E}[f(S)]$ in option pricing problems. In the case of European options, it is possible to directly sample the final value of the underlying Lévy process, but in the case of Asian, lookback and barrier options the option value depends on functionals of the Lévy process and so it is necessary to approximate those. In the case of a VG model with a lookback option, the convergence results in [13] show that to achieve an $\mathcal{O}(\varepsilon)$ root mean square (RMS) error using a standard Monte Carlo method with a uniform timestep discretisation requires $\mathcal{O}(\varepsilon^{-2})$ paths, each with $\mathcal{O}(\varepsilon^{-1})$ timesteps, leading to a computational complexity of $\mathcal{O}(\varepsilon^{-3})$.

In the case of simple Brownian diffusion, Giles [16,17] introduced a multilevel Monte Carlo (MLMC) method, reducing the computational complexity from $\mathcal{O}(\varepsilon^{-3})$ to $\mathcal{O}(\varepsilon^{-2})$ for a variety of payoffs. The objective of this paper is to investigate whether similar benefits can be obtained for exponential Lévy processes.

Various researchers have investigated simulation methods for the running maximum of Lévy processes. Reference [15] develops an adaptive Monte Carlo method for functionals of killed Lévy processes with a controlled bias. Small-time asymptotic expansions of the exit probability are given with computable error bounds. For evaluating the exit probability when the barrier is close to the starting point of the process, this algorithm outperforms a uniform discretisation significantly. Reference [20] develops a novel Wiener-Hopf Monte-Carlo method to generate the joint distribution of $\left(X_T, \sup_{0\le t\le T} X_t\right)$ which is further extended to MLMC in [14], obtaining an RMS error ε with a computational complexity of $\mathcal{O}\left(\varepsilon^{-3}\right)$ for Lévy processes with bounded variation and $\mathcal{O}\left(\varepsilon^{-4}\right)$ for processes with infinite variation. The method currently cannot be directly applied to VG, NIG and $\alpha-$stable processes. References [12,11] adapt MLMC to Lévy-driven SDEs with payoffs which are Lipschitz w.r.t. the supremum norm. If the Lévy process does not incorporate a Brownian process, reference [11] obtains an $\mathcal{O}\left(\varepsilon^{-(6\beta)/(4-\beta)}\right)$ upper bound on the worst case computational complexity, where β is the BG index which will be defined later.

In contrast to those advanced techniques, we take the discretely monitored maximum based on a uniform timestep discretisation of the Lévy process as the approximation. The outline of the work is as follows. First we review the Multilevel Monte

Carlo method and present the three Lévy processes we will consider in our numerical experiments. To prepare for the analysis of the multilevel variance of lookback and barrier, we bound the convergence rate of the discretely monitored running maximum for a large class of Lévy processes whose Lévy measures have a power law behavior for small jumps, and have exponential tails. Based on this, we conclude by bounding the variance of the multilevel estimators. Numerical results are then presented for the multilevel Monte Carlo applied to Asian, lookback and barrier options using the three different exponential Lévy models.

2 Multilevel Monte Carlo (MLMC) method

For a path-dependent payoff P based on an exponential Lévy model on the time interval $[0,T]$, let $\widehat{P}_\ell$ denote its approximation using a discretisation with M^ℓ uniform timesteps of size $h_\ell = M^{-\ell}T$ on level ℓ; in the numerical results reported later, we use $M=2$. Due to the linearity of the expectation operator, we have the following identity:

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]. \tag{2.1}$$

Let $\widehat{Y}_0$ denote the standard Monte Carlo estimate for $\mathbb{E}[\widehat{P}_0]$ using N_0 paths, and for $\ell > 0$, we use N_ℓ independent paths to estimate $\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ using

$$\widehat{Y}_\ell = N_\ell^{-1} \sum_{i=1}^{N_\ell} \left(\widehat{P}_\ell^{(i)} - \widehat{P}_{\ell-1}^{(i)} \right). \tag{2.2}$$

For a given path generated for $\widehat{P}_\ell^{(i)}$, we can calculate $\widehat{P}_{\ell-1}^{(i)}$ using the same underlying Lévy path. The multilevel method exploits the fact that $V_\ell := \mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ decreases with ℓ, and adaptively chooses N_ℓ to minimise the computational cost to achieve a desired RMS error. This is summarized in the following theorem in [18, 19]:

Theorem 2.1 *Let P denote a functional of S_t, and let $\widehat{P}_\ell$ denote the corresponding approximation using a discretisation with uniform timestep $h_\ell = M^{-\ell}T$. If there exist independent estimators $\widehat{Y}_\ell$ based on N_ℓ Monte Carlo samples, each with complexity C_ℓ, and positive constants $\alpha, \beta, c_1, c_2, c_3$ such that $\alpha \geq \frac{1}{2}\min(1,\beta)$ and*

i) $\left|\mathbb{E}[\widehat{P}_\ell - P]\right| \leq c_1 h_\ell^\alpha$

ii) $\mathbb{E}[\widehat{Y}_\ell] = \begin{cases} \mathbb{E}[\widehat{P}_0], & \ell = 0 \\ \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], & \ell > 0 \end{cases}$

iii) $\mathbb{V}[\widehat{Y}_\ell] \leq c_2 N_\ell^{-1} h_\ell^\beta$

iv) $C_\ell \leq c_3 N_\ell h_\ell^{-1}$,

then there exists a positive constant c_4 such that for any $\varepsilon < e^{-1}$ there are values L and N_ℓ for which the multilevel estimator

$$\widehat{Y} = \sum_{\ell=0}^{L} \widehat{Y}_\ell,$$

has a mean-square-error with bound

$$MSE \equiv \mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}[P]\right)^2\right] < \varepsilon^2$$

with a computational complexity C with bound

$$C \leq \begin{cases} c_4\,\varepsilon^{-2}, & \beta > 1, \\ c_4\,\varepsilon^{-2}(\log\varepsilon)^2, & \beta = 1, \\ c_4\,\varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

We will focus on the multilevel variance convergence rate β in the following numerical results and analysis since it is crucial in determining the computational complexity.

3 Lévy models

The numerical results to be presented later use the following three models.

3.1 Variance Gamma (VG)

The VG process with parameter set (σ, θ, κ) is the Lévy process X_t with characteristic function $\mathbb{E}[\exp(iuX_t)] = (1 - iu\theta\kappa + \frac{1}{2}\sigma^2 u^2 \kappa)^{-t/\kappa}$. The Lévy measure of the VG process is ([10] p117)

$$\nu(x) = \frac{1}{\kappa\,|x|} e^{A-B|x|} \;\text{ with }\; A = \frac{\theta}{\sigma^2} \;\text{ and }\; B = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}.$$

One advantage of the VG process is that its additional parameters make it possible to fit the skewness and kurtosis of the stock returns ([10]). Another is that it is easily simulated as we have a subordinator representation $X_t = \theta G_t + \sigma B_{G_t}$ in which B_t is a Brownian process and the subordinator G_t is a Gamma process with parameters $(1/\kappa, 1/\kappa)$.

For the ease of computation, we follow the mean-correcting pricing measure in [25], with risk-free interest rate $r = 0.05$. This results in the drift being

$$m = r + \kappa^{-1}\log(1 + \theta\kappa - \tfrac{1}{2}\sigma^2\kappa).$$

The calibration in the same book gives $\sigma = 0.1213, \theta = -0.1436, \kappa = 0.1686$.

3.2 Normal Inverse Gaussian (NIG)

The NIG process with parameter set (σ,θ,κ) is the Lévy process X_t with characteristic function $\mathbb{E}[\exp(iuX_t)] = \exp\left(\frac{t}{\kappa} - \frac{t}{\kappa}\sqrt{1-2iu\theta\kappa+\kappa\sigma^2u^2}\right)$ and Lévy measure

$$\nu(x) = \frac{C}{\kappa|x|}e^{Ax}K_1(B|x|) \text{ with } A = \frac{\theta}{\sigma^2},\; B = \frac{\sqrt{\theta^2+\sigma^2/\kappa}}{\sigma^2},\; C = \frac{\sqrt{\theta^2+2\sigma^2/\kappa}}{2\pi\sigma\sqrt{\kappa}}.$$

$K_n(x)$ is the modified Bessel function of the second kind ([10] p117). As $x \to 0$, $K_1(x) \sim \frac{1}{x} + \mathscr{O}(1)$, while as $x \to \infty$, $K_1(x) \sim e^{-x}\sqrt{\frac{\pi}{2|x|}}\left(1+\mathscr{O}\left(\frac{1}{|x|}\right)\right)$.

In terms of simulation, the NIG process can be represented as $X_t = \theta I_t + \sigma B_{I_t}$, where the subordinator I_t is an Inverse Gaussian process with parameters $(\frac{1}{\kappa},1)$. Algorithm in p184 of [10] can be used to generate Inverse Gaussian samples.

Using the mean-correcting pricing measure leads to

$$m = r - \kappa^{-1} + \pi CB\kappa^{-1}\sqrt{B^2-(A+1)^2}.$$

Following the calibration in [25] we use the parameters $\sigma = 0.1836, \theta = -0.1313, \kappa = 1.2819$, and again use risk-free interest rate $r = 0.05$.

3.3 Spectrally negative α-stable process

The scalar spectrally negative α-stable process has a Lévy measure of the form ([10]):

$$\nu(x) = \frac{B}{|x|^{\alpha+1}}1_{\{x<0\}}$$

for $0 < \alpha < 2$ and some non-negative B. We follow the reference to discuss another parameterisation of α-stable process with characteristic function

$$\begin{aligned} \mathbb{E}[\exp(iuX_t)] &= \exp\left\{-tB^\alpha|u|^\alpha\left(1+i\,\mathrm{sgn}(u)\tan\tfrac{\pi\alpha}{2}\right)\right\}, \text{ if } \alpha \neq 1,\\ \mathbb{E}[\exp(iuX_t)] &= \exp\left\{-tB|u|\left(1+i\tfrac{2}{\pi}\mathrm{sgn}(u)\log|u|\right)\right\}, \text{ if } \alpha = 1, \end{aligned} \tag{3.1}$$

where $\mathrm{sgn}(u) = |u|/u$ if $u \neq 0$ and $\mathrm{sgn}(0) = 0$; see [23].There are no positive jumps for the spectrally negative process, which has a finite exponential moment $\mathbb{E}[\exp(uX_t)]$ [4].

For this case, the mean-correcting drift is

$$m = r + B^\alpha \sec\frac{\alpha\pi}{2}.$$

Sample paths of α-stable processes can be generated by the algorithm in [5]. Following [4], we use the parameters $\alpha = 1.5597$ and $B = 0.1486$.

4 Key numerical analysis results

The variance of the multilevel correction, $V_\ell = \mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ depends on the behavior of the difference between the continuously and discretely monitored suprema of X_t, defined for a unit time interval as

$$D_n = \sup_{0\le t\le 1} X_t \; - \max_{i=0,1,\ldots,n} X_{i/n}.$$

To derive the order of weak convergence for lookback-type payoffs, we are concerned with $\mathbb{E}[D_n]$, which is extensively studied in the literature. For example, [13], [8] and [9] derive asymptotic expansions for jump-diffusion, VG, NIG processes, as well as estimates for general Lévy processes, by using Spitzer's identity [26].

A key result due to Chen [8] is the following:

Theorem 4.1 *Suppose X_t is a scalar Lévy process with triple (m,σ,ν), with finite first moment, i.e.*

$$\int_{|x|>1} |x| \;\nu(\mathrm{d}x) < \infty.$$

Then $D_n = \sup_{0\le t\le 1} X_t - \max_{i=0,1,\ldots,n} X_{i/n}$ satisfies

1. *If $\sigma > 0$*
$$\mathbb{E}[D_n] = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right);$$
2. *If $\sigma = 0$ and X_t is of finite variation, i.e. $\int_{|x|<1} |x|\,\nu(\mathrm{d}x) < \infty$*
$$\mathbb{E}[D_n] = \mathcal{O}\left(\frac{\log n}{n}\right);$$
3. *If $\sigma = 0$ and X_t is of infinite variation, then*
$$\mathbb{E}[D_n] = o\left(n^{-1/\beta+\delta}\right),$$
where
$$\beta = \inf\left\{\alpha > 0 : \int_{|x|<1} |x|^\alpha \,\nu(\mathrm{d}x) < \infty\right\}$$
is the Blumenthal-Getoor index of X_t, and $\delta > 0$ is an arbitrarily small strictly positive constant.

