Stochastic Numerical Analysis

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MLMC variance analysis

Today we are looking at the error analysis in

M.B. Giles, D.J. Higham and X. Mao. 'Analysing multilevel Monte Carlo for options with non-globally Lipschitz payoff'. *Finance and Stochastics*, 13(3):403-413, 2009

This is based on

- scalar SDE satisfying usual conditions
- Euler-Maruyama discretisation
- various different financial options

and is really doing the numerical analysis for the testcases in the original MLMC paper in *Operations Research*

MLMC variance analysis

- European option with Lipschitz payoff:

$$P = f(S_T)$$

- Lookback option with Lipschitz payoff:

$$P = f\left(\inf_{[0,T]} S_t,\ S_T\right)$$

- Digital call option:

$$P = H(S_T - K)$$

- Barrier down-and-out option:

$$P = f(S_T)\ H(\inf_{[0,T]} S_t - B)$$

MLMC variance analysis

The objective is to bound $\mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$.

Since

$$\widehat{P}_\ell - \widehat{P}_{\ell-1} = (\widehat{P}_\ell - P) - (\widehat{P}_{\ell-1} - P)$$

we will do this by using

$$\begin{aligned}\mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}] &\leq \mathbb{E}[(\widehat{P}_\ell - \widehat{P}_{\ell-1})^2] \\ &\leq 2\,\mathbb{E}[(\widehat{P}_\ell - P)^2] + 2\,\mathbb{E}[(\widehat{P}_{\ell-1} - P)^2]\end{aligned}$$

and then bounding $\mathbb{E}[(\widehat{P}_\ell - P)^2]$.

European Lipschitz payoff

This case is easy.

If the payoff function is L-Lipschitz, so that

$$|f(x_1) - f(x_2)| \leq L \, |x_1 - x_2|$$

then

$$\begin{aligned} \mathbb{E}[(\widehat{P}-P)^2] &\leq L^2 \, \mathbb{E}[(\widehat{S}(T)-S(T))^2] \\ &\leq C \, h \end{aligned}$$

due to $O(h^{1/2})$ strong convergence.

This extends to $\mathbb{E}[(\widehat{P}-P)^2] = O(h^2)$ when using the Milstein approximation.

Lookback Lipschitz payoff

This case is very similar

If the payoff function is L-Lipschitz, so that

$$|f(x_1,y_1) - f(x_2,y_2)| \leq L\ (|x_1 - x_2| + |y_1 - y_2|)$$

then

$$\mathbb{E}[(\widehat{P} - P)^2] \leq 2\, L^2 \left(\mathbb{E}[(\widehat{S}_{min} - \inf_{[0,T]} S_t)^2] + \mathbb{E}[(\widehat{S}(T) - S_T)^2] \right)$$

The questions are:

- how is $\widehat{S}_{min}$ defined?
- what is the bound on $\mathbb{E}[(\widehat{S}_{min} - \inf S_t)^2]$?

Lookback Lipschitz payoff

In this paper we chose to use the minimum of the discrete timestep values

$$\widehat{S}_{min} \;\equiv\; \min_n \widehat{S}_n \;\equiv\; \inf_t \widehat{S}(t)$$

where $\widehat{S}(t)$ is defined by piecewise linear interpolation between the discrete values $\widehat{S}_n$ (not the usual Kloeden-Platen interpolant).

There is a theoretical result by Müller-Gronbach (2002) which says that

$$\mathbb{E}\left[\sup_t |\widehat{S}(t) - S_t|^p\right] \leq C\; |h \log h|^{p/2}$$

Lookback Lipschitz payoff

For any two processes A_t, B_t,

$$\begin{aligned} \sup_t A_t &\leq \sup_t B_t + \sup_t (A_t - B_t) \\ \sup_t B_t &\leq \sup_t A_t + \sup_t (B_t - A_t) \end{aligned}$$

Hence

$$\left| \sup_t A_t - \sup_t B_t \right| \leq \sup_t |A_t - B_t|$$

and also, by considering $C_t = -A_t, \ D_t = -B_t$,

$$\left| \inf_t A_t - \inf_t B_t \right| \leq \sup_t |A_t - B_t|$$

Lookback Lipschitz payoff

Therefore

$$\Longrightarrow \quad \left(\inf_t \widehat{S}(t) - \inf_t S_t\right)^2 \leq \mathbb{E}\left[\sup_t |\widehat{S}(t) - S_t|^2\right]$$

and hence

$$\mathbb{E}[(\widehat{S}_{min} - \inf_t S_t)^2] \leq C\ h\,|\log h|$$

which gives us the final bound

$$\mathbb{E}[(\widehat{P} - P)^2] = O(h\,|\log h|)$$

Digital payoff

The digital payoff is

$$P = H(S_T > K)$$

and the numerical approximation is

$$\widehat{P} = H(\widehat{S}(T) > K).$$

The exact and numerical payoffs differ only when one solution exceeds K and the other does not, so

$$\begin{aligned} \mathbb{E}(|\widehat{P} - P|^2) &= \mathbb{P}(\{S_T > K\} \cap \{\widehat{S}(T) \leq K\}) \\ &+ \mathbb{P}(\{S_T \leq K\} \cap \{\widehat{S}(T) > K\}) \end{aligned}$$

