Stochastic Numerical Analysis

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Wiener process

A scalar Wiener process W_t (also known as a Brownian motion) is characterised by the following properties:

- $W_0 = 0$
- continuity: W_t is almost surely continuous
- independent increments: if $r \leq s \leq t \leq u$ then $W_u - W_t$ is independent of $W_s - W_r$
- Normal increments: $W_u - W_t \sim N(0, u - t)$

Wiener process

Three notes:

- The last two conditions are compatible: if $s < t < u$ then
  $W_u - W_t \sim N(0, u-t)$ and
  $W_t - W_s \sim N(0, t-s)$ so
  $W_u - W_s \sim N(0, u-s)$

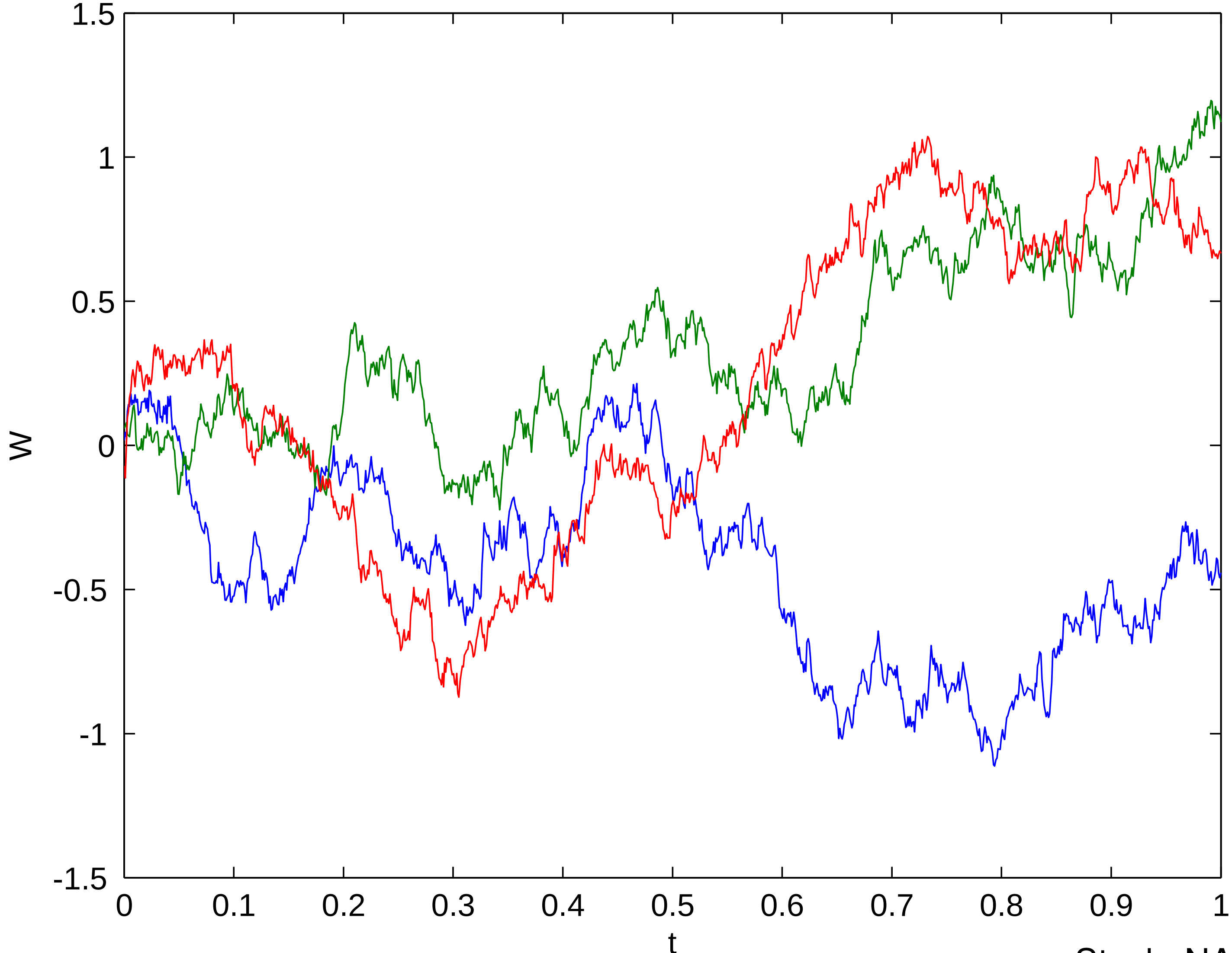
- Over a small time interval of size h, $\Delta W = O(h^{1/2})$, whereas it would be $O(h)$ if W_t were differentiable.

