# Multilevel quasi-Monte Carlo path simulation

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# Outline

Long-term objective is faster Monte Carlo simulation of path dependent options to estimate values and Greeks.

Several ingredients, not yet all combined:

- multilevel method
- quasi-Monte Carlo
- adjoint pathwise Greeks
- parallel computing on NVIDIA graphics cards

Emphasis in this presentation is on multilevel method

# Generic Problem

Stochastic differential equation with general drift and volatility terms:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t)$$

We want to compute the expected value of an option dependent on $S(t)$. In the simplest case of European options, it is a function of the terminal state

$$P = f(S(T))$$

with a uniform Lipschitz bound,

$$|f(U) - f(V)| \;\leq c\,\|U - V\|\,, \quad \forall\, U, V.$$

# Simplest MC Approach

Euler discretisation with timestep $h$:

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n$$

Estimator for expected payoff is an average of $N$ independent path simulations:

$$\widehat{Y} = N^{-1} \sum_{i=1}^{N} f(\widehat{S}^{(i)}_{T/h})$$

- weak convergence – $O(h)$ error in expected payoff
- strong convergence – $O(h^{1/2})$ error in individual path

# Simplest MC Approach

Mean Square Error is $O\left(N^{-1}+h^2\right)$

- first term comes from variance of estimator
- second term comes from bias due to weak convergence

To make this $O(\varepsilon^2)$ requires

$$N=O(\varepsilon^{-2}), \quad h=O(\varepsilon) \quad \Longrightarrow \quad \text{cost}=O(N\,h^{-1})=O(\varepsilon^{-3})$$

Aim is to improve this cost to $O\left(\varepsilon^{-p}\right)$, with $p$ as small as possible, ideally close to 1.

Note: for a relative error of $\varepsilon=0.001$, the difference between $\varepsilon^{-3}$ and $\varepsilon^{-1}$ is huge.

# Standard MC Improvements

- variance reduction techniques (e.g. control variates, stratified sampling) improve the constant factor in front of $\varepsilon^{-3}$, sometimes spectacularly
- improved second order weak convergence (e.g. through Richardson extrapolation) leads to $h = O(\sqrt{\varepsilon})$, giving $p=2.5$
- quasi-Monte Carlo reduces the number of samples required, at best leading to $N \approx O(\varepsilon^{-1})$, giving $p \approx 2$ with first order weak methods

Multilevel method gives $p=2$ without QMC, and at best $p \approx 1$ with QMC.

# Other Related Research

- In Dec. 2005, Ahmed Kebaier published an article in *Annals of Applied Probability* describing a two-level method which reduces the cost to $O\left(\varepsilon^{-2.5}\right)$.
- Also in Dec. 2005, Adam Speight wrote a working paper describing a very similar multilevel use of control variates.
- There are also close similarities to a multilevel technique developed by Stefan Heinrich for parametric integration (*Journal of Complexity*, 1998)

# Multilevel MC Approach

Consider multiple sets of simulations with different timesteps $h_l = 2^{-l}\, T,\ l=0,1,\ldots,L$, and payoff $\widehat{P}_l$

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{l=1}^{L} \mathbb{E}[\widehat{P}_l - \widehat{P}_{l-1}]$$

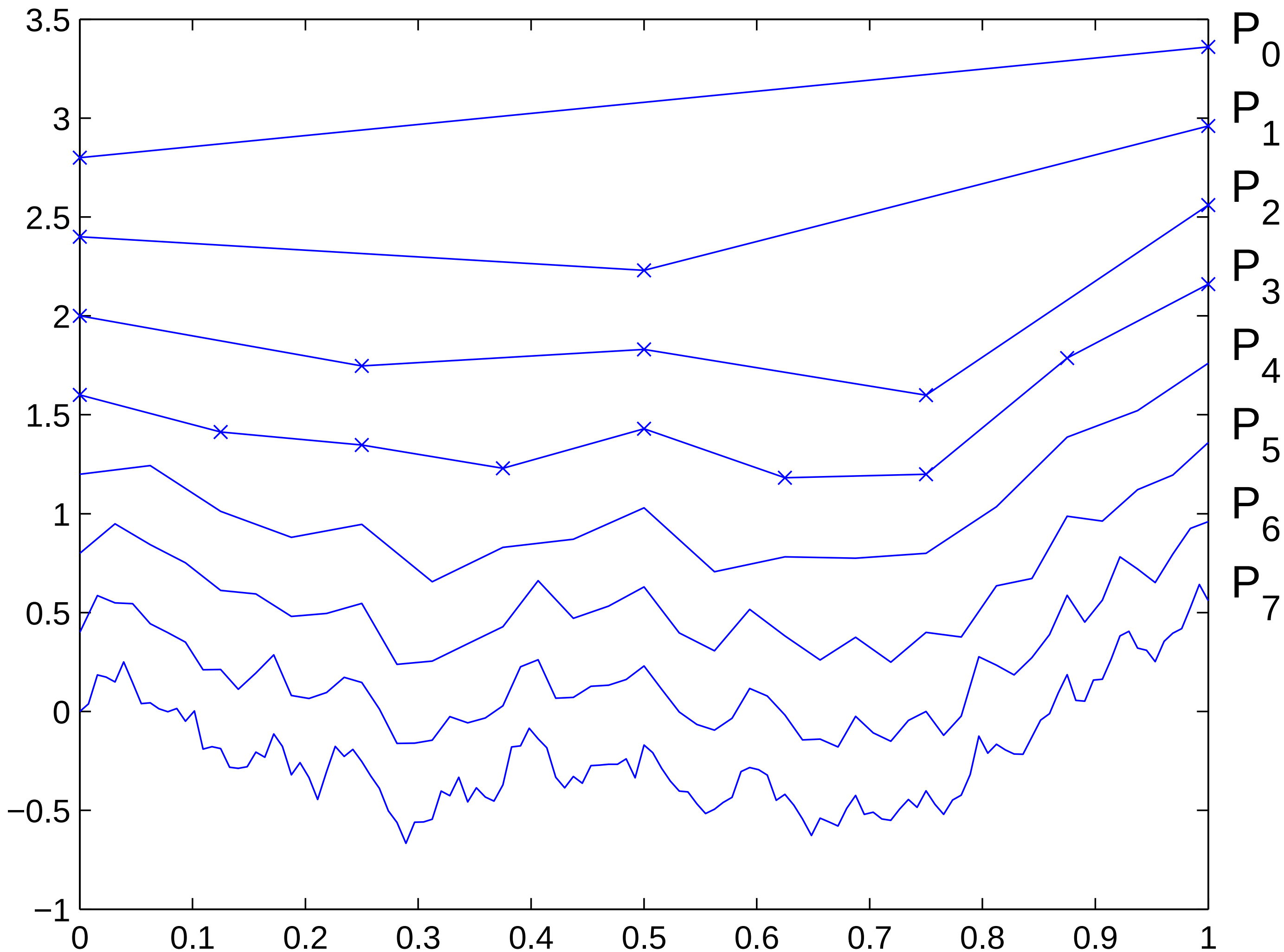
Expected value is same – aim is to reduce variance of estimator for a fixed computational cost.

Key point: approximate $\mathbb{E}[\widehat{P}_l - \widehat{P}_{l-1}]$ using $N_l$ simulations with $\widehat{P}_l$ and $\widehat{P}_{l-1}$ obtained using <u>same</u> Brownian path.

$$\widehat{Y}_l = N_l^{-1} \sum_{i=1}^{N_l} \left( \widehat{P}_l^{(i)} - \widehat{P}_{l-1}^{(i)} \right)$$

# Multilevel MC Approach

Discrete Brownian path at different levels



# Multilevel MC Approach

Using independent paths for each level, the variance of the combined estimator is

$$\mathbb{V}\left[\sum_{l=0}^{L} \widehat{Y}_l\right] = \sum_{l=0}^{L} N_l^{-1} V_l, \qquad V_l \equiv \mathbb{V}[\widehat{P}_l - \widehat{P}_{l-1}],$$

and the computational cost is proportional to $\displaystyle\sum_{l=0}^{L} N_l \, h_l^{-1}$.