Digital payoff

For any $0 < \delta < \frac{1}{2}$, we have

$$\begin{aligned}
&\mathbb{P}(\{S_T > K\} \cap \{\widehat{S}(T) \leq K\}) \\
&\quad = \quad \mathbb{P}\left(\{K + h^{1/2-\delta/2} > S_T > K\} \cap \{\widehat{S}(T) \leq K\}\right) \\
&\qquad\quad + \mathbb{P}\left(\{S_T \geq K + h^{1/2-\delta/2}\} \cap \{\widehat{S}(T) \leq K\}\right) \\
&\quad < \quad \mathbb{P}\left(K + h^{1/2-\delta/2} > S_T > K\right) \\
&\qquad\quad + \mathbb{P}\left(S_T - \widehat{S}(T) \geq h^{1/2-\delta/2}\right)
\end{aligned}$$

We assume that S_T has a bounded p.d.f., so then

$$\mathbb{P}\left(K + h^{1/2-\delta/2} > S_T > K\right) = O(h^{1/2-\delta/2}) = o(h^{1/2-\delta})$$

Digital payoff

Remember from Markov inequality (lecture 5) that

$$\mathbb{E}[|h^{-1/2}(\widehat{S}(T)-S_T)|^p] < \infty$$

for all $p>0$ implies that for any $\delta>0$ can prove that

$$\mathbb{P}\left(|\widehat{S}(T)-S_T| \geq h^{1/2-\delta/2}\right) = o(h^q)$$

for any $q>0$.

The other term is treated similarly, and hence
$\mathbb{E}(|\widehat{P}-P|^2) = o(h^{1/2-\delta})$ for any $\delta>0$.

Using the Milstein approximation, this would change to
$\mathbb{E}(|\widehat{P}-P|^2) = o(h^{1-\delta})$

Digital payoff

In later research I found a simple proof of a generalisation previously derived by Rainer Avikainen:

Thm: if a scalar r.v. τ has a p.d.f. with maximum density ρ_{sup}, and $\widehat{\tau}$ is an approximation to τ, then for any s

$$\mathbb{E}[(\mathbf{1}_{\tau<s} - \mathbf{1}_{\widehat{\tau}<s})^2] \leq c_p \, \rho_{sup}^{p/(p+1)} \, \mathbb{E}[\, |\tau - \widehat{\tau}|^p]^{1/(p+1)}$$

Proof: define

$$\begin{aligned}
\Omega_1 &= \{|\tau - s| \leq X\}\,, \\
\Omega_2 &= \{|\tau - \widehat{\tau}| \geq X\} \cap \Omega_1^c, \\
\Omega_3 &= \Omega_1^c \cap \Omega_2^c,
\end{aligned}$$

then if $\omega \in \Omega_3$ we have $\mathbf{1}_{\tau<s} - \mathbf{1}_{\widehat{\tau}<s} = 0$.

Digital payoff

Hence,

$$\mathbb{E}[(\mathbf{1}_{\tau<s} - \mathbf{1}_{\widehat{\tau}<s})^2] \;\leq\; P_1 + P_2 \;\leq\; 2\,\rho_{sup}\,X + X^{-p}\,\mathbb{E}[\,|\tau-\widehat{\tau}|^p]$$

with the second step using the Markov inequality.

Differentiating the upper bound w.r.t. X, we find that it is minimised by choosing

$$X^{p+1} = \frac{p}{2\,\rho_{sup}}\,\mathbb{E}[\,|\tau-\widehat{\tau}|^p]$$

and we then get the bound

$$\mathbb{E}[(\mathbf{1}_{\tau<s} - \mathbf{1}_{\widehat{\tau}<s})^2] \;\leq\; c_p\;\rho_{sup}^{p/(p+1)}\,\mathbb{E}[\,|\tau-\widehat{\tau}|^p]^{1/(p+1)}$$

Barrier payoff

Payoff function:

$$P = f(S_T)\ H(\inf_{[0,T]} S_t - B)$$

with $|f(x) - f(y)| \leq |x-y|$.