The VG process has finite variation with Blumenthal-Getoor index 0; the NIG process has infinite variation with Blumenthal-Getoor index 1. They correspond to the second and third cases of Theorem 4.1 respectively.

For the multilevel variance analysis we require higher moments of D_n. In the pure Brownian case, Asmussen *et al* ([1]) obtain the asymptotic distribution of D_n, which in turn gives the asymptotic behavior of $\mathbb{E}\left[D_n^2\right]$. [13] extends the result to finite activity jump processes with non-zero diffusion.

However, in this paper we are looking at infinite activity jump processes. Our main new result is therefore concerned with the L^p convergence rate of D_n for pure jump Lévy processes. This will be used later to bound the variance of the Multilevel Monte Carlo correction term V_ℓ for both lookback and barrier options.

Theorem 4.2 *Let X_t be a scalar pure jump Lévy process, and suppose its Lévy measure $\nu(x)$ satisfies*

$$\begin{aligned} C_2\,|x|^{-1-\alpha} &\le \nu(x) \le C_1\,|x|^{-1-\alpha}, \text{ for } |x| \le 1; \\ \nu(x) &\le \exp\left(-C_3\,|x|\right), \text{ for } |x| > 1, \end{aligned} \tag{4.1}$$

where $C_1, C_2, C_3 > 0$, $0 \le \alpha < 2$ are constants. Then for $p \ge 1$

$$D_n = \sup_{0\le t\le 1} X_t \;-\; \max_{i=0,1,\dots,n} X_{i/n}$$

satisfies

$$\mathbb{E}\left[D_n^p\right] = \begin{cases} \mathscr{O}\left(\frac{1}{n}\right), & p > 2\alpha; \\ \mathscr{O}\left(\left(\frac{\log n}{n}\right)^{\frac{p}{2\alpha}}\right), & p \le 2\alpha. \end{cases}$$

If, in addition, X_t is spectrally negative, i.e. $\nu(x) = 0$ for $x > 0$, then

$$\mathbb{E}\left[D_n^p\right] = \begin{cases} \mathscr{O}\left(n^{-p}\right), & 0 \le \alpha < 1; \\ o\left(n^{-p/\alpha+\delta}\right), & 1 \le \alpha < 2; \end{cases}$$

for any $\delta > 0$.

We will give the proof of this result later in Section 7.6. Note that for $p=1$, the general bound in Theorem 4.2 is slightly sharper than Chen's result for $\alpha < \frac{1}{2}$, is the same for $\alpha = \frac{1}{2}$, and is not as tight as Chen's result for $\frac{1}{2} < \alpha < 2$; the spectrally negative bound is slightly sharper than Chen's result for $\alpha < 1$, and the bound is the same for $1 \le \alpha < 2$.

5 MLMC analysis

5.1 Asian options

We consider the analysis for a Lipschitz arithmetic Asian payoff $P = P\left(\overline{S}\right)$ where

$$\overline{S} = S_0\, T^{-1} \int_0^T \exp\left(X_t\right) \mathrm{d}t\ .$$

and P is Lipschitz such that $|P(S_1) - P(S_2)| \le L_K\,|S_1 - S_2|$.

We approximate the integral using a trapezoidal approximation:

$$\widehat{\overline{S}} := S_0\, T^{-1} \sum_{j=0}^{n-1} \tfrac{1}{2} h\ \left(\exp\left(X_{jh}\right) + \exp\left(X_{(j+1)h}\right)\right), \tag{5.1}$$

and the approximated payoff is then $\widehat{P} = P\left(\widehat{\overline{S}}\right)$.

Proposition 5.1 *Let X_t be a scalar Lévy process underlying an exponential Lévy model. If $\widehat{\overline{S}}, \overline{S}$ are as defined above, and $\int_{|z|>1} e^{2z}\, \nu(\mathrm{d}z) < \infty$, then*

$$\mathbb{E}\left[\left(\widehat{\overline{S}} - \overline{S}\right)^2\right] = \mathscr{O}(h^2).$$

The proof will be given later in Section 7.1. Using the Lipschitz property, the weak convergence for the numerical approximation is given by

$$\left|\mathbb{E}\left[\widehat{P}_\ell - P\right]\right| \le L_K \mathbb{E}\left[\left|\widehat{\overline{S}}_\ell - \overline{S}\right|\right] \le L_K \left(\mathbb{E}\left[\left(\widehat{\overline{S}} - \overline{S}\right)^2\right]\right)^{1/2},$$

while the convergence of the MLMC variance follows from

$$\begin{aligned} V_\ell &\le \mathbb{E}\left[\left(\widehat{P}_\ell - \widehat{P}_{\ell-1}\right)^2\right] \\ &\le 2\,\mathbb{E}\left[\left(\widehat{P}_\ell - P\right)^2\right] + 2\,\mathbb{E}\left[\left(\widehat{P}_{\ell-1} - P\right)^2\right] \\ &\le 2L_K^2\,\mathbb{E}\left[\left(\widehat{\overline{S}}_\ell - \overline{S}\right)^2\right] + 2L_K^2\,\mathbb{E}\left[\left(\widehat{\overline{S}}_{\ell-1} - \overline{S}\right)^2\right]. \end{aligned}$$

5.2 Lookback options

In exponential Lévy models, the moment generating function $\mathbb{E}\left[\exp\left(q \sup_{0\le t\le T} X_t\right)\right]$ can be infinite for large value of q. To avoid problems due to this, we consider a lookback put option which has a bounded payoff

$$P = \exp(-rT)\,(K - S_0 \exp(m))^+, \tag{5.2}$$

where $m = \sup_{0\le t\le T} X_t$. Note that P is a Lipschitz function of m, since we have $|P'(x)| \le K$. Without loss of generality, we assume $T = 1$ in the following.

Because of the Lipschitz property, we have $\left|\mathbb{E}[P - \widehat{P}_\ell]\right| \le K\,\mathbb{E}[D_n]$ where $n = M^\ell = h_\ell^{-1}$. Therefore we obtain weak convergence for the processes covered by Theorem 4.1, with the convergence rate given by the Theorem.

To analyse the variance, $V_\ell = \mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$, we first note that

$$0 \le \max_{0\le i\le M^\ell} X_{i/M^\ell} - \max_{0\le i\le M^{\ell-1}} X_{i/M^{\ell-1}} \le \sup_{0\le t\le 1} X_t - \max_{0\le i\le M^{\ell-1}} X_{i/M^{\ell-1}} = D_n$$

where $n = M^{\ell-1}$. Hence, we have

$$V_\ell \;\le\; \mathbb{E}\left[\left(\widehat{P}_\ell - \widehat{P}_{\ell-1}\right)^2\right] \;\le\; K^2\,\mathbb{E}[D_n^2].$$

Theorem 4.2 then provides the following bounds on the variance for the VG, NIG and spectrally negative α-stable processes.

Proposition 5.2 *Let X_t be a scalar Lévy process underlying an exponential Lévy model. For the Lipschitz lookback put payoff (5.2), we have the following multilevel variance convergence rate results:*

1. *If X_t is a Variance Gamma (VG) process, then $V_\ell = \mathcal{O}(h_\ell)$;*
2. *If X_t is a Normal Inverse Gaussian (NIG) process, then $V_\ell = \mathcal{O}(h_\ell |\log h_\ell|)$;*
3. *If X_t is a spectrally negative α-stable process with $\alpha > 1$, then $V_\ell = o\left(h_\ell^{2/\alpha-\delta}\right)$, for any small $\delta > 0$.*

5.3 Barrier options

We consider a bounded up-and-out barrier option with discounted payoff

$$P \;=\; \exp(-rT)\, f(S_T)\, \mathbf{1}_{\{\sup_{0<t<T} S_t < B\}} \;=\; \exp(-rT)\, f(S_T)\, \mathbf{1}_{\{m<\log(B/S_0)\}}, \qquad (5.3)$$

where again $m = \sup_{0<t<T} X_t$, and $|f(x)| \leq F$ is bounded. On level ℓ, the numerical approximation is

$$\widehat{P}_\ell = \exp(-rT)\; f(S_T)\; \mathbf{1}_{\{\widehat{m}_\ell<\log(B/S_0)\}}. \qquad (5.4)$$

where $\widehat{m}_\ell = \max_{0\leq i\leq M^\ell} X_{ih_\ell}$.

Our analysis for NIG and the spectrally negative α-stable processes requires the following quite general result.

Proposition 5.3 *If m is a random variable with a locally bounded density in a neighbourhood of B, and $\widehat{m}$ is a numerical approximation to m, then for any $p>0$ there exists a constant $C_p(B)$ such that*

$$\mathbb{E}\left[\,\left|\mathbf{1}_{\{m<B\}} - \mathbf{1}_{\{\widehat{m}<B\}}\right|\,\right] < C_p(B)\; \|m-\widehat{m}\|_p^{p/(p+1)}.$$

Proof This result was first proved by Avikainen (Lemma 3.4 in [2]), but we give here a simpler proof. If, for some fixed $X>0$, we have $|m-B| > X$ and $|m-\widehat{m}| < X$, then $\mathbf{1}_{m<B} - \mathbf{1}_{\widehat{m}<B} = 0$. Hence,

$$\begin{aligned}\mathbb{E}[\,|\mathbf{1}_{m<B} - \mathbf{1}_{\widehat{m}<B}|\,] &\leq \mathbb{P}(|m-B| \leq X) + \mathbb{P}(|m-\widehat{m}| \geq X)\\ &\leq 2\,\rho_{sup}(B)\,X + X^{-p}\; \|m-\widehat{m}\|_p^p\end{aligned}$$

with the first term being due to the local bound $\rho_{sup}(B)$ of m's density and the second term due to the Markov inequality. Differentiating the upper bound w.r.t. X, we find that it is minimised by choosing $X^{p+1} = \frac{p}{2\rho_{sup}(B)}\; \|m-\widehat{m}\|_p^p$, and we then get the desired bound.

Our analysis for the Variance Gamma process requires a sharper result customised to the properties of Lévy processes.

Proposition 5.4 *If X_t is a scalar pure jump Lévy process satisfying the conditions of Theorem 4.2 with $0 \le \alpha \le \frac{1}{2}$, and m and $\widehat{m}_n$ are the continuously and discretely monitored suprema of X_t and m has a locally bounded density in a neighbourhood of B, then*

$$\mathbb{E}\left[\left|\mathbf{1}_{\{m<B\}} - \mathbf{1}_{\{\widehat{m}<B\}}\right|\right] = o(n^{-1/(1+2\alpha)+\delta}),$$

for any $\delta > 0$.

The proof is given later in Section 7.7.

Both of the above propositions require the condition that the supremum m has a locally bounded density for all strictly positive values. There is considerable current research on the supremum of Lévy processes [6,7,21,22]. In particular, the comments following Proposition 2 in [7] indicate that the condition is satisfied by stable processes, and by a wide class of symmetric subordinated Brownian motions. Unfortunately, the VG and NIG processes in the current paper are not symmetric, so at present they lie outside the range of current theory, but new theory under development [3] will extend the property to a larger class of Lévy processes including both VG and NIG.

We now bound the weak convergence of the estimator and the multilevel variance convergence.

Proposition 5.5 *Let X_t be a scalar Lévy process underlying an exponential Lévy model. For the up-and-out barrier option payoff (5.3), with the numerical approximation (5.4), we have the following rates of convergence for the multilevel correction variance and the weak error, assuming that m has a bounded density:*

– *If X_t is a Variance Gamma (VG) process, then*

$$V_\ell = o(h_\ell^{1-\delta});$$
$$\left|\mathbb{E}\left[\widehat{P} - P\right]\right| = o(h_\ell^{1-\delta})$$

where δ is an arbitrary positive number.

– *If X_t is a NIG process, then*

$$V_\ell = o(h_\ell^{1/2-\delta});$$
$$\left|\mathbb{E}\left[\widehat{P} - P\right]\right| = o(h_\ell^{1/2-\delta})$$

where δ is an arbitrary positive number.

– *If X_t is a spectrally negative α-stable process with $\alpha > 1$, then*

$$V_\ell = o\left(h_\ell^{\frac{1}{\alpha}-\delta}\right);$$
$$\left|\mathbb{E}\left[\widehat{P} - P\right]\right| = o\left(h_\ell^{\frac{1}{\alpha}-\delta}\right)$$

where δ is an arbitrary positive number.