- For future reference, $\mathbb{E}[\Delta W] = 0, \mathbb{V}[\Delta W] = h$, and

$$
\begin{aligned}
\mathbb{E}[\,(\Delta W)^2 - h\,] &= 0 \\
\mathbb{V}[\,(\Delta W)^2 - h\,] &= 2\,h^2
\end{aligned}
$$

Wiener process

Illustration of 3 Wiener paths:



Lévy process

A Lévy process is a generalisation of a Wiener process – non-overlapping intervals still have independent increments, but now $L_u - L_t$ has a non-Normal distribution which depends solely on $u - t$.

Note that because of the CLT

$$L_N = \sum_{n=1}^{N} (L_n - L_{n-1})$$

is approximately Normally distributed with variance $N\,\mathbb{V}[L_1]$, as $N \to \infty$.

Hence, over large enough time increments, any Lévy process ends up looking like a Wiener process.

Itô integral

The definition and solution of an SDE (stochastic differential equation) involves Itô integrals of the form

$$\int_0^T f_t \, \mathrm{d}W_t$$

These are trickier than "standard" Riemann integrals – must be very careful.

For example, you might think that

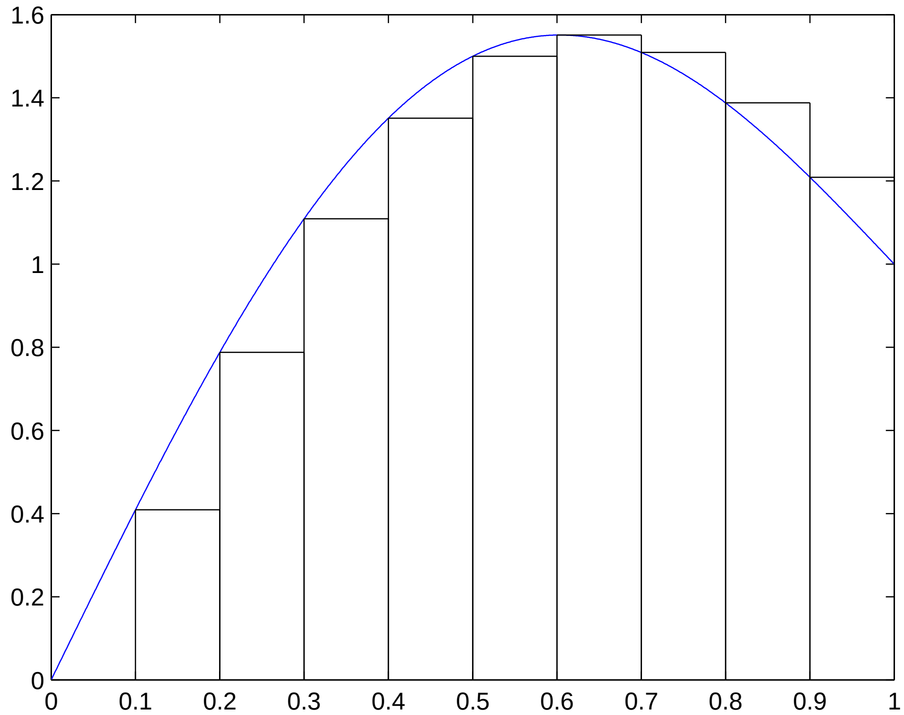
$$\int_0^T W_t \, \mathrm{d}W_t = \left[\tfrac{1}{2} W_t^2\right]_0^T$$

but you'd be wrong!

Riemann integral

The formal definition of a Riemann integral is:

$$\int_0^T f_t \, \mathrm{d}t = \lim_{N\to\infty} \sum_{n=0}^{N-1} f_{Tn/N} \, \frac{T}{N}$$



Itô integral

The formal definition of the Itô integral is very similar:

$$\int_0^T f_t \, \mathrm{d}W_t = \lim_{N\to\infty} \sum_{n=0}^{N-1} f_{Tn/N} \, \Delta W_n$$

where $\Delta W_n = W_{T(n+1)/N} - W_{Tn/N}$

When integrating $\int_0^T W_t \, \mathrm{d}W_t$ we have

$$W_{Tn/N} = \tfrac{1}{2}(W_{T(n+1)/N} + W_{Tn/N}) - \tfrac{1}{2}\Delta W_n$$

and therefore

$$\sum_{n=0}^{N-1} f_{Tn/N} \Delta W_n \;=\; \tfrac{1}{2} \sum_{n=0}^{N-1} (W^2_{T(n+1)/N} - W^2_{Tn/N}) - (\Delta W_n)^2$$