Hence, the variance is minimised for a fixed computational cost by choosing $N_l$ to be proportional to $\sqrt{V_l \, h_l}$.

The constant of proportionality can be chosen so that the combined variance is $O(\varepsilon^2)$.

# Multilevel MC Approach

For the Euler discretisation and a Lipschitz payoff function

$$\mathbb{V}[\widehat{P}_l - P] = O(h_l) \quad \Longrightarrow \quad \mathbb{V}[\widehat{P}_l - \widehat{P}_{l-1}] = O(h_l)$$

and the optimal $N_l$ is asymptotically proportional to $h_l$.

To make the combined variance $O(\varepsilon^2)$ requires

$$N_l = O(\varepsilon^{-2} L\, h_l).$$

To make the bias $O(\varepsilon)$ requires

$$L = \log_2 \varepsilon^{-1} + O(1) \quad \Longrightarrow \quad h_L = O(\varepsilon).$$

Hence, we obtain an $O(\varepsilon^2)$ MSE for a computational cost which is $O(\varepsilon^{-2} L^2) = O(\varepsilon^{-2} (\log \varepsilon)^2)$.

# Multilevel MC Approach

**Theorem:** *Let $P$ be a functional of the solution of a stochastic o.d.e., and $\widehat{P}_l$ the discrete approximation using a timestep $h_l = M^{-l}\, T$.*

**If** *there exist independent estimators $\widehat{Y}_l$ based on $N_l$ Monte Carlo samples, and positive constants $\alpha \geq \frac{1}{2}, \beta, c_1, c_2, c_3$ such that*

*i)* $\mathbb{E}[\widehat{P}_l - P] \leq c_1\, h_l^\alpha$

*ii)* $\mathbb{E}[\widehat{Y}_l] = \begin{cases} \mathbb{E}[\widehat{P}_0], & l = 0 \\ \mathbb{E}[\widehat{P}_l - \widehat{P}_{l-1}], & l > 0 \end{cases}$

*iii)* $\mathbb{V}[\widehat{Y}_l] \leq c_2\, N_l^{-1} h_l^\beta$

*iv)* $C_l$*, the computational complexity of* $\widehat{Y}_l$*, is bounded by*

$$C_l \leq c_3\, N_l\, h_l^{-1}$$

# Multilevel MC Approach

**then** *there exists a positive constant* $c_4$ *such that for any* $\varepsilon \! < \! e^{-1}$ *there are values* $L$ *and* $N_l$ *for which the multi-level estimator*

$$\widehat{Y} = \sum_{l=0}^{L} \widehat{Y}_l,$$

*has Mean Square Error* $MSE \equiv \mathbb{E}\left[ \left( \widehat{Y} - \mathbb{E}[P] \right)^2 \right] < \varepsilon^2$

*with a computational complexity* $C$ *with bound*

$$C \leq \begin{cases} c_4 \, \varepsilon^{-2}, & \beta > 1, \\ \\ c_4 \, \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ \\ c_4 \, \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

# Milstein Scheme

The theorem suggests use of Milstein scheme —
better strong convergence, same weak convergence

Generic scalar SDE:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t), \quad 0<t<T.$$

Milstein scheme:

$$\widehat{S}_{n+1} = \widehat{S}_n + a\,h + b\,\Delta W_n + \tfrac{1}{2}\,b'\,b\left((\Delta W_n)^2 - h\right).$$

# Milstein Scheme

In scalar case:

- $O(h)$ strong convergence
- $O(\varepsilon^{-2})$ complexity for Lipschitz payoffs – trivial
- $O(\varepsilon^{-2})$ complexity for Asian, lookback, barrier and digital options using carefully constructed estimators based on Brownian interpolation or extrapolation

# Milstein Scheme

Brownian interpolation: within each timestep, model the behaviour as simple Brownian motion conditional on the two end-points

$$\begin{aligned}\widehat{S}(t) &= \widehat{S}_n + \lambda(t)(\widehat{S}_{n+1} - \widehat{S}_n) \\ &\quad + b_n \Big(W(t) - W_n - \lambda(t)(W_{n+1} - W_n)\Big),\end{aligned}$$

where

$$\lambda(t) = \frac{t - t_n}{t_{n+1} - t_n}$$

There then exist analytic results for the distribution of the min/max/average over each timestep.

# Milstein Scheme

Brownian extrapolation for final timestep:

$$\widehat{S}_N = \widehat{S}_{N-1} + a_{N-1} h + b_{N-1} \Delta W_N$$

– considering all possible $\Delta W_N$ gives Gaussian distribution, for which a digital option has a known conditional expectation (Glasserman)

This payoff smoothing can be generalised to multivariate cases, and leads to a “vibrato” Monte Carlo technique which is suitable for both efficient multilevel analysis and the computation of Greeks

# Results

Geometric Brownian motion:

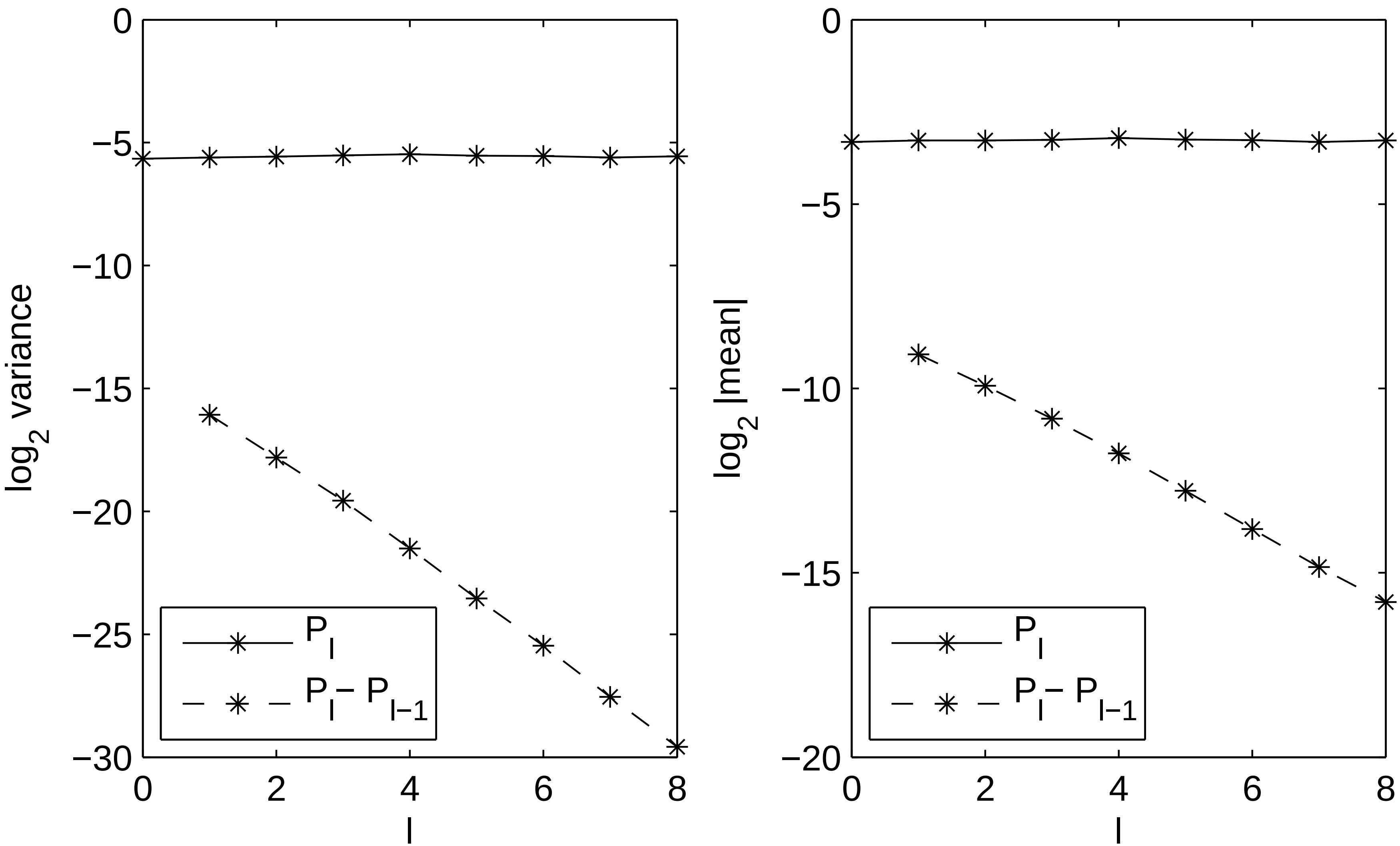
$$\mathrm{d}S = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W, \qquad 0 < t < T,$$

with parameters $T\!=\!1, \;\; S(0)\!=\!1, \;\; r\!=\!0.05, \;\; \sigma\!=\!0.2$

- European call option: $\exp(-rT)\ \max(S(T) - 1, 0)$
- European digital call: $\exp(-rT)\,\mathbf{1}_{S(T)>1}$
- Down-and-out barrier option: same as call provided $S(t)$ stays above $B\!=\!0.9$

