Numerical Methods II
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Problem sheet 4

1. (a) If a and b are random variables with zero expectation, prove that

$$\sqrt{\mathbb{V}[a+b]} \;\leq\; \sqrt{\mathbb{V}[a]} + \sqrt{\mathbb{V}[b]},$$

and hence prove that this result remains true even when a and b have non-zero expectation.

Determine the most general circumstances under which

$$\sqrt{\mathbb{V}[a+b]} = \sqrt{\mathbb{V}[a]} + \sqrt{\mathbb{V}[b]}.$$

(b) As a corollary, prove the lower bound

$$\sqrt{\mathbb{V}[a+b]} \;\geq\; \sqrt{\mathbb{V}[a]} - \sqrt{\mathbb{V}[b]}.$$

and determine the most general circumstances under which

$$\sqrt{\mathbb{V}[a+b]} = \sqrt{\mathbb{V}[a]} - \sqrt{\mathbb{V}[b]}.$$

(c) Also as a corollary, prove that

$$\sqrt{\mathbb{V}\left[\sum_{n=1}^{N} a_n\right]} \leq \sum_{n=1}^{N} \sqrt{\mathbb{V}[a_n]},$$

where the a_n are all random variables.

2. Given the values of a scalar Brownian motion at two times t_n and t_{n+1}, determine the conditional probability density function for $W(t)$ at an arbitrary intermediate time $t_n < t < t_{n+1}$.

Hence, explain how one can implement a Brownian Bridge construction in the case in which the number of timesteps is not a power of 2.

3. Given that a scalar Brownian motion has end values $W(0) = W(1) = 0$, use the result from the first part of the previous question to prove that the covariance matrix for the discrete Brownian values $W_n = W(n/N)$, $n = 1, 2, \ldots, N-1$ is

$$\Omega_{jk} = \min(t_j, t_k) - t_j\, t_k.$$

Prove that

$$\Omega^{-1} = N \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}.$$

and hence that the eigenvalues and unit eigenvectors of Ω are

$$\lambda_m = \frac{1}{4N} \left(\sin\left(\frac{m\pi}{2N}\right)\right)^{-2}$$

$$(V_m)_n = \frac{2}{\sqrt{2N}} \sin\left(\frac{mn\pi}{N}\right).$$

(This gives an interesting hybrid between the PCA and Brownian Bridge constructions, with an efficient FFT implementation when N is a power of 2.)

4. Suppose two stocks satisfy the Geometric Brownian Motion SDEs

$$\begin{aligned} \mathrm{d}S_1 &= r\, S_1\, \mathrm{d}t + \sigma_1\, S_1\, \mathrm{d}W_1 \\ \mathrm{d}S_2 &= r\, S_2\, \mathrm{d}t + \sigma_2\, S_2\, \mathrm{d}W_2 \end{aligned}$$

with correlation ρ between the two driving Brownian motions, i.e. $\mathrm{d}W_1\, \mathrm{d}W_2 = \rho\, \mathrm{d}t$.

An option pays 1 unit at maturity $T = 2$, if the minimum of the two stocks is below a barrier value B at 3 or more of the 8 times $t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$.

(a) Giving full details, explain how you could estimate the option value numerically, using exact numerical integration of the SDEs so that each timestep has size $h = \frac{1}{4}$.

(b) Explain how you could compute the two Deltas (i.e. the option value's derivatives w.r.t. $S_1(0)$ and $S_2(0)$) using the Likelihood Ratio Method.

(c) Explain the problem that prevents the use of the pathwise sensitivity approach, and how this could be solved, approximately, by smoothing the payoff. Give an example of a smoothing which over-estimates the value of the option, and an example of a smoothing which underestimates it.