Itô integral

Note that

$$\mathbb{E}\left[\sum_{n=0}^{N-1} (\Delta W_n)^2\right] \;=\; \sum_{n=0}^{N-1} \mathbb{E}[(\Delta W_n)^2] \;=\; \sum_{n=0}^{N-1} \frac{T}{N} \;=\; T$$

and by CLT it is asymptotically Normally distributed as $N \to \infty$ with variance

$$\mathbb{V}\left[\sum_{n=0}^{N-1} (\Delta W_n)^2\right] \;=\; \sum_{n=0}^{N-1} \mathbb{V}[(\Delta W_n)^2] \;=\; \sum_{n=0}^{N-1} 2\left(\frac{T}{N}\right)^2 \;=\; \frac{2\,T^2}{N} \to 0$$

Hence,

$$\int_0^T W_t \, \mathrm{d}W_t = \tfrac{1}{2}\,(W_T^2 - T)$$

Stratonovich integral

The Stratonovich integral is defined differently

$$\int_0^T f_t \circ \mathrm{d}W_t = \lim_{N\to\infty} \sum_{n=0}^{N-1} \tfrac{1}{2}(f_{Tn/N} + f_{T(n+1)/N})\, \Delta W_n$$

where $\Delta W_n = W_{T(n+1)/N} - W_{Tn/N}$.

This definition leads to

$$\int_0^T W_t \circ \mathrm{d}W_t = \tfrac{1}{2} W_T^2$$

Note use of $\circ$ notation to distinguish Stratonovich integral from Itô integral.

Stratonovich integral

Why do we not use this definition?

This really comes back to the mathematical modelling.

The Itô integral corresponds to the idea that we cannot anticipate what will happen in the future.

In financial applications, we choose now whether we want to do something (e.g. buy or sell stocks) and then wait to see what random events happen later.

Itô isometry

One key result with Itô integrals is that

$$\mathbb{E}\left[\left(\int_0^T f_t \,\mathrm{d}W_t\right)^2\right] = \mathbb{E}\left[\int_0^T f_t^2 \,\mathrm{d}t\right]$$

Informal proof:

$$\mathbb{E}\left[\left(\sum_{n=0}^{N-1} f_{Tn/N}\ \Delta W_n\right)^2\right] = \mathbb{E}\left[\sum_{m,n=0}^{N-1} f_{Tm/N}\ \Delta W_m\ f_{Tn/N}\ \Delta W_n\right]$$

However, for $m < n$, we know that ΔW_n is independent of the earlier evolution of the Wiener path and hence

$$\mathbb{E}[f_{Tm/N}\Delta W_m\ f_{Tn/N}\Delta W_n] = \mathbb{E}[f_{Tm/N}\Delta W_m\ f_{Tn/N}]\ \mathbb{E}[\Delta W_n] = 0$$

Itô isometry

Similar applies for $m > n$, and hence

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{n=0}^{N-1} f_{Tn/N}\, \Delta W_n\right)^2\right] &= \mathbb{E}\left[\sum_{n=0}^{N-1} f_{Tn/N}^2\, (\Delta W_n)^2\right] \\
&= \sum_{n=0}^{N-1} \mathbb{E}[f_{Tn/N}^2]\; \mathbb{E}[(\Delta W_n)^2] \\
&= \mathbb{E}\left[\sum_{n=0}^{N-1} f_{Tn/N}^2\, \frac{T}{N}\right]
\end{aligned}
$$

Then take the limit $N \to \infty$ to get the result.

SDE

The differential form of a scalar Itô stochastic differential equation is:

$$\mathrm{d}S_t = a(S_t, t)\ \mathrm{d}t + b(S_t, t)\ \mathrm{d}W_t$$

which is just a shorthand for the integral form:

$$S_t = S_0 + \int_0^t a(S_u, u)\ \mathrm{d}u + \int_0^t b(S_u, u)\ \mathrm{d}W_u$$

with the second integral being an Itô integral.

Itô lemma

The SDE satisfied by $f(S_t, t)$ is

$$\mathrm{d}f = \frac{\partial f}{\partial S}\,\mathrm{d}S_t + \left(\frac{1}{2}\,\frac{\partial^2 f}{\partial S^2}\,b^2(S_t,t) + \frac{\partial f}{\partial t}\right)\mathrm{d}t$$

The $\frac{\partial^2 f}{\partial S^2}$ term is an extra term due to Itô calculus – you wouldn't get this with an ODE.