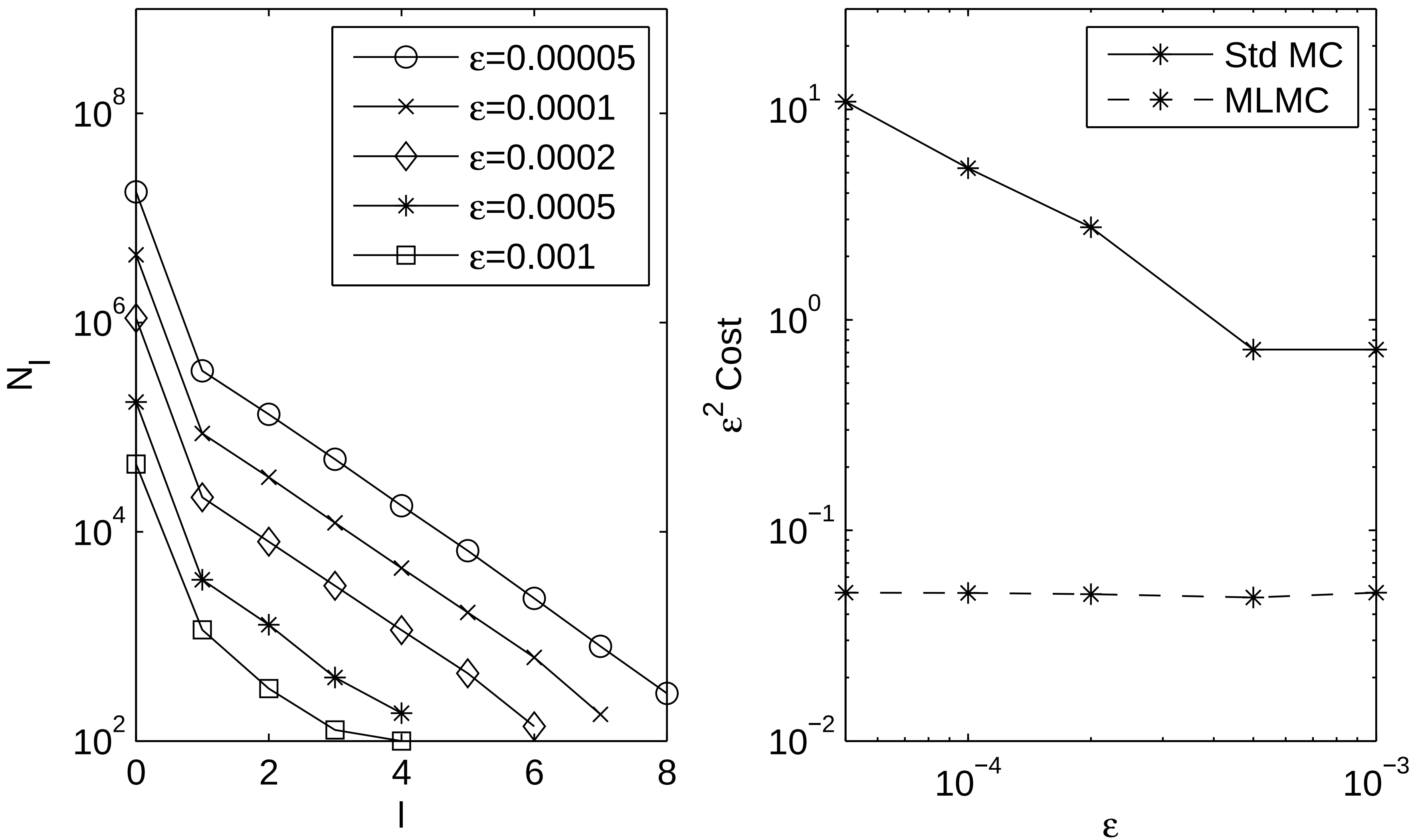
# MLMC Results

GBM: European call



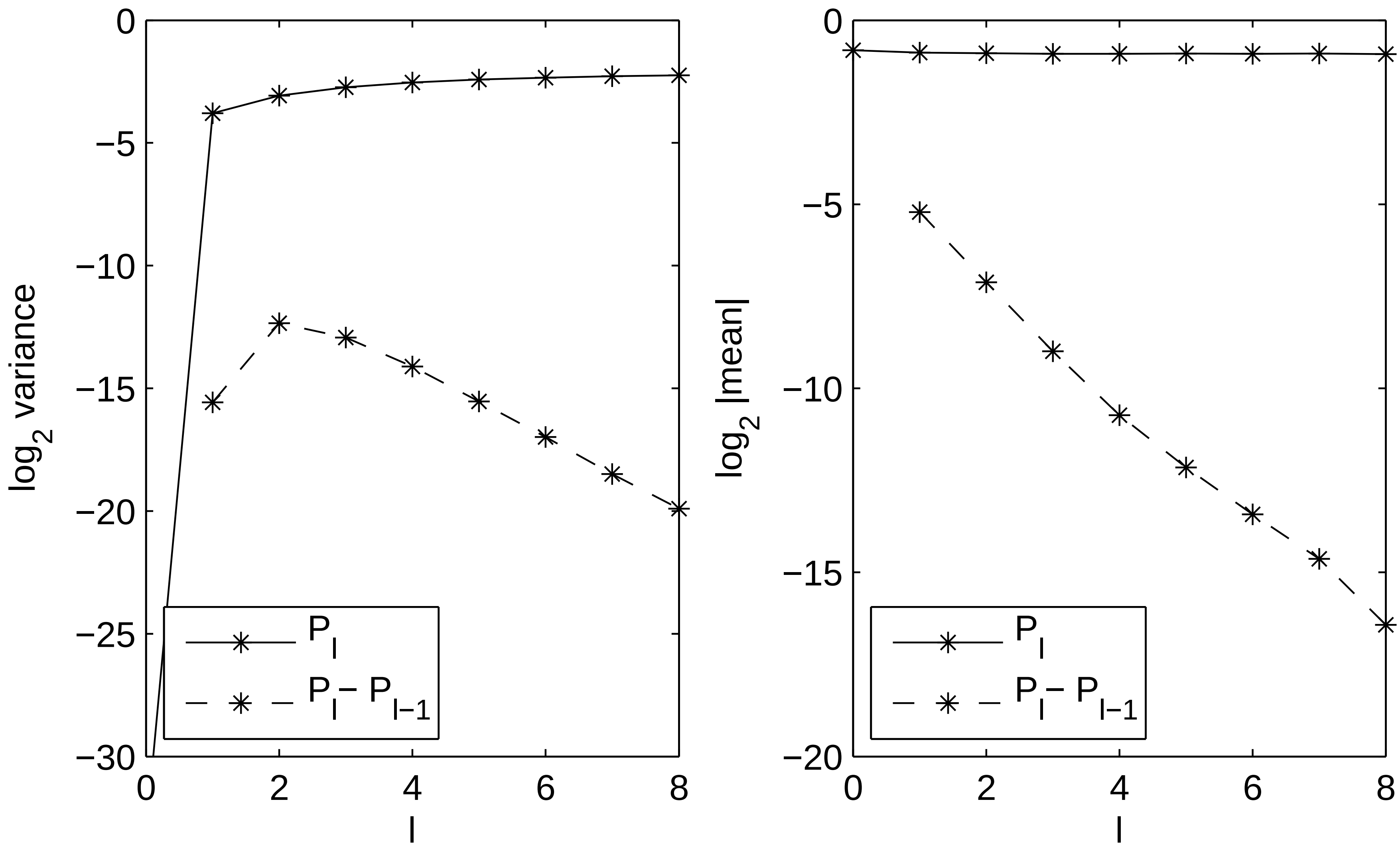
# MLMC Results

GBM: European call



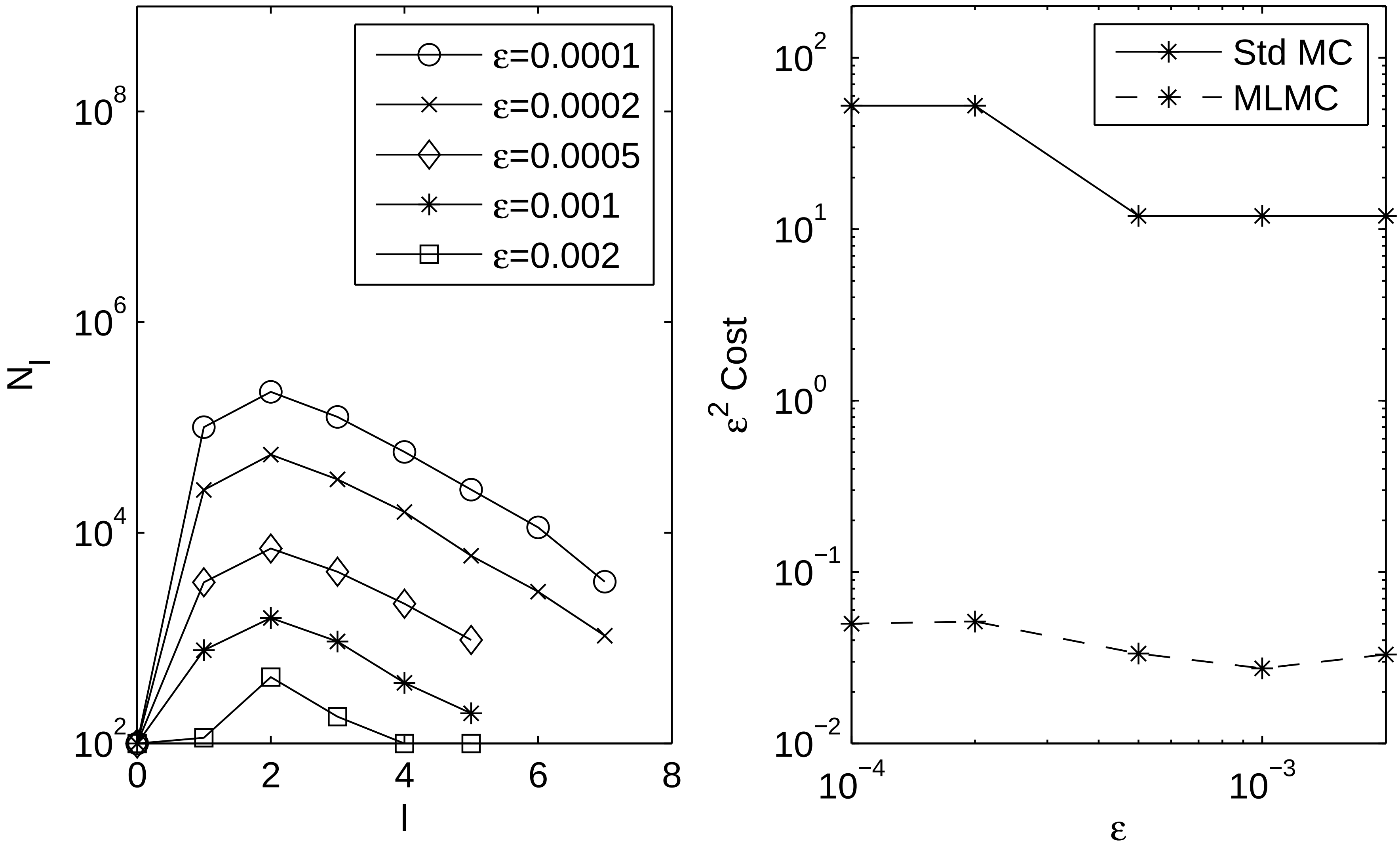
# MLMC Results

GBM: digital call



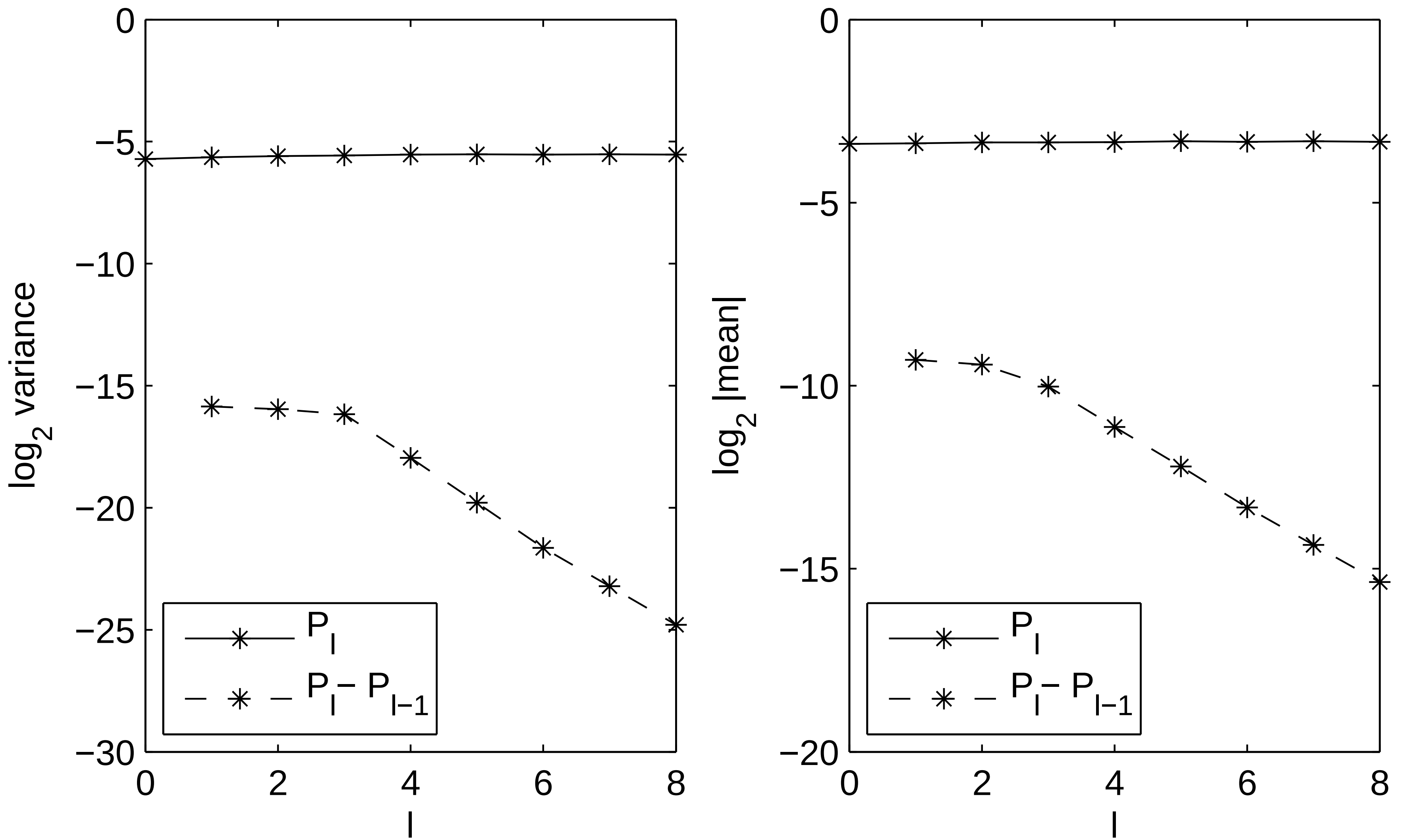
# MLMC Results

GBM: digital call



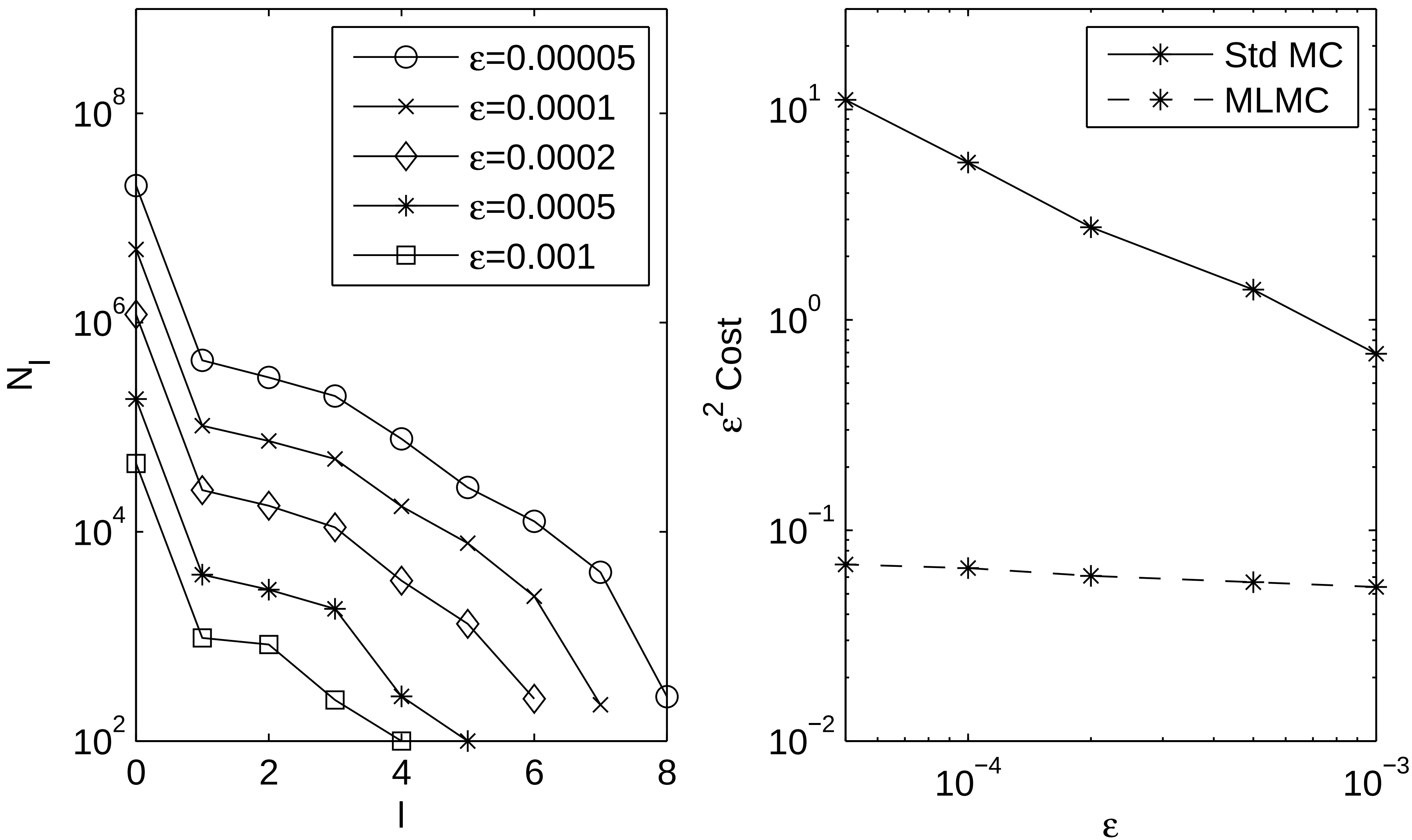
# MLMC Results

GBM: barrier option



# MLMC Results

GBM: barrier option



# Milstein Scheme

Generic vector SDE:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t), \quad 0<t<T,$$