Numerical approximation:

$$\widehat{P} = f(\widehat{S}(T))\ H(\widehat{S}_{min} - B)$$

with $\widehat{S}_{min}$ as defined before.

What is the difficulty?

For some paths, an $O(h^{1/2})$ fraction, S_t crosses B but $\widehat{S}(t)$ does not, or vice versa, giving $\widehat{P} - P = O(1)$

Barrier payoff

First the analysis when $f(S)$ is bounded, so $|f(S)| < f_{max}$.

Define $F = \{\inf_t S_t \geq B\}$, $G = \{\inf_t \widehat{S}(t) \geq B\}$, and then

$$
\begin{aligned}
\mathbb{E}(|P - \widehat{P}|^2) &= \mathbb{E}(|f(S_T)\mathbf{1}_F - f(\widehat{S}(T))\mathbf{1}_G|^2) \\
&= \mathbb{E}(|f(S_T) - f(\widehat{S}(T))|^2 \mathbf{1}_{\{F\cap G\}}) \\
&\quad + \mathbb{E}(|f(S_T)|^2 \mathbf{1}_{\{F\cap G^c\}}) + \mathbb{E}(|f(\widehat{S}(T))|^2 \mathbf{1}_{\{G\cap F^c\}}) \\
&\leq \mathbb{E}(|S(T) - \widehat{S}(T)|^2 \mathbf{1}_{\{F\cap G\}}) \\
&\quad + f_{max}^2\, \mathbb{P}(F\cap G^c) + f_{max}^2\, \mathbb{P}(G\cap F^c) \\
&\leq \mathbb{E}(|S(T) - \widehat{S}(T)|^2) + f_{max}^2 \left[\mathbb{P}(F\cap G^c) + \mathbb{P}(G\cap F^c)\right] \\
&\leq O(h) + f_{max}^2 [\mathbb{P}(F\cap G^c) + \mathbb{P}(G\cap F^c)]
\end{aligned}
$$

Barrier payoff

$\omega \in F \cap G^c$ requires

$\inf_t S_t \;\in\; [B, B{+}h^{1/2-\delta/2}]$ or

$\inf_t S_t - \inf_t \widehat{S}(t) > h^{1/2-\delta/2}$

Hence

$$\mathbb{P}(F \cap G^c) \;\; \leq \;\; \mathbb{P}\left(\inf_t S_t \in [B, B{+}h^{1/2-\delta/2}]\right) + \mathbb{P}\left(\inf_t S_t - \inf_t \widehat{S}(t) > h^{1/2-\delta/2}\right) \;\; \leq \;\; O(h^{1/2-\delta/2})$$

It's similar for $\mathbb{P}(F^c \cap G)$ so for any $\delta > 0$ we get

$$\mathbb{E}(|\widehat{P} - P|^2) = o(h^{1/2-\delta}).$$

Barrier payoff

What happends if $f(S)$ is not bounded?

There is a simpler analysis than given in the paper.

We know that $\mathbb{E}[|f(S_T)|^p],\ \mathbb{E}[|f(\widehat{S}(T))|^p]$ are bounded for all $p>2$. How do we use this?

Answer: Hölder inequality.

Barrier payoff

$$\begin{aligned}\mathbb{E}\left[|f(S_T)|^2 \mathbf{1}_{\{F\cap G^c\}}\right] &\leq \left(\mathbb{E}[|f(S_T)|^{2p}]\right)^{1/p} \left(\mathbb{E}[(\mathbf{1}_{\{F\cap G^c\}})^q]\right)^{1/q} \\ &\leq \left(\mathbb{E}[|f(S_T)|^{2p}]\right)^{1/p} \left(\mathbb{E}[\mathbf{1}_{\{F\cap G^c\}}]\right)^{1/q} \\ &\leq O(h^{(1/2-\delta/2)/q})\end{aligned}$$

where $1/p + 1/q = 1$.

Choose q very close to 1 so that $(1/2 - \delta/2)\,/\,q > 1/2 - \delta$ which needs $q < (1-\delta)/(1-2\,\delta)$ and then we get

$$\mathbb{E}\left[|f(S_T)|^2 \mathbf{1}_{\{F\cap G^c\}}\right] = o(h^{1/2-\delta})$$

as before, and we can do the same for $\mathbb{E}\left[|f(S_T)|^2 \mathbf{1}_{\{F^c\cap G\}}\right]$ to obtain $\mathbb{E}(|\widehat{P}-P|^2) = o(h^{1/2-\delta})$.