Proof The variance of the multilevel correction term is bounded by

$$V_\ell \;\leq\; \mathbb{E}\left[\left(\widehat{P}_\ell - \widehat{P}_{\ell-1}\right)^2\right] \;\leq\; 2\,\mathbb{E}\left[\left(\widehat{P}_\ell - P\right)^2\right] + 2\,\mathbb{E}\left[\left(\widehat{P}_{\ell-1} - P\right)^2\right].$$

For an up-and-out Barrier option, since the payoff is bounded we have

$$\mathbb{E}\left[\left(\widehat{P}_\ell - P\right)^2\right] \leq F^2\,\mathbb{E}\left[\mathbf{1}_{\{\widehat{m}_n<\log(B/S_0)\}} - \mathbf{1}_{\{m<\log(B/S_0)\}}\right],$$
$$\left|\mathbb{E}\left[\widehat{P}_\ell - P\right]\right| \leq F\,\mathbb{E}\left[\mathbf{1}_{\{\widehat{m}_n<\log(B/S_0)\}} - \mathbf{1}_{\{m<\log(B/S_0)\}}\right],$$

where $n = M^\ell$.

The bounds for the VG process come from Proposition 5.4 together with the results from Theorem 4.2.

The bounds for the NIG come from taking $p=1$ in Proposition 5.3 together with Chen's result Theorem 4.1.

The bounds for the spectrally negative α-stable process come from Proposition 5.3 together with the results from Theorem 4.2. Theorem 4.2 gives

$$\|m-\widehat{m}\|_p^{p/(p+1)} \;\equiv\; (\mathbb{E}\,[\,|m-\widehat{m}|^p])^{1/(p+1)} \;=\; o(h^{\frac{p}{(p+1)\alpha}-\frac{\delta}{p+1}}).$$

We then obtain the desired bound by taking p to be sufficiently large.

6 Numerical results

We have numerical results for three different Lévy models: Variance Gamma, Normal Inverse Gaussian and α-stable processes, and three different options: Asian, lookback and barrier.

The current code is based on Giles' MATLAB code [17], using which we generate standardised numerical results and a set of four figures. The top two plots correspond to a set of experiments to investigate how the variance and mean for both $\widehat{P}_\ell$ and $\widehat{P}_\ell - \widehat{P}_{\ell-1}$ vary with level ℓ. The top left plot shows the values for $\log_2(\text{variance})$, so that the absolute value of the slope of the line for $\log_2 \mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ indicates the convergence rate β of V_ℓ in condition i) of Thereom 2.1. Similarly, the absolute value of the slope of the line for $\log_2 |\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]|$ in the top right plot indicates the weak convergence rate α in the condition i) of Thereom 2.1.

The bottom two plots correspond to a set of MLMC calculations for different values of the desired accuracy ε. Each line in the bottom left plot corresponds to one multilevel calculation and displays the number of samples N_ℓ on each level. Note that as ε is varied, the MLMC algorithm automatically decides how many levels are required to reduce the weak error appropriately. The optimal number of samples on each level is based on an empirical estimation of the multilevel correction variance V_ℓ, together with the use of a Lagrange multiplier to determine how best to minimise the overall computational cost for a given target accuracy. A complete description of the algorithm is given in [19]. The bottom right plots show the variation of the computational complexity C with the desired accuracy ε. In the best cases, the MLMC

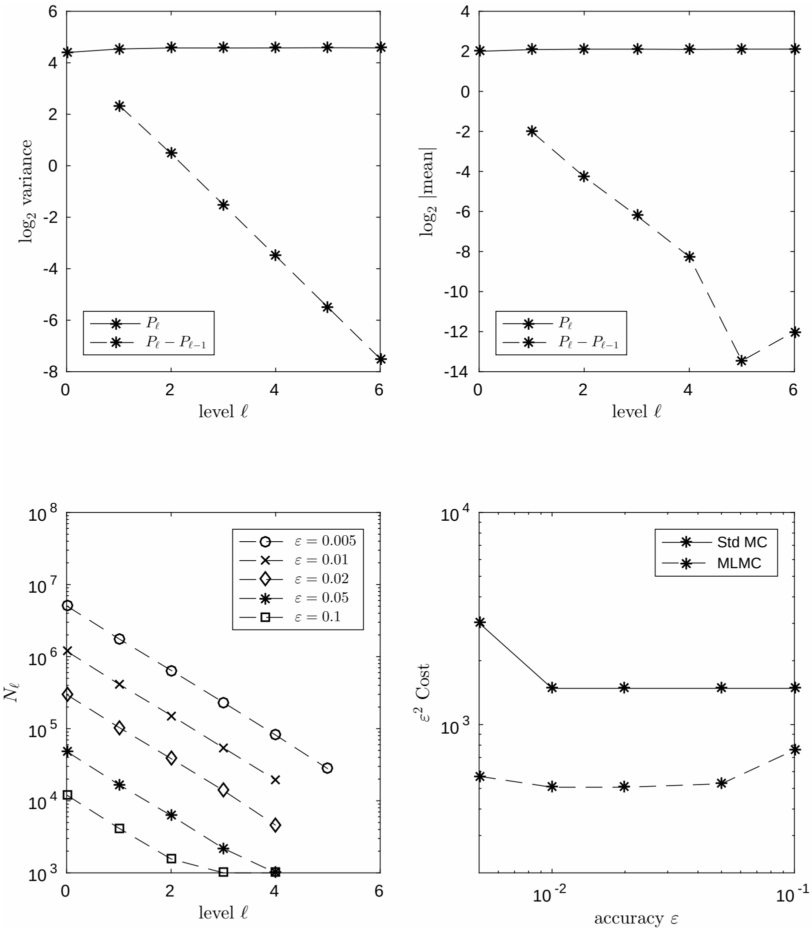

Fig. 1: Asian option in variance gamma model

complexity is $\mathscr{O}(\varepsilon^{-2})$, and therefore the plot is of $\varepsilon^2 C$ versus ε so that we can see whether this is achieved, and compare the complexity to that of the standard Monte Carlo method.

6.1 Asian option

The Asian option we consider is an arithmetic Asian call option with discounted payoff

$$P = \exp(-rT)\ \max\left(0, \overline{S} - K\right),$$

where $T=1$, $r=0.05$, $S_0=100$, $K=100$ and

$$\overline{S} = S_0\ T^{-1} \int_0^T \exp\left(X_t\right) \mathrm{d}t\ .$$

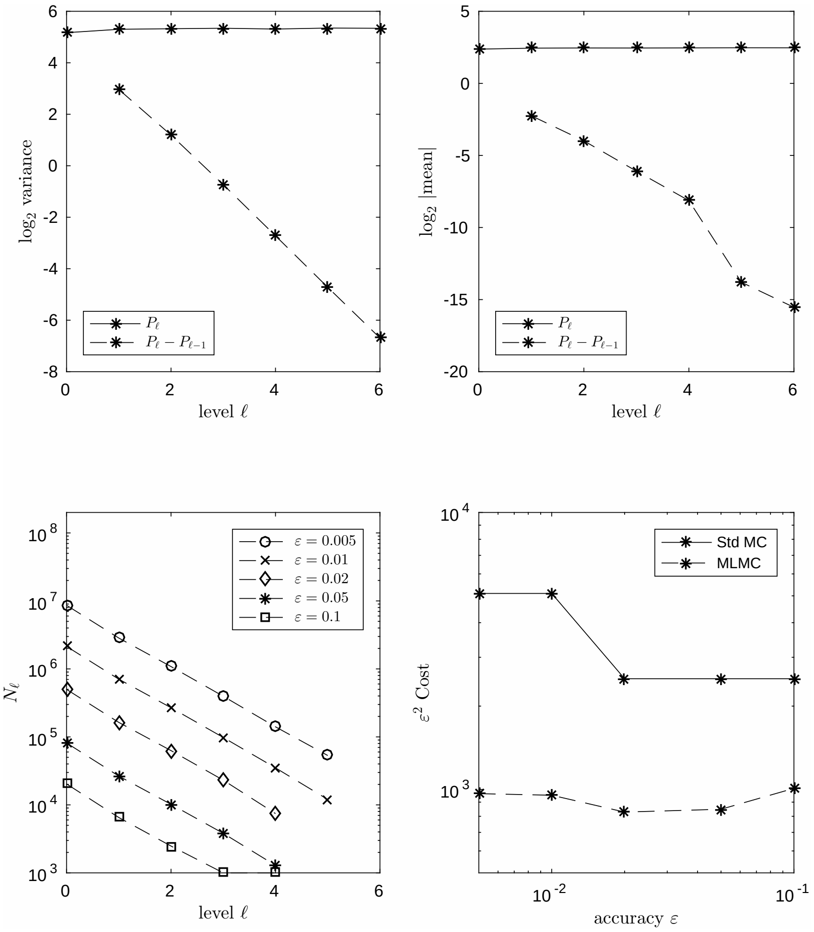

Fig. 2: Asian option in Normal Inverse Gaussian model

For a general Lévy process it is not easy to directly sample the integral process. We use the trapezoidal approximation

$$\widehat{\overline{S}} := S_0\, T^{-1} \sum_{j=0}^{n-1} \tfrac{1}{2}\, h\, \left(\exp\left(X_{jh}\right) + \exp\left(X_{(j+1)h}\right)\right),$$

where $n = T/h$ is the number of timesteps. The payoff approximation is then

$$\widehat{P} = \exp(-rT)\ \max\left(0, \widehat{\overline{S}} - K\right).$$

In the multilevel estimator, the approximation $\widehat{P}_\ell$ on level ℓ is obtained using $n_\ell := 2^\ell$ timesteps.

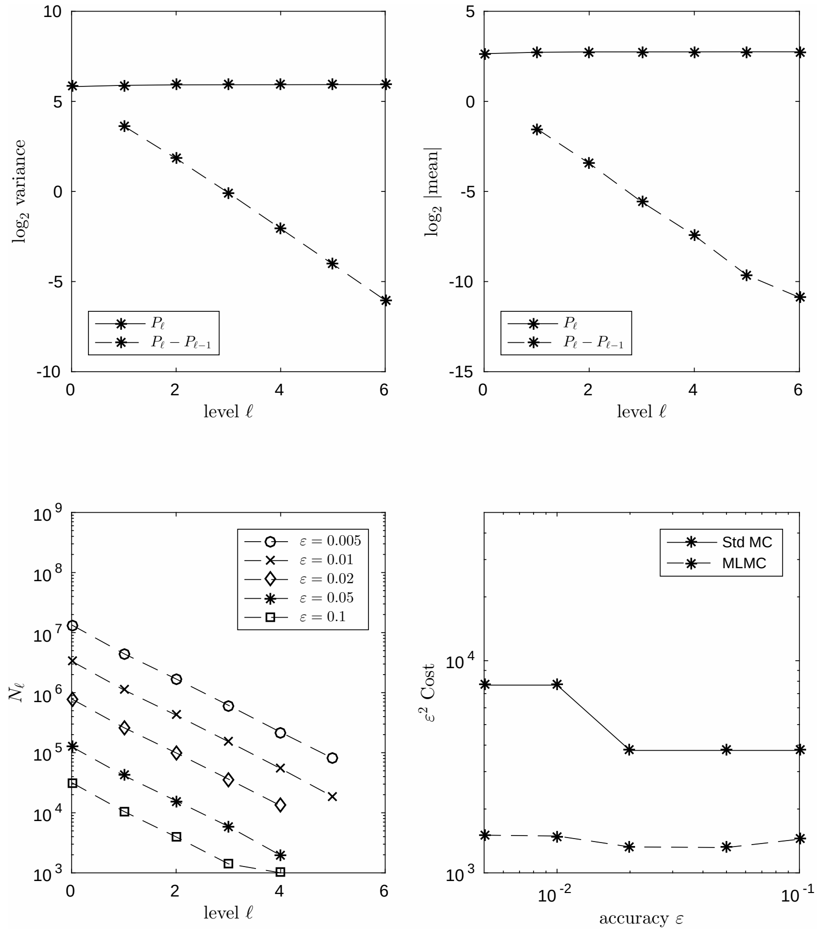

Fig. 3: Asian option in spectrally negative α-stable model

Figures 1, 2, 3 are for the VG, NIG and α-stable models respectively. The numerical results in the top right plots indicate approximately second order weak convergence. With the standard Monte Carlo method, the top left plots show that the variance is approximately independent, and therefore, the standard Monte Carlo calculation has computational cost $\mathscr{O}(\varepsilon^{-2}n_\ell) = \mathscr{O}(\varepsilon^{-2.5})$. Multiplying this cost by ε^2 to create the bottom right complexity plots, the scaled cost is $\mathscr{O}(n_\ell)$ and therefore goes up in steps as ε is reduced, when decreasing ε requires an increase in the value of the finest level L. On the other hand, the convergence rate of the variance of the MLMC estimator is approximately 1.2 for VG, 2.0 for NIG and 2 for the α-stable model. Since in all three cases we have $\beta > 1$, the MLMC theorem gives a complexity which is $\mathscr{O}(\varepsilon^{-2})$ which is consistent with the results in the bottom right plots which show little variation in $\varepsilon^2 C$ for the MLMC estimator.