with correlation matrix $\Omega(S,t)$ between elements of $\mathrm{d}W(t)$.

Milstein scheme:

$$\begin{aligned}
\widehat{S}_{i,n+1} &= \widehat{S}_{i,n} + a_i\,h + b_{ij}\,\Delta W_{j,n} \\
&\quad + \tfrac{1}{2}\,\frac{\partial b_{ij}}{\partial S_l}\,b_{lk}\left(\Delta W_{j,n}\,\Delta W_{k,n} - h\,\Omega_{jk} - A_{jk,n}\right)
\end{aligned}$$

with implied summation, and Lévy areas defined as

$$A_{jk,n} = \int_{t_n}^{t_{n+1}} (W_j(t)-W_j(t_n))\,\mathrm{d}W_k - (W_k(t)-W_k(t_n))\,\mathrm{d}W_j.$$

# Milstein Scheme

In vector case:

- $O(h)$ strong convergence if Lévy areas are simulated correctly – expensive
- $O(h^{1/2})$ strong convergence in general if Lévy areas are omitted, except if a certain commutativity condition is satisfied (useful for a number of real cases)
- Lipschitz payoffs can be handled well using antithetic variables
- Other cases may require approximate simulation of Lévy areas

# Results

Heston model:

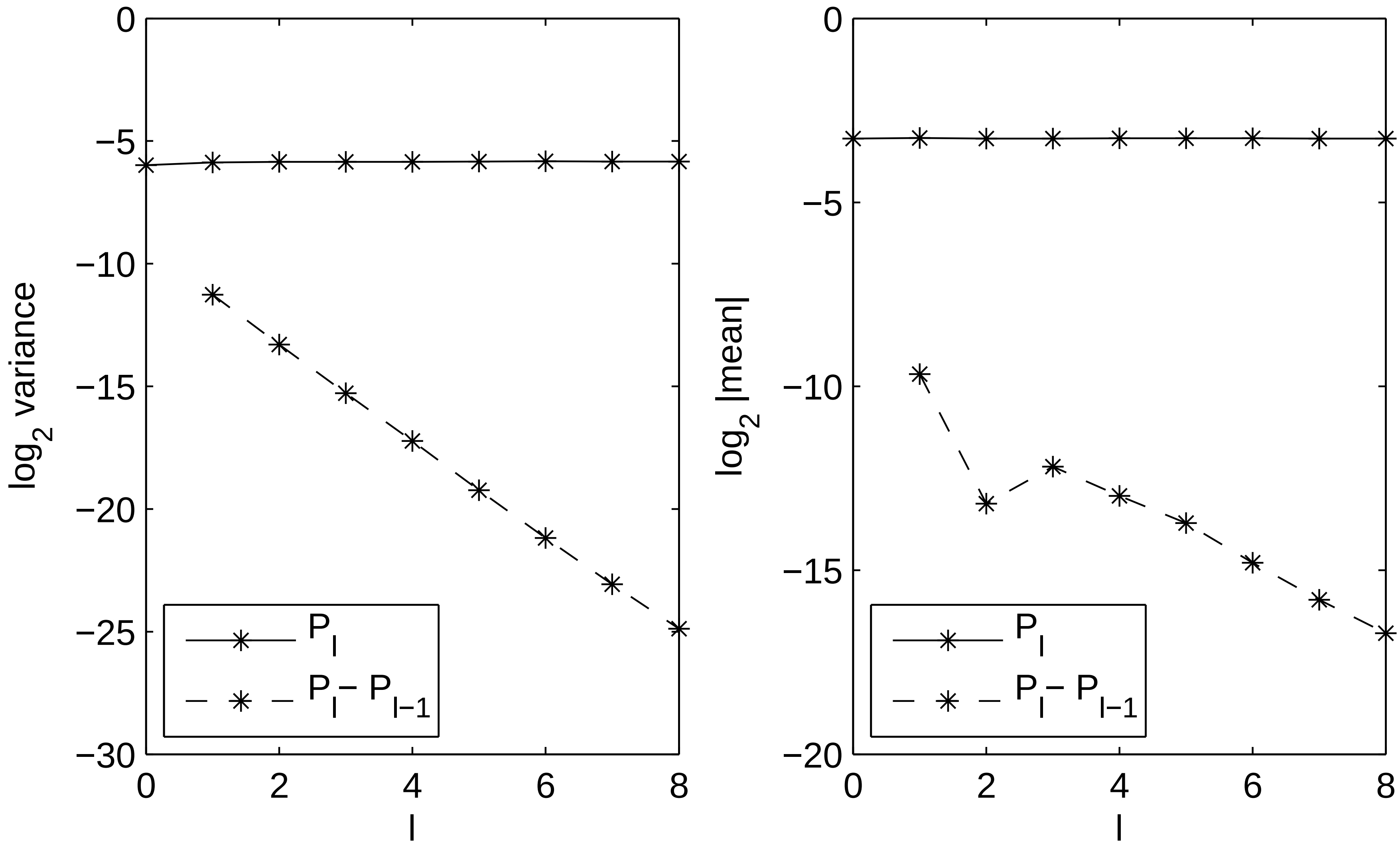
$$
\begin{aligned}
\mathrm{d}S &= r\,S\,\mathrm{d}t + \sqrt{V}\,S\,\mathrm{d}W_1, \qquad\quad 0<t<T \\
\mathrm{d}V &= \lambda\,(\sigma^2 - V)\,\mathrm{d}t + \xi\,\sqrt{V}\;\mathrm{d}W_2,
\end{aligned}
$$