Informal proof: using N intervals of size $h = T/N$, the discrete approximation to the integrals gives

$$\begin{aligned} S_n &= S_0 + \sum_{m=0}^{n-1} \left(a(S_m, t_m)\,h + b(S_m, t_m)\,\Delta W_m\right) \\ S_{n+1} &= S_n + a_n\,h + b_n\,\Delta W_n \end{aligned}$$

Itô lemma

Performing a Taylor series expansion gives us

$$f_{n+1} = f_n + \frac{\partial f}{\partial S}\,(a_n h + b_n \Delta W_n) + \tfrac{1}{2}\,\frac{\partial^2 f}{\partial S^2}\,(b_n \Delta W_n)^2 + \frac{\partial f}{\partial t}\,h + O(h^{3/2})$$

Summing over n, then taking the limit as $N \to \infty$, we get

$$\sum_n \frac{\partial f}{\partial S}\,(a_n\, h + b_n\, \Delta W_n) \to \int_0^T \frac{\partial f}{\partial S}\,(a\,\mathrm{d}t + b\,\mathrm{d}W_t)$$

$$\sum_n \frac{\partial f}{\partial t}\,h \to \int_0^T \frac{\partial f}{\partial t}\,\mathrm{d}t$$

Itô lemma

The final term we split into two:

$$\frac{\partial^2 f}{\partial S^2}\,(b_n\,\Delta W_n)^2 = \frac{\partial^2 f}{\partial S^2}\;b_n^2\;h + \frac{\partial^2 f}{\partial S^2}\;b_n^2\,\Big((\Delta W_n)^2 - h\Big)$$

The first part gives

$$\sum_n \frac{\partial^2 f}{\partial S^2}\;b_n^2\;h \to \int_0^T \frac{\partial^2 f}{\partial S^2}\;b^2\;\mathrm{d}t$$

For the second part we have

$$\mathbb{E}\left[\sum_n \frac{\partial^2 f}{\partial S^2}\;b_n^2\,\Big((\Delta W_n)^2 - h\Big)\right] = 0$$

and its variance is . . .

Itô lemma

$$\mathbb{E}\left[\left(\sum_n \frac{\partial^2 f}{\partial S^2}\, b_n^2\left((\Delta W_n)^2 - h\right)\right)^2\right]$$

$$\begin{aligned}
&= \sum_n \mathbb{E}\left[\left(\frac{\partial^2 f}{\partial S^2}\, b_n^2\left((\Delta W_n)^2 - h\right)\right)^2\right] \\
&= \sum_n \mathbb{E}\left[\left(\frac{\partial^2 f}{\partial S^2}\, b_n^2\right)^2\right] \mathbb{E}\left[\left((\Delta W_n)^2 - h\right)^2\right] \\
&= \frac{2T^2}{N^2} \sum_n \mathbb{E}\left[\left(\frac{\partial^2 f}{\partial S^2}\, b_n^2\right)^2\right] \\
&\to 0 \quad \text{as} \quad N \to \infty
\end{aligned}$$

so this second part disappears in the limit.

Itô lemma

Trivial example: if we define

$$f(W_t) \equiv \tfrac{1}{2}\, W_t^2$$

then

$$\begin{aligned} \mathrm{d}f &= \frac{\partial f}{\partial W}\, \mathrm{d}W_t + \tfrac{1}{2}\frac{\partial^2 f}{\partial W^2}\, \mathrm{d}t \\ &= W_t\, \mathrm{d}W_t + \tfrac{1}{2}\, \mathrm{d}t \end{aligned}$$

Integrating over the interval $[0, T]$ then gives

$$\tfrac{1}{2}\, W_T^2 = \int_0^T W_t\, \mathrm{d}W_t + \tfrac{1}{2}\, T \quad \Longrightarrow \quad \int_0^T W_t\, \mathrm{d}W_t = \tfrac{1}{2}\, \left(W_T^2 - T\right)$$

Itô lemma

Geometric Brownian Motion (Black-Scholes model)

$$\mathrm{d}S_t = r\, S_t\, \mathrm{d}t + \sigma\, S_t\, \mathrm{d}W_t$$

Let $X = \log S$, then

$$\mathrm{d}X_t = (r - \tfrac{1}{2}\sigma^2)\, \mathrm{d}t + \sigma\, \mathrm{d}W_t$$

which can be integrated to give

$$X_T = X_0 + (r - \tfrac{1}{2}\sigma^2)\, T + \sigma\, W_T$$

and hence

$$S_T = S_0 \exp\left((r - \tfrac{1}{2}\sigma^2)\, T + \sigma\, W_T\right)$$