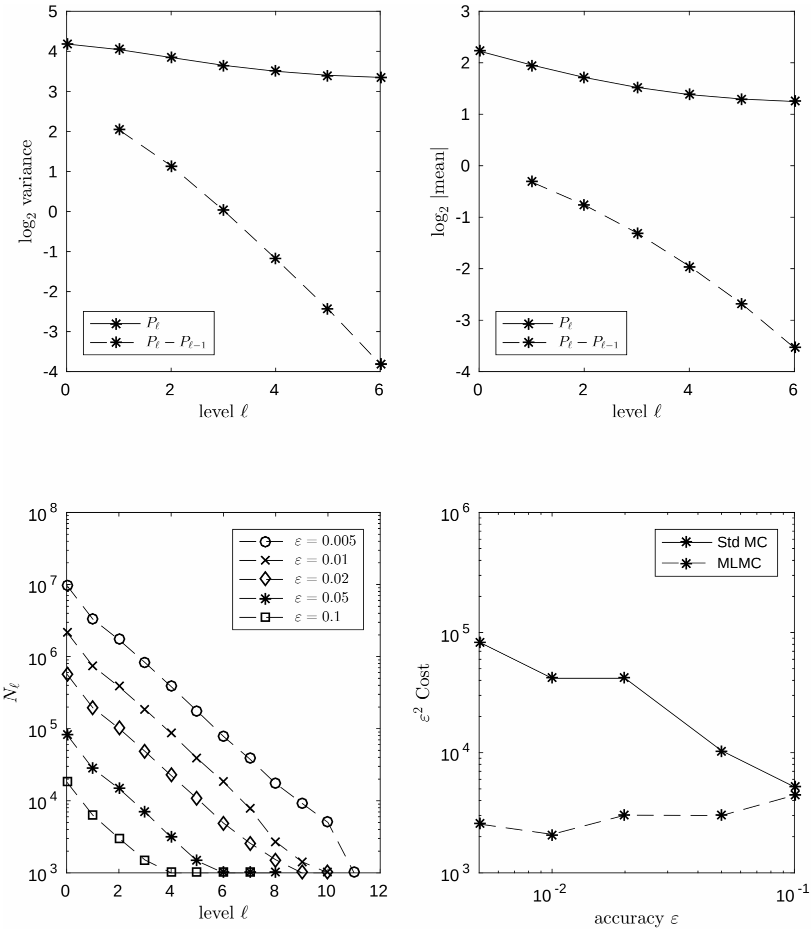

Fig. 4: Lookback option with Variance Gamma model

For this Asian option, MLMC is 3-8 times more efficient than standard MC. The gains are modest because the high rate of weak convergence means that only 4 levels of refinement are required in most cases, so there is only a $2^4 = 16$ difference in cost between each MC path calculation on the finest level, and each of the MLMC path calculations on the coarsest level.

6.2 Lookback option

The lookback option we consider is a put option on the floating underlying,

$$P \;=\; \exp(-rT)\left(K - \sup_{0\le t\le T} S_t\right)^{+} \;=\; \exp(-rT)\,(K - S_0\exp(m)\,)^{+},$$

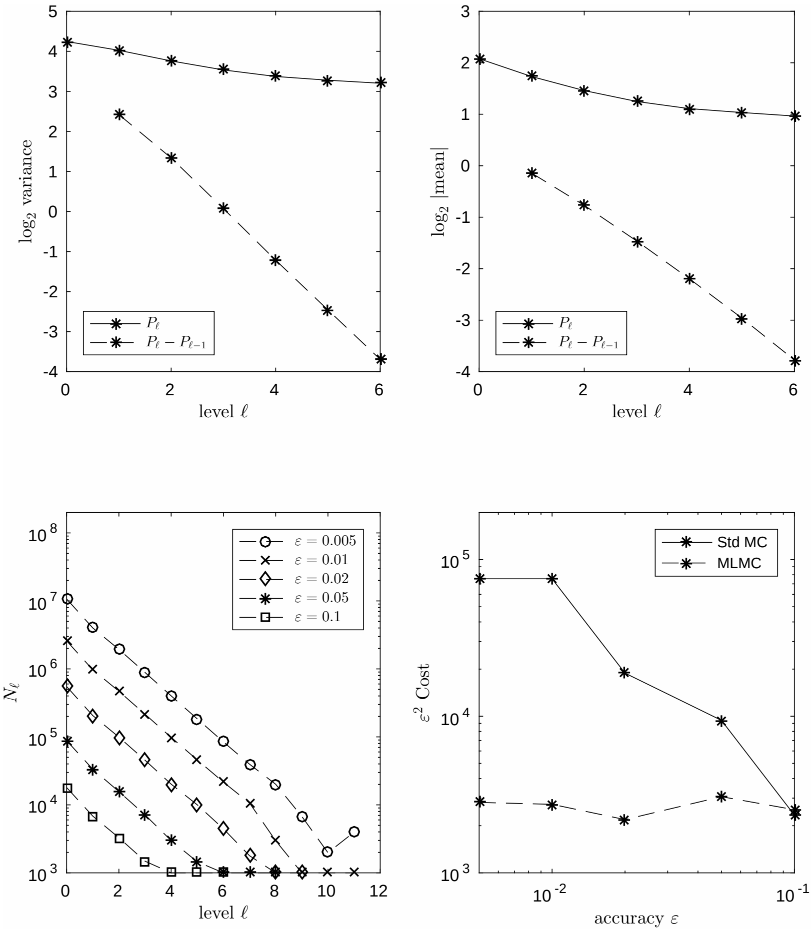

Fig. 5: Lookback option with Normal Inverse Gaussian model

where $m = \sup_{0\leq t\leq T} X_t$, with $T\!=\!1$, $r\!=\!0.05$, $S_0\!=\!100$, $K\!=\!110$. We use the discretely monitored maximum as the approximation, so that

$$\widehat{P}_\ell = \exp(-rT)\,(K - S_0 \exp(\widehat{m}_\ell))^+, \qquad \widehat{m}_\ell = \max_{0\leq j\leq n_\ell} X_{jh_\ell}.$$

Figures 4, 5, 6 show the numerical results for the VG, NIG and α-stable models. The most obvious difference compared to the Asian option is a greatly reduced order of weak convergence, approximately 1, 0.8 and 0.6 in the respective cases. This reduced weak convergence leads to a big increase in the finest approximation level, which in turn greatly increases the standard MC cost but doesn't significantly change the MLMC cost. Hence, the computational savings are much greater than for the Asian option, with savings of up to a factor of 30.

The small erratic fluctuation in N_ℓ on levels greater than 5 is due to poor estimates of the variance due to a limited number of samples. This also appears later for the barrier option.

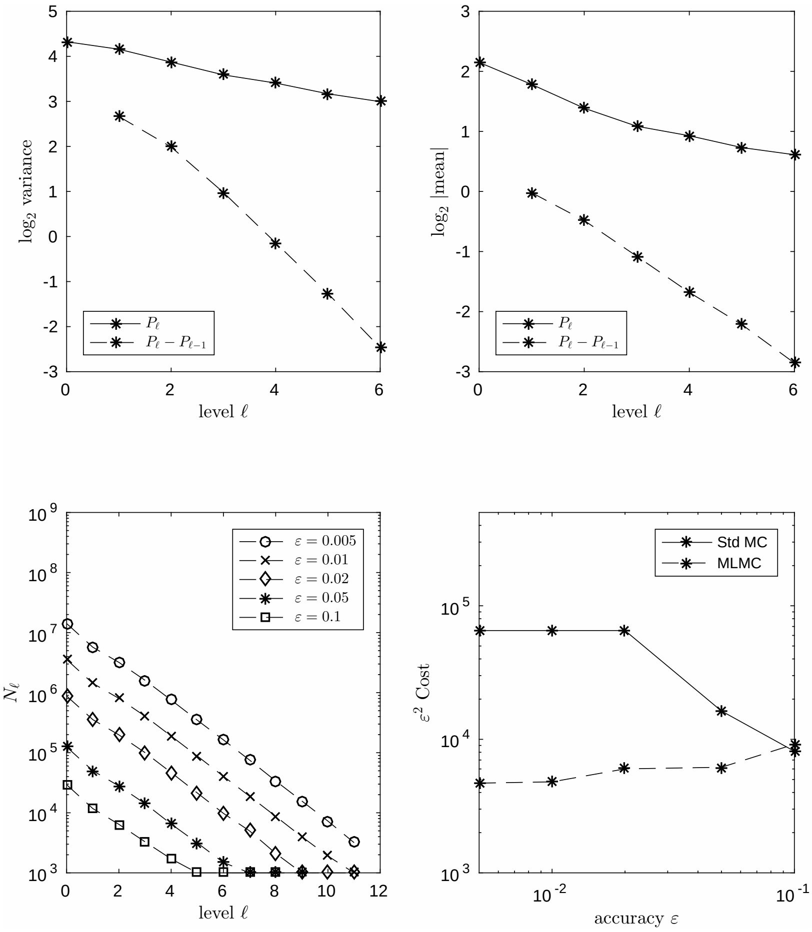

Fig. 6: Lookback option with spectrally negative α-stable model

6.3 Barrier option

The barrier option is an up-and-out call with payoff

$$P \;=\; \exp(-rT)\,(S_T - K)^+\,\mathbf{1}_{\left\{\sup_{0\le t\le T} S(t)<B\right\}} \;=\; \exp(-rT)\,(S_T - K)^+\,\mathbf{1}_{\{m<\log(B/S_0)\}},$$

with $T=1$, $r=0.05$, $S_0=100$, $K=100$, $B=115$. The discretely monitored approximation is

$$\widehat{P}_\ell = \exp(-rT)\,(S_T - K)^+\,\mathbf{1}_{\{\hat{m}_\ell<\log(B/S_0)\}}, \qquad \widehat{m}_\ell = \max_{0\le j\le n_\ell} X_{jh_\ell}$$

With the barrier option, the most noticeable change from the previous options is a reduction in the rate of convergence β of the MLMC variance, with $\beta \approx 0.75, 0.5, 0.6$ in the three cases. For $\beta < 1$, the MLMC theorem proves a complexity which is $\mathscr{O}(\varepsilon^{-2-(1-\beta)/\alpha})$, with α here being the rate of weak convergence. The fact that the

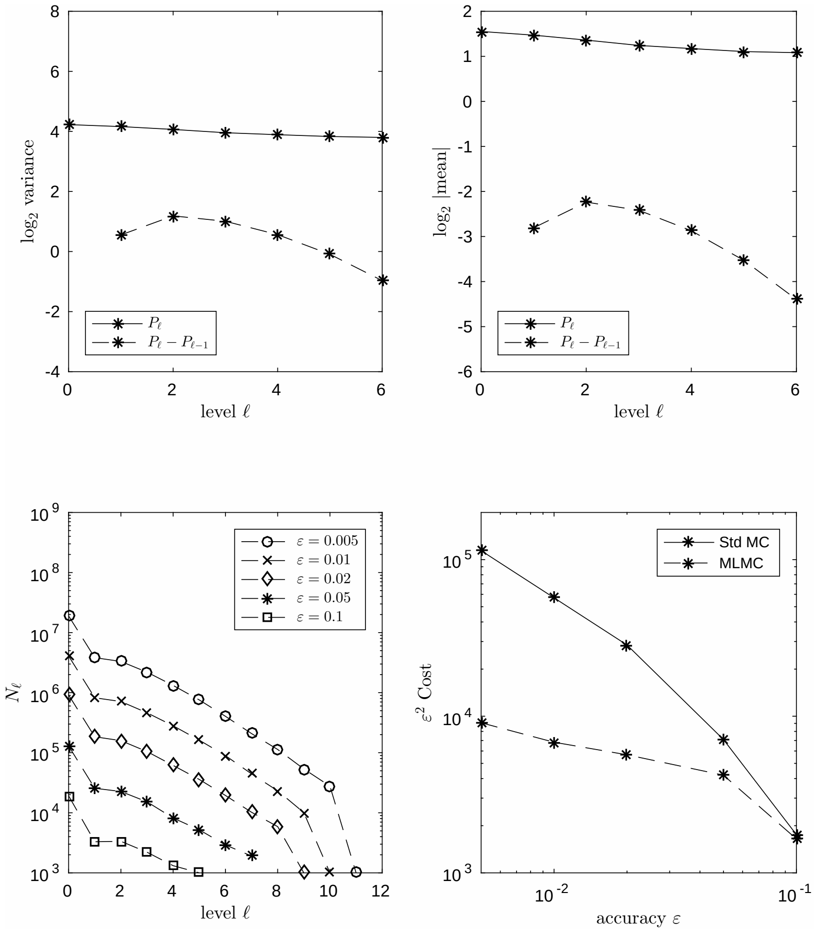

Fig. 7: Barrier option in variance gamma model

MLMC complexity is not $\mathscr{O}(\varepsilon^{-2})$ is clearly visible from the bottom right complexity plots, but there are still significant savings compared to the standard MC computations.