$T=1,\ \ S(0)=1,\ \ V(0)=0.04,\ \ r=0.05,$

$\sigma=0.2,\ \ \lambda=5,\ \ \xi=0.25,\ \ \rho=-0.5$

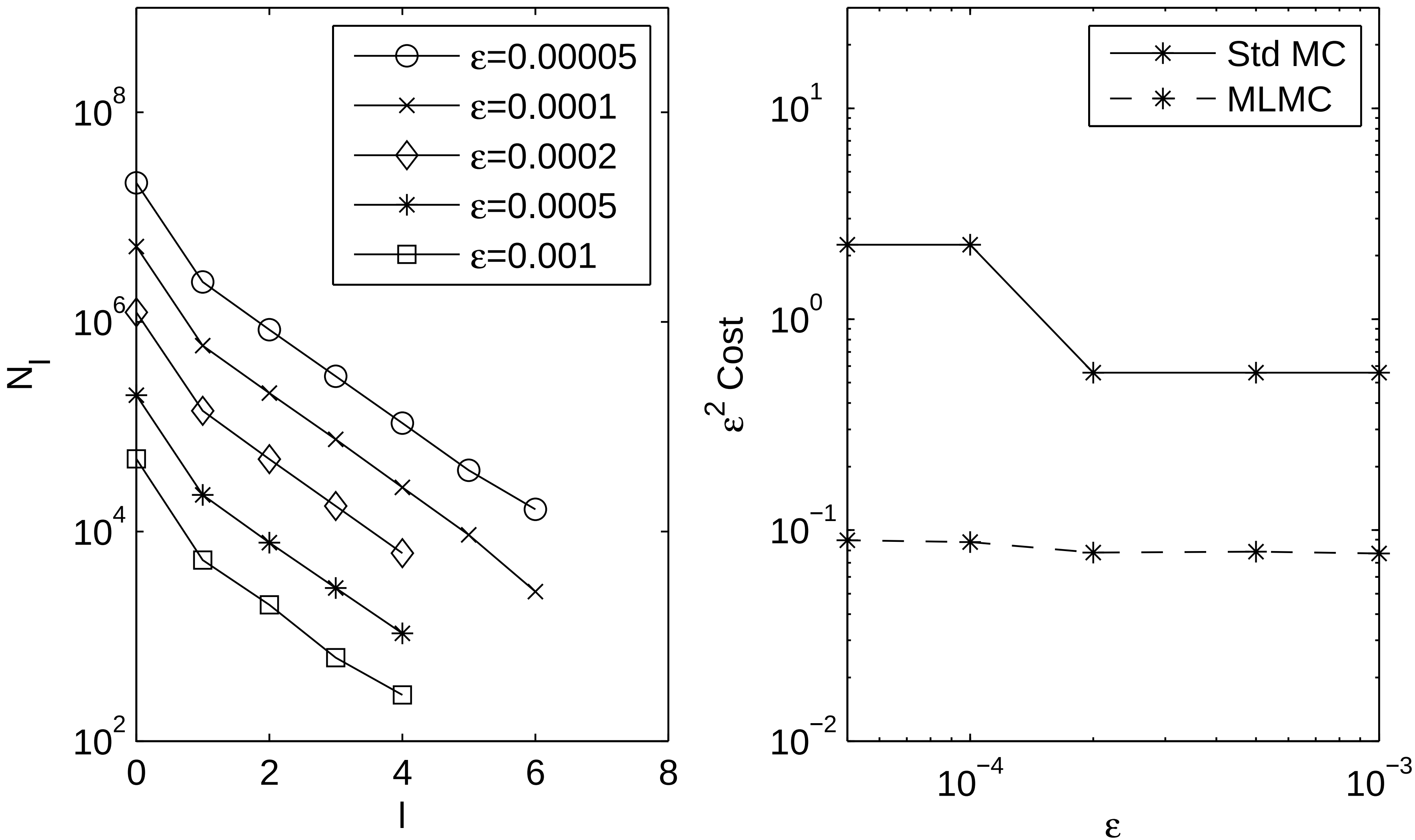
# MLMC Results

Heston model: European call



# MLMC Results

Heston model: European call



# Quasi-Monte Carlo

- well-established technique for approximating high-dimensional integrals
- for finance applications see papers by l'Ecuyer and book by Glasserman
- Sobol sequences are perhaps most popular; we use lattice rules (Sloan & Kuo)
- two important ingredients for success:
  - randomized QMC for confidence intervals
  - good identification of "dominant dimensions" (Brownian Bridge and/or PCA)

# Quasi-Monte Carlo

Approximate high-dimensional hypercube integral

$$\int_{[0,1]^d} f(x)\ \mathrm{d}x$$

by

$$\frac{1}{N} \sum_{i=0}^{N-1} f(x^{(i)})$$

where

$$x^{(i)} = \left[ \frac{i}{N}\, z \right]$$

and $z$ is a $d$-dimensional “generating vector”.

# Quasi-Monte Carlo

In the best cases, error is $O(N^{-1})$ instead of $O(N^{-1/2})$ but without a confidence interval.

To get a confidence interval, let

$$x^{(i)} = \left[ \frac{i}{N} z \; + \; x_0 \right].$$

where $x_0$ is a random offset vector.

Using 32 different random offsets gives a confidence interval in the usual way.

# Quasi-Monte Carlo

For the path discretisation we can use

$$\Delta W_n = \sqrt{h}\, \Phi^{-1}(x_n),$$

where $\Phi^{-1}$ is the inverse cumulative Normal distribution.