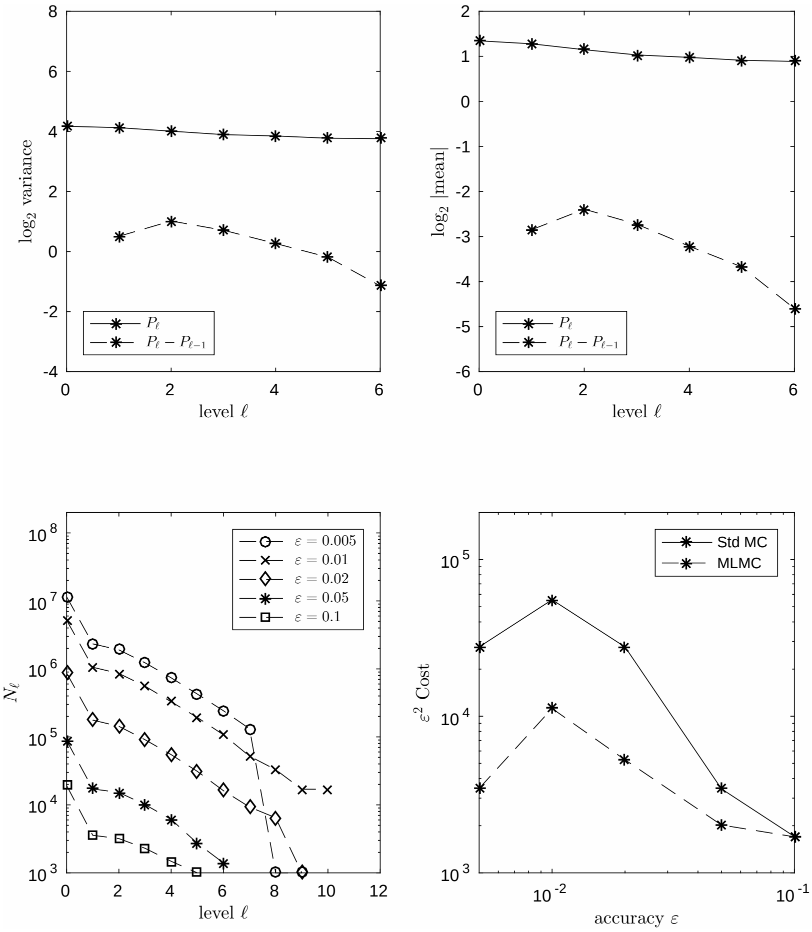

Fig. 8: Barrier option in Normal Inverse Gaussian model

6.4 Summary and discussion

Table 1 summarizes the convergence rates for the weak error $\mathbb{E}[\widehat{P}_\ell - P]$ and the MLMC variance $V_\ell = \mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ given by Propositions 5.1, 5.2, 5.5, and the empirical convergence rates observed in the numerical experiments.

In general, the agreement between the analysis and the numerical rates of convergence is quite good, suggesting that in most cases the analysis may be sharp. The most obvious gap between the two is with the weak order of convergence for the Asian option with all three models; the analysis proves an $\mathcal{O}(h)$ bound, whereas the numerical results suggest it is actually $\mathcal{O}(h^2)$. The numerical results are perhaps not surprising as $\mathcal{O}(h^2)$ is the order of convergence of trapezoidal integration of a smooth function, and therefore it is the order one would expect if the payoff was simply a multiple of $\overline{S}$.

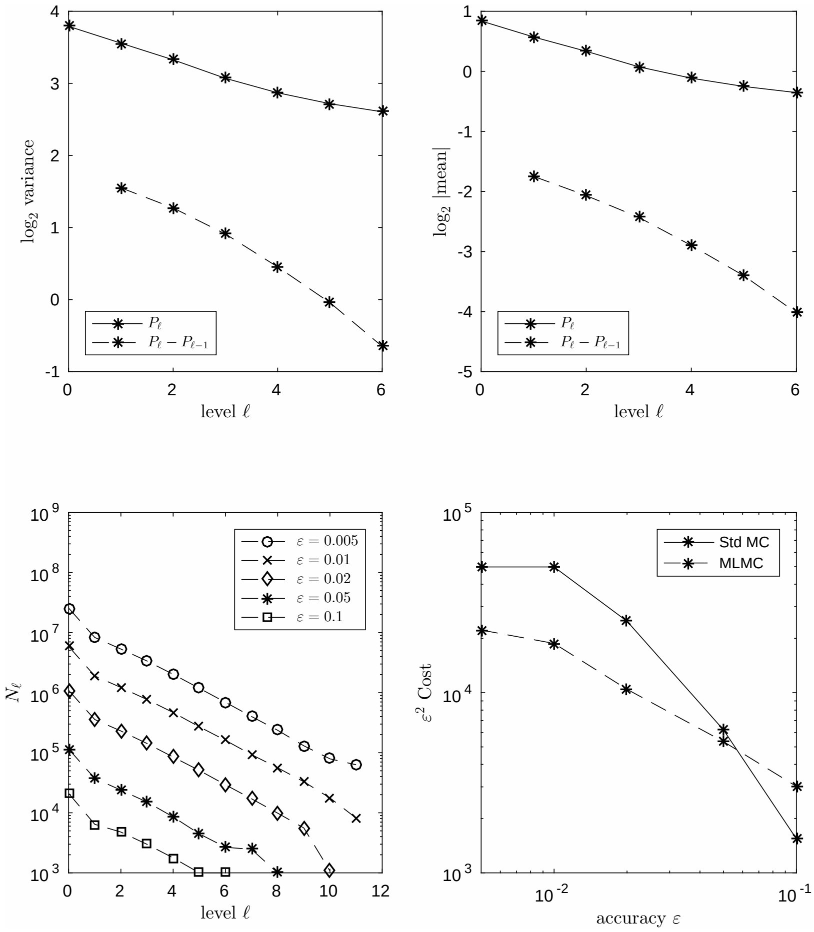

Fig. 9: Barrier option in spectrally negative α-stable model

7 Proofs

7.1 Proof of Proposition 5.1

Proof We decompose the difference between the true value and approximation into parts which we can bound separately:

$$\begin{aligned}\left|\overline{S}-\widehat{\overline{S}}\right| &= S_0\, T^{-1}\left|\int_0^T \exp\left(X_t\right) \mathrm{d}t \;-\; \sum_{j=0}^{n-1} \tfrac{1}{2}\, h\, \left(\exp\left(X_{jh}\right)+\exp\left(X_{(j+1)h}\right)\right)\right| \\ &= S_0\, T^{-1}\left|\sum_{j=0}^{n-1} \exp\left(X_{jh}\right) \int_{jh}^{(j+1)h} \left(\exp\left(X_t - X_{jh}\right) - 1\right) \mathrm{d}t - \tfrac{1}{2} h \exp\left(X_T\right) + \tfrac{1}{2} h\right|.\end{aligned}$$

Table 1: Convergence rates of weak error and variance V_ℓ for VG, NIG and α-stable processes; δ can be any small positive constant. The numerical values are estimates based on the numerical experiments.

	VG			
	numerical		analysis	
option	weak	var	weak	var
Asian	$\mathscr{O}(h^2)$	$\mathscr{O}(h^2)$	$\mathscr{O}(h)$	$\mathscr{O}(h^2)$
lookback	$\mathscr{O}(h)$	$\mathscr{O}(h^{1.2})$	$\mathscr{O}(h\lvert\log h\rvert)$	$\mathscr{O}(h)$
barrier	$\mathscr{O}(h^{0.8})$	$\mathscr{O}(h^{0.9})$	$o\left(h^{1-\delta}\right)$	$o\left(h^{1-\delta}\right)$

	NIG			
	numerical		analysis	
option	weak	var	weak	var
Asian	$\mathscr{O}(h^2)$	$\mathscr{O}(h^2)$	$\mathscr{O}(h)$	$\mathscr{O}(h^2)$
lookback	$\mathscr{O}(h^{0.8})$	$\mathscr{O}(h^{1.2})$	$o\left(h^{1-\delta}\right)$	$\mathscr{O}(h\lvert\log h\rvert)$
barrier	$\mathscr{O}(h^{0.4})$	$\mathscr{O}(h^{0.5})$	$o\left(h^{0.5-\delta}\right)$	$o\left(h^{0.5-\delta}\right)$

	spectrally negative α-stable with $\alpha>1$			
	numerical for $\alpha=1.5597$		analysis	
option	weak	var	weak	var
Asian	$\mathscr{O}(h^2)$	$\mathscr{O}(h^2)$	$\mathscr{O}(h)$	$\mathscr{O}(h^2)$
lookback	$\mathscr{O}(h^{0.6})$	$\mathscr{O}(h^{1.6})$	$o\left(h^{1/\alpha-\delta}\right)$	$o\left(h^{2/\alpha-\delta}\right)$
barrier	$\mathscr{O}(h^{0.5})$	$\mathscr{O}(h^{0.6})$	$o\left(h^{1/\alpha-\delta}\right)$	$o\left(h^{1/\alpha-\delta}\right)$

If we define

$$\begin{aligned} b_j &= \exp\left(X_{jh}\right), \\ I_j &= \int_{jh}^{(j+1)h} \left(\exp\left(X_t - X_{jh}\right) - 1\right)\, \mathrm{d}t, \\ R_A &= -\tfrac{1}{2} h \exp\left(X_T\right) + \tfrac{1}{2} h, \end{aligned}$$

then

$$\mathbb{E}\left[\left(\widehat{\overline{S}} - \overline{S}\right)^2\right] = T^{-2} S_0^2\, \mathbb{E}\left[\left|\sum_{j=0}^{n-1} b_j I_j + R_A\right|^2\right] \leq 2T^{-2} S_0^2 \left(\mathbb{E}\left[\left|\sum_{j=0}^{n-1} b_j I_j\right|^2\right] + \mathbb{E}\left[R_A^2\right]\right).$$

We have $\mathbb{E}\left[R_A^2\right] = \mathscr{O}\left(h^2\right)$, and due to the independence of b_j and I_j we obtain

$$\begin{aligned} \mathbb{E}\left[\left|\sum_{j=0}^{n-1} b_j I_j\right|^2\right] &= \mathbb{E}\left[\sum_{j=0}^{n-1} b_j^2 I_j^2 + 2 \sum_{m=1}^{n-1} \sum_{j=0}^{m-1} b_m I_m b_j I_j\right] \\ &= \sum_{j=0}^{n-1} \mathbb{E}\left[b_j^2\right] \mathbb{E}\left[I_j^2\right] + 2 \sum_{m=1}^{n-1} \sum_{j=0}^{m-1} \mathbb{E}\left[b_m I_m b_j I_j\right]. \end{aligned} \tag{7.1}$$

Defining $A = 2m + \int \left(e^{2z} - 1 - 2z1_{|z|<1}\right) \nu(\mathrm{d}z)$, we have $\mathbb{E}\left[b_j^2\right] = e^{Ajh}$. Furthermore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}\left[I_j^2\right] &\le h\, \mathbb{E}\left[\int_{jh}^{(j+1)h} \left(\exp\left(X_t - X_{jh}\right) - 1\right)^2 \mathrm{d}t\right] \\ &= h \int_0^h \mathbb{E}\left[\left(\exp\left(X_t\right) - 1\right)^2\right] \mathrm{d}t \\ &= h\left(\frac{1}{A}\left(e^{Ah} - 1 - Ah\right) - 2\frac{1}{r}\left(e^{rh} - 1 - rh\right)\right)\end{aligned}$$

Note that $1 + x < e^x < 1 + x + x^2$ for $0 < x < 1$, and therefore for $h < 1/A$ we have $\mathbb{E}\left[I_j^2\right] < Ah^3$ and hence

$$\sum_{j=0}^{n-1} \mathbb{E}\left[b_j^2\right] \mathbb{E}\left[I_j^2\right] \;<\; A\,h^3 \sum_{j=0}^{n-1} e^{Ajh} \;=\; A\, \frac{e^{AT} - 1}{e^{Ah} - 1}\, h^3 \;<\; \left(e^{AT} - 1\right) h^2.$$

Now we calculate the second term in (7.1). Note that for $m > j$, I_m is independent of $b_m b_j I_j$, and b_m / b_{j+1} is independent of $b_{j+1} b_j I_j$, so

$$\sum_{m=1}^{n-1} \sum_{j=0}^{m-1} \mathbb{E}\left[b_m I_m b_j I_j\right] = \sum_{m=1}^{n-1} \mathbb{E}\left[I_m\right] \sum_{j=0}^{m-1} \mathbb{E}\left[b_m / b_{j+1}\right] \mathbb{E}\left[b_{j+1} b_j I_j\right].$$

Firstly, for $h < 1/r$,

$$\mathbb{E}\left[I_m\right] \;=\; \int_0^h \left(e^{rt} - 1\right) \mathrm{d}t \;=\; r^{-1}\left(e^{rh} - 1 - rh\right) \;<\; r h^2.$$

Moreover, we have $\mathbb{E}\left[b_m / b_{j+1}\right] = e^{r(m-j-1)h}$ and

$$\begin{aligned}\mathbb{E}\left[b_{j+1} b_j I_j\right] &= \mathbb{E}\left[\exp\left(2X_{jh}\right) \exp\left(X_{(j+1)h} - X_{jh}\right) \int_{jh}^{(j+1)h} \left(\exp\left(X_t - X_{jh}\right) - 1\right) \mathrm{d}t\right] \\ &= \mathbb{E}\left[\exp\left(2X_{jh}\right)\right]\; \mathbb{E}\left[\exp\left(X_h\right) \int_0^h \left(\exp\left(X_t\right) - 1\right) \mathrm{d}t\right] \\ &= e^{Ajh} \int_0^h \left(\mathbb{E}\left[\exp\left(X_h - X_t\right)\right] \mathbb{E}\left[\exp\left(2X_t\right)\right] - \mathbb{E}\left[\exp\left(X_h\right)\right]\right) \mathrm{d}t \\ &= e^{Ajh} \int_0^h \left(e^{r(h-t)} e^{At} - e^{rh}\right) \mathrm{d}t \\ &= e^{Ajh} e^{rh}\, \frac{e^{(A-r)h} - 1 - (A-r)h}{A - r}.\end{aligned}$$

Thus, for $h < 1/(A-r)$,

$$\begin{aligned}\sum_{m=1}^{n-1}\sum_{j=0}^{m-1}\mathbb{E}\left[b_m/b_{j+1}\right]\mathbb{E}\left[b_{j+1}b_jI_j\right] &= \frac{e^{(A-r)h}-1-(A-r)}{A-r}\sum_{m=1}^{n-1}\sum_{j=0}^{m-1}e^{r(m-j)h}e^{Ajh}\\ &= \frac{e^{(A-r)h}-1-(A-r)h}{(A-r)\left(e^{(A-r)h}-1\right)}\sum_{m=1}^{n-1}\left(e^{Amh}-e^{rmh}\right)\\ &< h\,\frac{e^{AT}-1}{e^{Ah}-1}\\ &< A^{-1}(e^{AT}-1).\end{aligned}$$

Hence,

$$\mathbb{E}\left[\sum_{m=1}^{n-1}\sum_{j=0}^{m-1}b_mI_mb_jI_j\right] = \sum_{m=1}^{n-1}\mathbb{E}\left[I_m\right]\sum_{j=0}^{m-1}\mathbb{E}\left[b_m/b_{j+1}\right]\mathbb{E}\left[b_{j+1}b_jI_j\right] = \mathscr{O}(h^2),$$

and we can therefore conclude that $\mathbb{E}\left[\left(\widehat{\overline{S}}-\overline{S}\right)^2\right] = \mathscr{O}(h^2)$.

7.2 Lévy process decomposition

The proofs rely on a decomposition of the Lévy process into a combination of a finite-activity pure jump part, a drift part, and a residual part consisting of very small jumps.

Let X_t be an $(m,0,\nu)$-Lévy process:

$$X_t = mt + \int_0^t\int_{|z|\geq 1} z\,J(\mathrm{d}z,\mathrm{d}s) + \int_0^t\int_{|z|<1} z\left(J(\mathrm{d}z,\mathrm{d}s)-\nu(\mathrm{d}z)\mathrm{d}s\right). \qquad (7.2)$$

The finite activity jump part is defined by

$$X_t^{\varepsilon} = \int_0^t\int_{\varepsilon<|z|} z\,J(\mathrm{d}z,\mathrm{d}s) = \sum_{i=1}^{N_t} Y_i$$

to be the compound Poisson process truncating the jumps of X_t smaller than ε which is assumed to satisfy $0<\varepsilon<1$. The intensity of N_t and the c.d.f. of Y_i are

$$\lambda_{\varepsilon} = \int_{\varepsilon<|z|}\nu(\mathrm{d}z). \qquad (7.3)$$

$$\mathbb{P}\left(Y_i < y\right) = \lambda_{\varepsilon}^{-1}\int_{z<y} 1_{\{\varepsilon<|z|\}}\nu(\mathrm{d}z);$$

The drift rate for the drift term is defined to be

$$\mu_{\varepsilon} = m - \int_{\varepsilon<|z|<1} z\,\nu(\mathrm{d}z), \qquad (7.4)$$

so that the residual term is then a martingale:

$$R_t^\varepsilon := \int_0^t \int_{|z|\le\varepsilon} z\,(J(\mathrm{d}z,\mathrm{d}s) - \nu(\mathrm{d}z)\mathrm{d}s)\,. \tag{7.5}$$

We define

$$\sigma_\varepsilon^2 = \int_{|z|\le\varepsilon} z^2 \nu(\mathrm{d}z), \tag{7.6}$$

so that $\mathbb{V}[R_t^\varepsilon] = \sigma_\varepsilon^2 t$.

These three quantities, μ_ε, λ_ε and σ_ε will all play a major role in the subsequent numerical analysis.

We bound D_n by the difference between continuous maxima and 2-point maxima over all timesteps:

$$D_n \;=\; \sup_{0\le t\le 1} X_t - \max_{i=0,1,\ldots,n} X_{\frac{i}{n}} \;\le\; \max_{i=0,\ldots,n-1} D_n^{(i)} \tag{7.7}$$

where the random variables

$$D_n^{(i)} \;=\; \sup_{[\frac{i}{n},\frac{i+1}{n}]} X_t - \max\left(X_{\frac{i+1}{n}}, X_{\frac{i}{n}}\right)$$

are independent and identically distributed. If we now define

$$\Delta^{(i)}X_t = X_{\frac{i}{n}+t} - X_{\frac{i}{n}},\quad \Delta^{(i)}X_t^\varepsilon = X^\varepsilon_{\frac{i}{n}+t} - X^\varepsilon_{\frac{i}{n}},\quad \Delta^{(i)}t = t - \frac{i}{n},\quad \Delta^{(i)}R_t^\varepsilon = R^\varepsilon_{\frac{i}{n}+t} - R^\varepsilon_{\frac{i}{n}},$$

then

$$\begin{aligned}
D_n^{(i)} &= \sup_{[0,\frac{1}{n}]} \Delta^{(i)}X_t - \left(\Delta^{(i)}X_{\frac{1}{n}}\right)^+ \\
&= \sup_{[0,\frac{1}{n}]} \left(\Delta^{(i)}X_t^\varepsilon + \Delta^{(i)}R_t^\varepsilon + \mu_\varepsilon \Delta^{(i)}t\right) - \left(\Delta^{(i)}X^\varepsilon_{\frac{1}{n}} + \Delta^{(i)}R^\varepsilon_{\frac{1}{n}} + \mu_\varepsilon \frac{1}{n}\right)^+ \\
&\le \sup_{[0,\frac{1}{n}]} \left(\Delta^{(i)}X_t^\varepsilon + \Delta^{(i)}R_t^\varepsilon\right) - \left(\Delta^{(i)}X^\varepsilon_{\frac{1}{n}} + \Delta^{(i)}R^\varepsilon_{\frac{1}{n}}\right)^+ + \frac{|\mu_\varepsilon|}{n} \\
&\le \sup_{[0,\frac{1}{n}]} \Delta^{(i)}X_t^\varepsilon - \left(\Delta^{(i)}X^\varepsilon_{\frac{1}{n}}\right)^+ + \frac{|\mu_\varepsilon|}{n} + \sup_{[0,\frac{1}{n}]} \Delta^{(i)}R_t^\varepsilon + \left(-\Delta^{(i)}R^\varepsilon_{\frac{1}{n}}\right)^+ \\
&\le \sup_{[0,\frac{1}{n}]} \Delta^{(i)}X_t^\varepsilon - \left(\Delta^{(i)}X^\varepsilon_{\frac{1}{n}}\right)^+ + \frac{|\mu_\varepsilon|}{n} + 2\sup_{[0,\frac{1}{n}]} \left|\Delta^{(i)}R_t^\varepsilon\right|
\end{aligned} \tag{7.8}$$

where we use $(a{+}b)^+ \le a^+ + b^+$ with $a = \Delta^{(i)}X^\varepsilon_{\frac{1}{n}} + \Delta^{(i)}R^\varepsilon_{\frac{1}{n}} + \mu_\varepsilon \frac{1}{n}$, $b = -\mu_\varepsilon \frac{1}{n}$ in the first inequality, and $a = \Delta^{(i)}X^\varepsilon_{\frac{1}{n}} + \Delta^{(i)}R^\varepsilon_{\frac{1}{n}}$, $b = -\Delta^{(i)}R^\varepsilon_{\frac{1}{n}}$ in the second inequality.

Let $Z_n^{(i)} := \sup_{[0,\frac{1}{n}]} \Delta^{(i)} X_t^\varepsilon - \left(\Delta X_{\frac{1}{n}}^\varepsilon\right)^+$ and $S_n^{(i)} := \sup_{[0,\frac{1}{n}]} \left|\Delta^{(i)} R_t^\varepsilon\right|$. Then, for $p \geq 1$, Jensen's inequality gives us

$$
\begin{aligned}
&\mathbb{E}\left[D_n^p\right] \\
&\leq \mathbb{E}\left[\max_{0\leq i<n} \left(Z_n^{(i)} + \frac{|\mu_\varepsilon|}{n} + 2\, S_n^{(i)}\right)^p\right] \\
&\leq 3^{p-1}\mathbb{E}\left[\max_{0\leq i<n} \left(Z_n^{(i)}\right)^p + \left(\frac{|\mu_\varepsilon|}{n}\right)^p + 2^p \max_{0\leq i<n} \left(S_n^{(i)}\right)^p\right] \\
&\leq 3^{p-1} n\, \mathbb{E}\left[\left(\sup_{[0,\frac{1}{n}]} X_t^\varepsilon - \left(X_{\frac{1}{n}}^\varepsilon\right)^+\right)^p\right] + 3^{p-1}\left(\frac{|\mu_\varepsilon|}{n}\right)^p + 3^{p-1} 2^p\, \mathbb{E}\left[\max_{0\leq i<n} \left(S_n^{(i)}\right)^p\right]
\end{aligned}
\tag{7.9}
$$

where in the final step we have used the fact that all of the $\Delta^{(i)} X_t^\varepsilon$ have the same distribution as X_t^ε.

The task now is to bound the first and third terms in the final line of (7.9).

7.3 Bounding moments of $\sup_{[0,\frac{1}{n}]} X_t^\varepsilon - (X_{\frac{1}{n}}^\varepsilon)^+$

Theorem 7.1 *Let X_t be a scalar Lévy process with a triple $(m, 0, \nu)$, and let X_t^ε, μ_ε, λ_ε and σ_ε be as defined in section 7.2.*

Then provided $\lambda_\varepsilon \leq n$, for any $p > 1$ there exists a constant K_p such that

$$
\mathbb{E}\left[\left(\sup_{[0,\frac{1}{n}]} X_t^\varepsilon - \left(X_{\frac{1}{n}}^\varepsilon\right)^+\right)^p\right] \leq K_p \left(\varepsilon^p + \frac{L_\varepsilon(p)}{\lambda_\varepsilon^2}\right) \frac{\lambda_\varepsilon^2}{n^2}, \tag{7.10}
$$

where $L_\varepsilon(p) = p \int_{x>\varepsilon} x^{p-1} \lambda_x^2 \,\mathrm{d}x$ is a function depending on the Lévy measure $\nu(x)$.