Much better to use a Brownian Bridge construction:

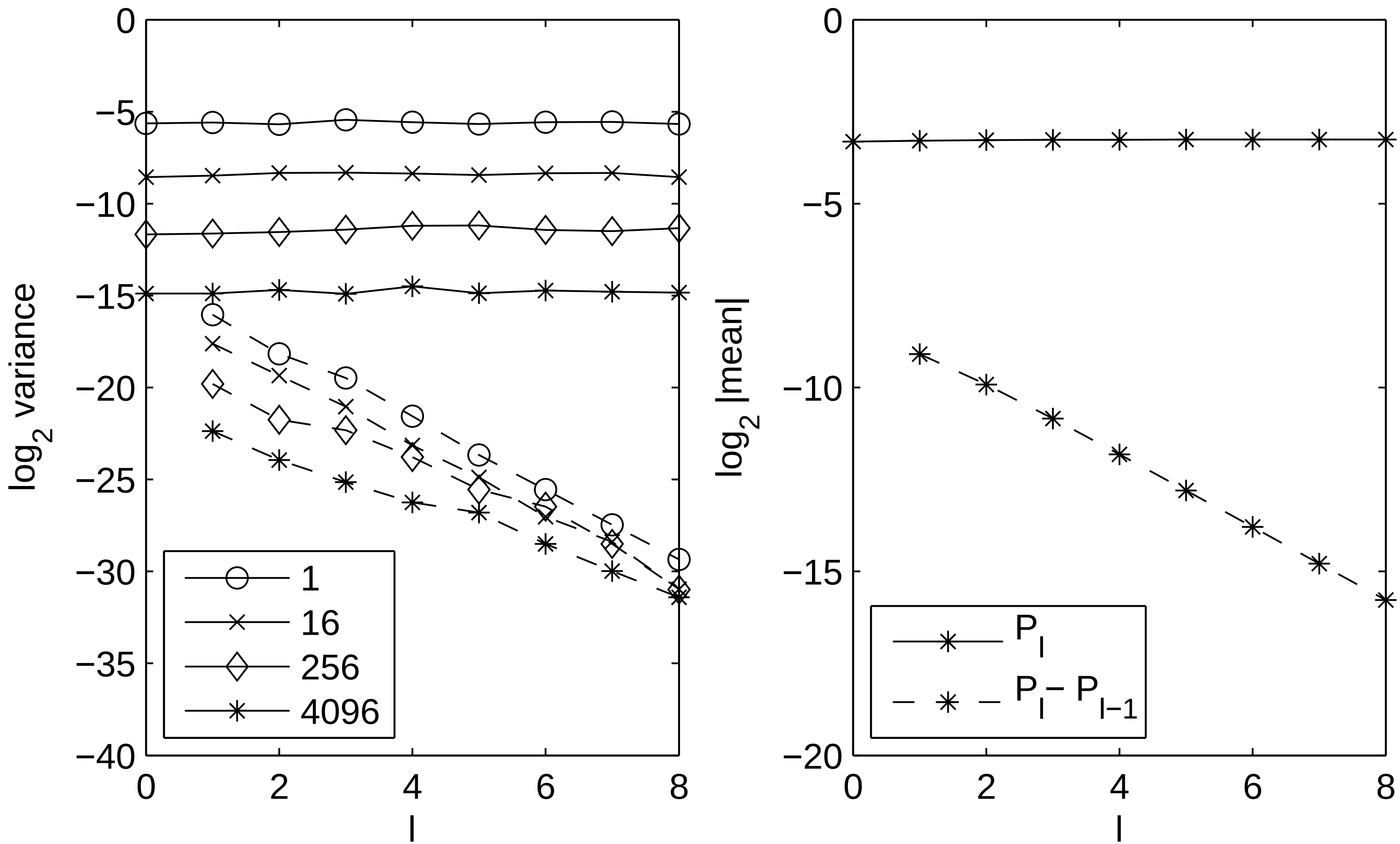
- $x_1 \quad \longrightarrow \quad W(T)$
- $x_2 \quad \longrightarrow \quad W(T/2)$
- $x_3, x_4 \quad \longrightarrow \quad W(T/4), W(3T/4)$
- $\ldots$ and so on by recursive bisection

# Multilevel QMC

- rank-1 lattice rule developed by Sloan, Kuo & Waterhouse at UNSW
- 32 randomly-shifted sets of QMC points
- number of points in each set increased as needed to achieved desired accuracy, based on confidence interval estimate
- results show QMC to be particularly effective on lowest levels with low dimensionality