Proof Let

$$
Z = \sup_{[0,\frac{1}{n}]} X_t^\varepsilon - \left(X_{\frac{1}{n}}^\varepsilon\right)^+ .
$$

We will determine an upper bound on $\mathbb{E}[Z^p]$ by analysing the jump behavior of the finite-activity process X_t^ε in a single interval $[0, \frac{1}{n}]$.

Let N be the number of jumps. If $N \leq 1$, then $Z = 0$, while if $N = 2$, then $Z \leq \min(|Y_1|, |Y_2|)$. This can be seen from the behavior of X_t^ε in the different scenarios illustrated in Figure 10. More generally, if $N = k$, $k \geq 2$, then

$$
\begin{aligned}
Z > x &\Longrightarrow \exists\, 1 \leq j \leq k-1 \text{ s.t. } \left|\sum_{l=1}^{j} Y_l\right| > x, \left|\sum_{l=j+1}^{k} Y_l\right| > x \\
&\Longrightarrow \exists\, j_1, j_2 \text{ s.t. } \left|Y_{j_1}\right| > \frac{x}{k-1}, \ \left|Y_{j_2}\right| > \frac{x}{k-1}.
\end{aligned}
$$

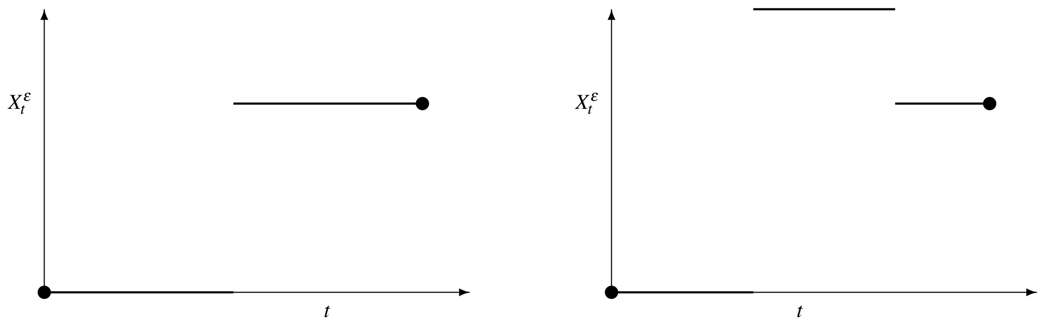

Fig. 10: Behavior of X_t^ε in the case of one or two jumps in the interval $[0, \frac{1}{n}]$.

Since

$$\begin{aligned}
&\mathbb{P}\left(\exists j_1, j_2 \text{ s.t. } \left|Y_{j_1}\right| > \frac{x}{k-1},\ \left|Y_{j_2}\right| > \frac{x}{k-1}\right) \\
&\qquad \le \sum_{(j_1,j_2)} \mathbb{P}\left(\left|Y_{j_1}\right| > \frac{x}{k-1},\ \left|Y_{j_2}\right| > \frac{x}{k-1}\right) \\
&\qquad = \frac{k\,(k-1)}{2}\mathbb{P}\left(|Y_1| > \frac{x}{k-1}\right)^2.
\end{aligned}$$

it follows that

$$\begin{aligned}
\mathbb{E}\left[Z^p \mid N\!=\!k\right] &= p\int x^{p-1}\mathbb{P}\left(Z > x \mid N\!=\!k\right)\mathrm{d}x \\
&\le \frac{k\,(k-1)}{2}\, p\int x^{p-1}\mathbb{P}\left(|Y_1| > \frac{x}{k-1}\right)^2 \mathrm{d}x \\
&= \frac{k\,(k-1)}{2}\,\frac{p}{\lambda_\varepsilon^2}\int x^{p-1}\left(\int_{|z|>x/(k-1)} 1_{\{\varepsilon<|z|\}}\,\nu(\mathrm{d}z)\right)^2 \mathrm{d}x \\
&= \frac{k\,(k-1)^{p+1}}{2}\,\frac{p}{\lambda_\varepsilon^2}\int x^{p-1}\left(\int_{|z|>x} 1_{\{\varepsilon<|z|\}}\,\nu(\mathrm{d}z)\right)^2 \mathrm{d}x \\
&\equiv d_{k,p}\left(\varepsilon^p + \frac{L_\varepsilon(p)}{\lambda_\varepsilon^2}\right),
\end{aligned} \tag{7.11}$$

where $d_{k,p} = \frac{1}{2}k\,(k-1)^{p+1}$. We then have

$$\begin{aligned}
\mathbb{E}\left[Z^p\right] &= \sum_{k=2}^{\infty}\mathbb{E}\left[Z^p \mid N\!=\!k\right]\ \mathbb{P}\left(N\!=\!k\right) \\
&\le \left(\varepsilon^p + \frac{L_\varepsilon(p)}{\lambda_\varepsilon^2}\right)\ \exp\left(-\frac{\lambda_\varepsilon}{n}\right)\ \sum_{k=2}^{\infty} d_{k,p}\left(\frac{\lambda_\varepsilon}{n}\right)^k \frac{1}{k!}
\end{aligned}$$

For $k_p = \lceil p \rceil + 2$ there exists C_p such that for any $k \geq k_p$, $d_{k,p} \leq C_p \frac{k!}{(k-k_p)!}$, so

$$\begin{aligned}\sum_{k=2}^{\infty} d_{k,p} \left(\frac{\lambda_\varepsilon}{n}\right)^k \frac{1}{k!} &\leq \sum_{k=2}^{k_p-1} d_{k,p} \left(\frac{\lambda_\varepsilon}{n}\right)^k \frac{1}{k!} + C_p \sum_{k=k_p}^{\infty} \left(\frac{\lambda_\varepsilon}{n}\right)^k \frac{1}{(k-k_p)!} \\ &\leq \sum_{k=2}^{k_p-1} d_{k,p} \left(\frac{\lambda_\varepsilon}{n}\right)^k \frac{1}{k!} + C_p \left(\frac{\lambda_\varepsilon}{n}\right)^{k_p} \exp\left(\frac{\lambda_\varepsilon}{n}\right) \\ &\leq K_p \left(\frac{\lambda_\varepsilon}{n}\right)^2\end{aligned}$$

for some constant K_p, where the last step uses the fact that $\lambda_\varepsilon \leq n$.

Therefore, we obtain the final result that

$$\mathbb{E}[Z^p] \leq K_p \left(\varepsilon^p + \frac{L_\varepsilon(p)}{\lambda_\varepsilon^2}\right) \left(\frac{\lambda_\varepsilon}{n}\right)^2.$$

7.4 Bounding moments of $\sup_{[0,T]} |R_t^\varepsilon|$

Proposition 7.2 *Let X_t be a scalar Lévy process with a triple $(m, 0, \nu)$ and let R_t^ε, μ_ε, λ_ε and σ_ε be as defined in section 7.2. Then R_t^ε satisfies*

$$\mathbb{E}\left[\sup_{[0,T]} |R_t^\varepsilon|^p\right] \leq \begin{cases} K_p \left(T^{p/2} \sigma_\varepsilon^p + T \int_{|z|\leq\varepsilon} |z|^p \,\nu(\mathrm{d}x)\right), & p > 2; \\ K_p T^{p/2} \sigma_\varepsilon^p, & 1 \leq p \leq 2, \end{cases} \tag{7.12}$$

where K_p is a constant depending on p.

Proof For any $1 \leq p \leq 2$, by Jensen's inequality and the Doob inequality (c.f. [24]),

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T} |R_t^\varepsilon|^p\right] &\leq \mathbb{E}\left[\sup_{0\leq t\leq T} |R_t^\varepsilon|^2\right]^{p/2} \\ &\leq 2^p \,\mathbb{E}\left[|R_T^\varepsilon|^2\right]^{p/2} \\ &= 2^p \,T^{p/2} \sigma_\varepsilon^p.\end{aligned}$$

For any $p > 2$, the Burkholder–Davis–Gundy inequality (c.f. [24]) gives

$$\mathbb{E}\left[\sup_{0\leq t\leq 1} |R_t^\varepsilon|^p\right] \leq \mathbb{E}\left[[R^\varepsilon]_1^{p/2}\right]$$

where $[R^\varepsilon]_t$ is the quadratic variation of R_t^ε. We can use the method in pages 347-348 of [24] to get

$$\begin{aligned}\mathbb{E}\left[[R^\varepsilon]_1^{p/2}\right] &\leq K_p \left[\left(\int_{|z|\leq\varepsilon} z^2 \nu(\mathrm{d}z)\right)^{p/2} + \int_{|z|\leq\varepsilon} |z|^p \,\nu(\mathrm{d}z)\right] \\ &= K_p \left(\sigma_\varepsilon^p + \int_{|z|\leq\varepsilon} |z|^p \,\nu(\mathrm{d}z)\right)\end{aligned}$$

where K_p is a constant depending on p.

To extend this result to an arbitrary time interval $[0,T]$ we use a change of time coordinate, $t'=t/T$, with associated changed Lévy measure $\nu'(\mathrm{d}z)=T\,\nu(\mathrm{d}z)$ to obtain

$$\mathbb{E}\left[\sup_{[0,T]}|R_t^\varepsilon|^p\right] \le K_p\left[T^{p/2}\sigma_\varepsilon^{p/2}+T\int_{|z|\le\varepsilon}|z|^p\,\nu(\mathrm{d}z)\right].$$

7.5 Bounding moments of $\max_{0\le i<n} S_n^{(i)}$

Proposition 7.3 *Let X_t be a scalar pure jump Lévy process, with Lévy measure $\nu(x)$ which satisfies*

$$C_2\,|x|^{-1-\alpha} \le \nu(x) \le C_1\,|x|^{-1-\alpha},\ \textit{as}\ |x|\le 1;$$

for constants $C_1, C_2>0$ and $0\le\alpha<2$. If $S_n^{(i)}$ is as defined in section 7.2, and $\lambda_\varepsilon\le n$, then for $p\ge 1$, and arbitrary $\delta>0$ there exists a constant $C_{p,\delta}$, which does not depend on n,ε such that

$$\mathbb{E}\left[\left(\max_{0\le i<n} S_n^{(i)}\right)^p\right] \le C_{p,\delta}\;\varepsilon^{p-\delta}.$$

In the particular case of $\alpha=0$, such a bound holds with $\delta=0$.

Proof By Proposition 7.2, for $q>2$,

$$\mathbb{E}\left[\left(\max_{0\le i<n} S_n^{(i)}\right)^q\right] \;\le\; n\,\mathbb{E}\left[\sup_{[0,\frac{1}{n}]}|R_t^\varepsilon|^q\right] \;\le\; K_q\left(n^{1-q/2}\sigma_\varepsilon^q+\int_{|z|\le\varepsilon}|z|^q\,\nu(\mathrm{d}x)\right).$$

Recalling the definition of σ_ε (7.6), due to the assumption on $\nu(x)$ we have

$$\sigma_\varepsilon^q \;\le\; \left(\frac{2C_1}{2-\alpha}\right)^{q/2}\varepsilon^{q-q\alpha/2},\qquad \int_{|z|\le\varepsilon}|z|^q\,\nu(\mathrm{d}x) \;\le\; \frac{2C_1}{q-\alpha}\;\varepsilon^{q-\alpha}.$$

Given $p\ge 1$, for any $q>\max(2,p)$, Jensen's inequality gives us

$$\mathbb{E}\left[\left(\max_{0\le i<n} S_n^{(i)}\right)^p\right] \le \mathbb{E}\left[\left(\max_{0\le i<n} S_n^{(i)}\right)^q\right]^{p/q}$$
$$\le K_q^{p/q}\left[\left(\frac{2C_1}{2-\alpha}\right)^{q/2}\left(\frac{\varepsilon^{-\alpha}}{n}\right)^{q/2-1}+\frac{2C_1}{q-\alpha}\right]^{p/q}\varepsilon^{p-\alpha p/q}.$$

If $\alpha=0$, then the desired bound is obtained immediately. On the other hand, if $0<\alpha<2$, then

$$\lambda_\varepsilon \;\ge\; C_2\int_{\varepsilon<|z|<1}\frac{1}{|z|^{\alpha+1}}\,\mathrm{d}z \;=\; \frac{2C_2}{\alpha}\left(\varepsilon^{-\alpha}-1\right).$$

Since $\lambda_\varepsilon\le n$ it implies that $\varepsilon^{-\alpha}\le\frac{K\alpha}{2C_2}n+1$, and thus $\varepsilon^{-\alpha}/n$ is bounded. Hence there exists a constant C such that

$$\mathbb{E}\left[\left(\max_{0\le i<n} S_n^{(i)}\right)^p\right] \le C\,\varepsilon^{p-\alpha p/q},$$

and by choosing q large enough so that $\alpha p/q\le\delta$ we obtain the desired bound.