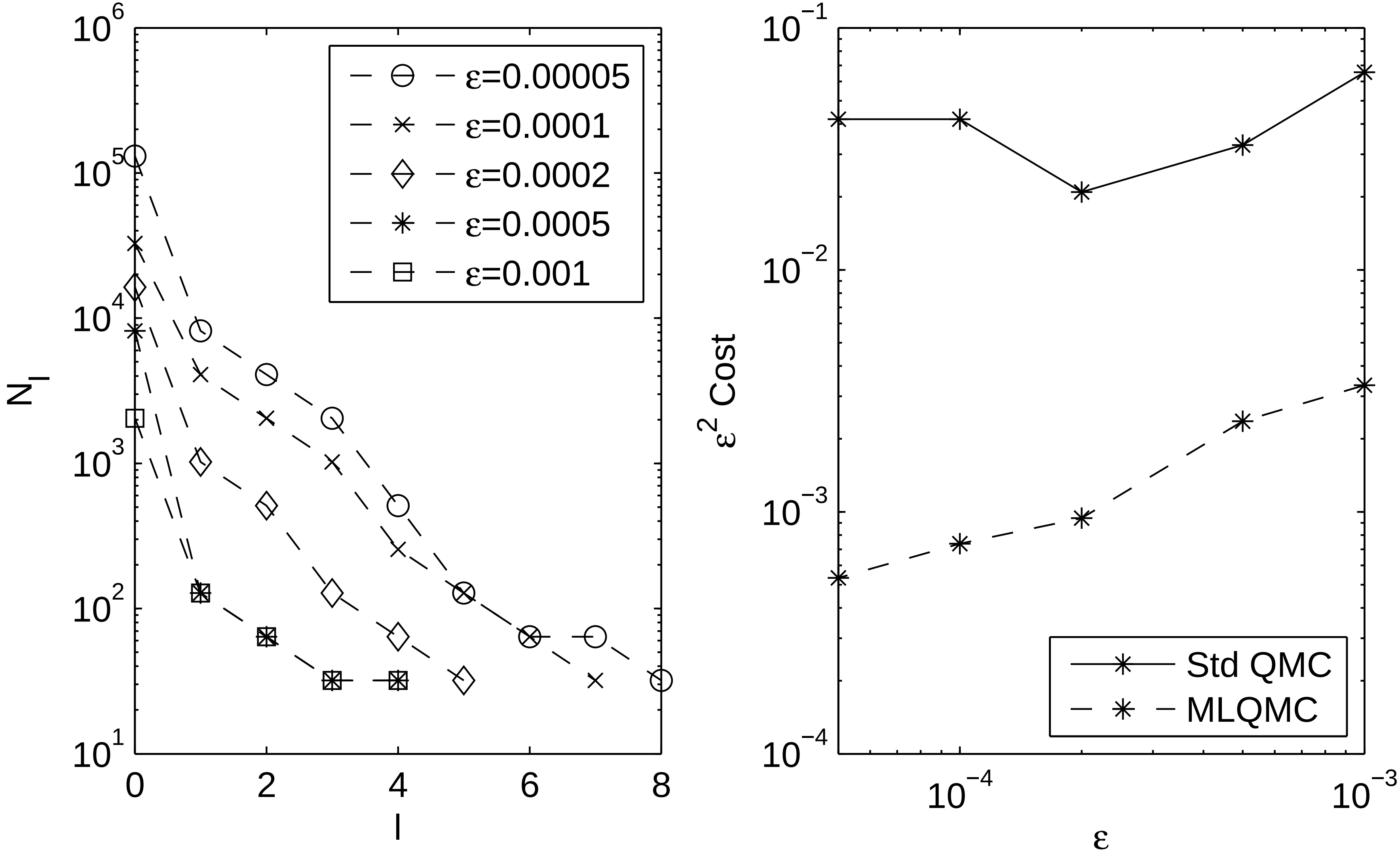
# MLQMC Results

GBM: European call



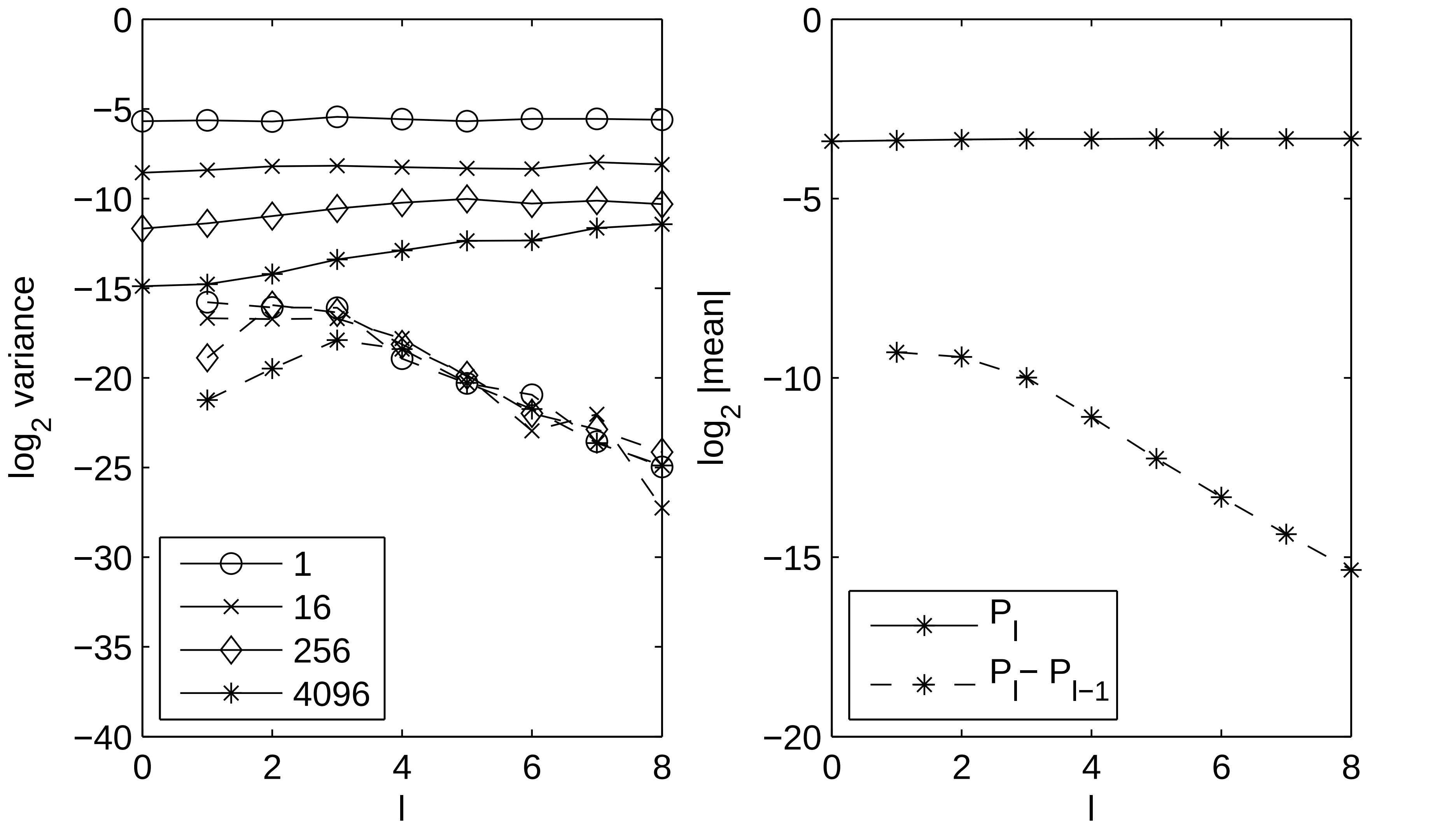
# MLQMC Results

GBM: European call



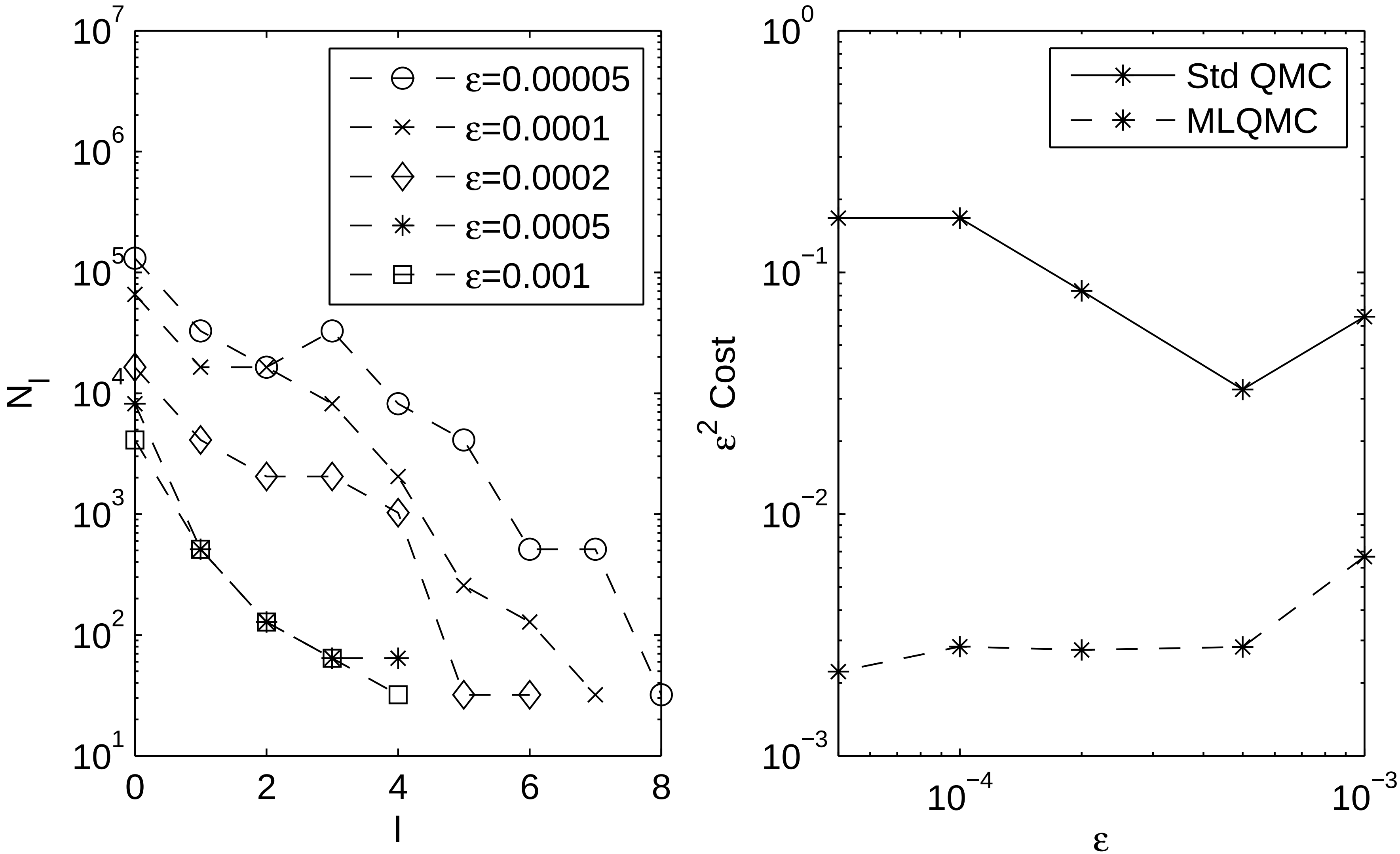
# MLQMC Results

GBM: barrier option



# MLQMC Results

GBM: barrier option



# Conclusions

Results so far:

- much improved order of complexity
- fairly easy to implement
- significant benefits for model problems

However:

- lots of scope for further development
  - multi-dimensional SDEs needing Lévy areas
  - adjoint Greeks and “vibrato” Monte Carlo
  - numerical analysis of algorithms
  - execution on NVIDIA graphics cards (128 cores)
- need to test ideas on real finance applications

# Papers

M.B. Giles, “Multilevel Monte Carlo path simulation”, to appear in *Operations Research*, 2007.

M.B. Giles, “Improved multilevel convergence using the Milstein scheme”, to appear in *MCQMC06* proceedings, Springer-Verlag, 2007.

M.B. Giles, “Multilevel quasi-Monte Carlo path simulation”, submitted to Journal of Computational Finance, 2007.

`www.comlab.ox.ac.uk/mike.giles/finance.html`

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