7.6 Proof of Theorem 4.2

Proof Provided $\lambda_\varepsilon \leq n$, by (7.9) and (7.10) we have

$$\mathbb{E}\left[D_n^p\right] \prec \underbrace{\mathbb{E}\left[\left(\max_{0\leq i<n} S_n^{(i)}\right)^p\right]}_{1)} + \underbrace{\varepsilon^p \frac{\lambda_\varepsilon^2}{n}}_{2)} + \underbrace{\frac{L_\varepsilon(p)}{n}}_{3)} + \underbrace{\left(\frac{|\mu_\varepsilon|}{n}\right)^p}_{4)}, \tag{7.13}$$

where the notation $u \prec v$ means there exists constant $c > 0$ independent of n such that $u < cv$.

We can now bound each term, given the specification of the Lévy measure, and if we can choose appropriately how $\varepsilon \to 0$ as $n \to \infty$ so that the RHS of (7.13) is convergent, then the convergence rate of $\mathbb{E}\left[D_n^p\right]$ can be bounded.

For $0<x<1$,

$$\begin{aligned}
\lambda_x &\leq C_1 \int_{x<|z|<1} \frac{1}{|z|^{\alpha+1}}\, \mathrm{d}z + \int_{1<|z|} \exp\left(-C_3\,|z|\right) \mathrm{d}z \\
&\leq \begin{cases} 2C_1 \log\frac{1}{x} + l_1, & \alpha = 0; \\ l_2\, x^{-\alpha}, & 0<\alpha<2. \end{cases}
\end{aligned} \tag{7.14}$$

where l_1, l_2 are constants with $l_2 \geq 2C_3^{-1}$, while for $x \geq 1$,

$$\lambda_x \;\leq\; \int_{x<|z|} \exp\left(-C_3\,|z|\right) \mathrm{d}z \;=\; 2C_3^{-1} \exp(-C_3\, x).$$

If $\alpha > 0$, then

$$\begin{aligned}
L_\varepsilon(p) &= p \int_{x>\varepsilon} x^{p-1} \lambda_x^2 \,\mathrm{d}x \\
&\leq l_2^2\, p \int_{x>\varepsilon} x^{p-1} \left(1_{\{x<1\}} x^{-2\alpha} + 1_{\{x>1\}} \exp\left(-2C_3 x\right) \right) \mathrm{d}x \\
&\leq \begin{cases} l_4, & p > 2\alpha; \\ l_4 \log\frac{1}{\varepsilon} + l_5, & p = 2\alpha; \\ l_4 \varepsilon^{-2\alpha+p} + l_5, & p < 2\alpha. \end{cases}
\end{aligned} \tag{7.15}$$

where l_3, l_4, l_5 are additional constants. If $\alpha = 0$, it is easily verified that $L_\varepsilon(p)$ is bounded for $p \geq 1$, so (7.15) applies equally to this case.

Given $0<\varepsilon<1$ we have

$$\begin{aligned}
|\mu_\varepsilon| &= \left| m - \int_{\varepsilon<|z|<1} z\, \nu(\mathrm{d}z) \right| \\
&\leq \begin{cases} |m| + |C_1 - C_2|\, \frac{\varepsilon^{1-\alpha}-1}{\alpha-1}, & \alpha \neq 1; \\ |m| + |C_1 - C_2| \log\frac{1}{\varepsilon}, & \alpha = 1. \end{cases}
\end{aligned} \tag{7.16}$$

Subject to the condition that $\lambda_\varepsilon \leq n$, we now have

1) By Proposition 7.3,

$$\mathbb{E}\left[\left(\max_{i=1,\dots,n} S_n^{(i)}\right)^p\right] \prec \varepsilon^{p-\delta}, \quad \text{for any } \delta > 0.$$

2) By (7.14),

$$\varepsilon^p \frac{\lambda_\varepsilon^2}{n} \prec n^{-1} \times \begin{cases} \varepsilon^p \log\frac{1}{\varepsilon}, & \alpha = 0; \\ \varepsilon^{p-2\alpha}, & 0 < \alpha < 2. \end{cases}$$

3) By (7.15),

$$\frac{L_\varepsilon(p)}{n} \prec n^{-1} \times \begin{cases} 1, & p > 2\alpha; \\ \log\frac{1}{\varepsilon}, & p = 2\alpha; \\ \varepsilon^{-2\alpha+p}, & p < 2\alpha. \end{cases}$$

4) By (7.16),

$$\left(\frac{|\mu_\varepsilon|}{n}\right)^p \prec n^{-p} \times \begin{cases} 1 + |C_1 - C_2|^p \, \varepsilon^{p(1-\alpha)}, & \alpha > 1; \\ 1 + \left(|C_1 - C_2| \log\frac{1}{\varepsilon}\right)^p, & \alpha = 1; \\ 1, & \alpha < 1. \end{cases}$$

In the following we assume $C_1 \neq C_2$.

1. $p \geq 2\alpha$.
   If we choose $\varepsilon = C n^{-2/p}$, then $\lambda_\varepsilon \prec \varepsilon^{-\alpha} \prec n^{2\alpha/p}$, and the constant C can be taken to be sufficiently small so that $\lambda_\varepsilon \leq n$ for sufficiently large n.
   Taking $\delta < p/2$, we find that the dominant contribution to (7.13) comes from 3), giving the desired result that

$$\mathbb{E}\,[D_n^p] \prec \begin{cases} n^{-1}, & p > 2\alpha; \\ \frac{\log n}{n}, & p = 2\alpha. \end{cases}$$

2. $1 \leq p < 2\alpha$.
   We can use Hölder's inequality to give $\mathbb{E}\left[D_n^p\right] \leq \mathbb{E}\left[D_n^{2\alpha}\right]^{\frac{p}{2\alpha}} \prec \left(\frac{\log n}{n}\right)^{\frac{p}{2\alpha}}$.

For a spectrally negative process, $\sup_{[0,\frac{1}{n}]} X_t^\varepsilon - \left(X_{\frac{1}{n}}^\varepsilon\right)^+ = 0$, since X_t doesn't have positive jumps, and hence

$$\mathbb{E}\,[D_n^p] \leq \mathbb{E}\left[\left(\max_{0\leq i<n} S_n^{(i)}\right)^p\right] + \left(\frac{|\mu_\varepsilon|}{n}\right)^p.$$

We can take $\varepsilon = C\, n^{-1/\alpha}$ with the constant C again chosen so that $\lambda_\varepsilon \leq n$ for sufficiently large n. We then obtain

$$\mathbb{E}\,[D_n^p] \prec \begin{cases} n^{-p/\alpha+\delta}, & \alpha \geq 1; \\ n^{-p}, & \alpha < 1. \end{cases}$$

for any $\delta > 0$.

7.7 Proof of Proposition 5.4

We decompose the term we want to bound into parts and then balance their asymptotic orders to get desired result.

Note that $\mathbf{1}_{\{\widehat{m}_n<B\}} - \mathbf{1}_{\{m<B\}} = 1$ only if either m is close to the barrier or the difference between discretely and continuously monitored maximum $D_n = m - \widehat{m}_n$ is large. More precisely,

$$\left\{\mathbf{1}_{\{\widehat{m}_n<B\}} - \mathbf{1}_{\{m<B\}} = 1\right\} \subset F \cup G,$$

where $F := \{B \le m \le B + n^{-r}\}$ and $G := \{D_n > n^{-r}\}$ for an $r > 0$ to be determined. Hence

$$\mathbb{E}\left[\mathbf{1}_{\{\widehat{m}_n<B\}} - \mathbf{1}_{\{m<B\}}\right] \le \mathbb{P}(F) + \mathbb{P}(G).$$

Due to the locally bounded density for m, $\mathbb{P}(F) = \mathscr{O}(n^{-r})$.

If we denote

$$Z_n^{(i)} = \sup_{[0,\frac{1}{n}]} \Delta^{(i)} X_t^{\varepsilon} - \left(\Delta^{(i)} X_{\frac{1}{n}}^{\varepsilon}\right)^{+}.$$

where $\Delta^{(i)} X_t$ is as defined previously in Section 7.2, then (7.8) gives

$$D_n \le \max_{0\le i<n} Z_n^{(i)} + \frac{|\mu_\varepsilon|}{n} + \max_{0\le i<n} S_n^{(i)}$$

For $\alpha < 1$, μ_ε is bounded, so $|\mu_\varepsilon| \le \frac{1}{2} n^{1-r}$, for sufficiently large n. Hence,

$$\begin{aligned}\mathbb{P}\left(D_n > n^{-r}\right) &\le \mathbb{P}\left(\max_{0\le i<n} Z_n^{(i)} + \max_{0\le i<n} S_n^{(i)} > \tfrac{1}{2} n^{-r}\right)\\ &\le \mathbb{P}\left(\max_{0\le i<n} Z_n^{(i)} > \tfrac{1}{4} n^{-r}\right) + \mathbb{P}\left(\max_{0\le i<n} S_n^{(i)} > \tfrac{1}{4} n^{-r}\right).\end{aligned}$$

Now, $\max_{0\le i<n} Z_n^{(i)} > 0$ requires that there are at least two jumps in one of the n intervals. The probability of two jumps in one particular interval is

$$1 - \exp\left(-\tfrac{\lambda_\varepsilon}{n}\right)\left(1 + \tfrac{\lambda_\varepsilon}{n}\right) \prec \left(\tfrac{\lambda_\varepsilon}{n}\right)^2$$

if $\lambda_\varepsilon \le n$, and hence

$$\mathbb{P}\left(\max_{0\le i<n} Z_n^{(i)} > \tfrac{1}{4} n^{-r}\right) \prec \frac{\lambda_\varepsilon^2}{n}.$$

We use the Markov inequality for the remaining term. According to Proposition 7.2, $\mathbb{E}\left[\max_{0\le i<n}\left(S_n^{(i)}\right)^p\right] \prec \varepsilon^{p-\delta}$ and so

$$\mathbb{P}\left(\max_{0\le i<n} S_n^{(i)} > \tfrac{1}{4} n^{-r}\right) \prec \mathbb{E}\left[\max_{0\le i<n}\left(S_n^{(i)}\right)^p\right] \Big/ \left(\tfrac{1}{4} n^{-r}\right)^p \prec \varepsilon^{p-\delta} n^{rp}.$$

Combining these elements, provided $\lambda_\varepsilon \le n$, we have

$$\mathbb{E}\left[\mathbf{1}_{\{\widehat{m}_n<B\}} - \mathbf{1}_{\{m<B\}}\right] \prec n^{-r} + \varepsilon^{p-\delta} n^{rp} + \frac{\lambda_\varepsilon^2}{n}.$$

Equating the first two terms on the right hand side gives $\varepsilon = n^{-r(1+p)/(p-\delta)}$.

If $\alpha = 0$, then $\lambda_\varepsilon \prec \log\frac{1}{\varepsilon} \prec \log n$, so $\lambda_\varepsilon = o(n)$ is satisfied. We also have $\frac{\lambda_\varepsilon^2}{n} \prec \frac{(\log n)^2}{n}$, and therefore for any $r<1$ we have $\mathbb{E}\left[\mathbf{1}_{\{\widehat{m}_n<B\}} - \mathbf{1}_{\{m<B\}}\right] \prec n^{-r}$.

If $0<\alpha<2$, then $\lambda_\varepsilon \prec \varepsilon^{-\alpha} \prec n^{r\alpha(1+p)/(p-\delta)}$, and hence $\frac{\lambda_\varepsilon^2}{n} \prec n^{-1+2r\alpha(1+p)/(p-\delta)}$. Balancing n^{-r} and $n^{-1+2r\alpha(1+p)/(p-\delta)}$, gives $\lambda_\varepsilon = o(n)$ and

$$r = \left(1+2\alpha\,\frac{1+p}{p-\delta}\right)^{-1}. \tag{7.17}$$

Since $r \to \frac{1}{1+2\alpha}$ as $\delta \to 0$, and $p \to \infty$, for any fixed value of $r < \frac{1}{1+2\alpha}$ it is possible to choose appropriate values of p and δ to satisfy (7.17) and thereby conclude that $\mathbb{E}\left[\mathbf{1}_{\{\widehat{m}_n<B\}} - \mathbf{1}_{\{m<B\}}\right] \prec n^{-r}$.

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