INFERRING FILTRATION LAWS FROM THE SPREADING OF A LIQUID MODELLED BY THE POROUS MEDIUM EQUATION

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Abstract. Motivated by modelling the spreading of a two-dimensional particle-laden gravity current on a porous membrane, we couple the porous medium equation with drainage with a blocking law for the pores of the membrane. The blocking law characterises a range of blocking phenomena through the choice of a single *blocking parameter* $\alpha \in [-\infty, \infty]$. We pose the question whether the value of the blocking parameter, and hence the blocking law, can be inferred by observing the position of the front of the current over time. We use two different strategies to determine the blocking parameter for almost the entire possible range of values. First, we show that the position of the front follows a power law when the current is fed by a constant influx at the center and that the exponent of the power law is unique for sufficiently large blocking parameters. Second, we show that the coupled system of the porous medium equation with absorption and the blocking law allows for a travelling wave solution if a suitable influx, dependent on the blocking parameter, is applied. For $\alpha < 1$, we show that the suitable influx is, to leading order, independent of α and that the corresponding travelling wave has a finite region in which fluid drains at the front and whose width can be used to infer the blocking parameter.

Key words. inverse problem, membrane filtration, nonlinear diffusion, porous medium equation, travelling wave solution

AMS subject classifications. 34A34, 35B40, 35G20, 35R30, 76S05

1. Introduction.

Consider a two-dimensional gravity current, formed by a fluid consisting of a liquid and small, neutrally buoyant particles, spreading on a prewetted, thin porous membrane (see Figure 1.1). As the fluid drains through the membrane, the particles are retained and thus reduce the permeability of the membrane. The reduction in permeability, which is quantified by a corresponding *filtration law*, will in turn affect the spreading rate of the gravity current. The question addressed in this paper is: Can we infer the underlying filtration law by observing the position of the front of the gravity current over time?

Determining filtration laws is an important problem, as they provide insight into how membranes block, or *foul*. This is of practical importance as membranes are used in a variety of industrial processes, including biotechnology [2], desalination [9], food processing [10], or oil removal [3], and understanding the fouling mechanism can help to improve the performance of the membrane. In this paper we show that, for a specific class of filtration laws, the filtration law can be inferred from observations of the speed of the front of the gravity current.

To model the gravity current, we neglect the effects of surface tension and assume that the particles are uniformly distributed throughout the liquid and that the particles do not change the rheology of the fluid. We take the dimensionless height of the fluid to be h and the dimensionless permeability of the membrane to be λ so that the gravity current then satisfies the *porous medium equation* with drainage,

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^n \frac{\partial h}{\partial x}\right) - \lambda h, \tag{1.1}$$

where $n = 3$ corresponds to the classical case of a gravity current in air [7]; see Appendix A for remarks on the nondimensionalisation. Here, the variable representing

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permeability is restricted to $\lambda \in [0,1]$ with $\lambda = 1$ being the fresh membrane. The term λh represents the flow through the membrane driven by a pressure difference proportional to h. The porous medium equation with drainage has been presented as a model for gravity currents in layered porous media [12] or for gravity currents in air on a permeable substrate [17], for the healing of wounds [15], and for population dynamics, where more general absorption terms ($-h^{1-q}$ for $q > 0$ instead of $-\lambda h$) are being considered [4].

The reduction in permeability due to particle retention depends on properties of both the membrane and the particles [13]. In this paper we will assume that $\lambda(x,t)$ satisfies

$$\frac{\partial \lambda}{\partial t} = -\lambda^{\alpha} h, \tag{1.2}$$

where we assume that initially no particles have been retained by the membrane and so $\lambda(x,0) = 1 \ \forall \ x \in \mathbb{R}$.

Equation (1.2) includes the behaviour of four classes of laws known as Hermia's filtration laws [5], where the different retention mechanisms correspond to different exponents $\alpha \in \{1, 1/2, 2, 3\}$, see Appendix B for a derivation. While Hermia's four filtration laws are not always appropriate to describe the observed reduction of permeability, they are sufficient in instances where only one retention mechanism dominates (see [8] for an overview). We may thus think of a filtration law as being characterised by the exponent α, and so inferring the filtration law reduces to the problem of determining α from observations of the position of the gravity current front defined as $x_f(t) := \min\{h(x) = 0 : x \geq 0\}$ (see Figure 1.1).

We assume that initially there is no liquid on the membrane, hence we obtain the initial condition $h(x,0) = 0 \ \forall x \in \mathbb{R}$. Further we assume that the gravity current has an influx from a point source of strength $2I(t)$ at the central point, $x = 0$. As (1.1) and the initial and influx conditions are symmetric in x, it suffices to consider the problem only for $x \geq 0$. We thus apply the boundary conditions

$$h^n \frac{\partial h}{\partial x} = -I(t) \text{ at } x = 0, \text{ as well as } h = 0 \text{ and } h^{n-1}\frac{\partial h}{\partial x} = \frac{\mathrm{d}x_f}{\mathrm{d}t} \text{ at } x = x_f(t), \tag{1.3}$$

where the conditions at the front state that the height of the current at the front must be zero and that the average speed of the current at the front is the speed of the front. The last two boundary conditions in (1.3) also imply $h^n \partial h/\partial x = 0$ at $x = x_f$.

In this paper, we will consider the problem (1.1)–(1.3) for the following two generalisations. First, we will consider a larger class of filtration laws than Hermia by allowing $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$. We identify the cases $\alpha = -\infty$ with $\lambda \equiv 0$, which corresponds to spreading on an impermeable surface, and $\alpha = \infty$ with $\lambda \equiv 1$, which corresponds to spreading on a porous substrate with uniform, constant permeability. Second, instead of only considering the porous medium equation with drainage (1.1) for $n = 3$, we will consider any $n \geq 1$. This generalisation is easily seen to be of physical relevance, as the simplest case of the porous medium equation without absorption ($\lambda \equiv 0$) can be derived by considering the flow of an ideal gas in a homogeneous porous medium, with exponents $n \geq 1$ arising from the constitutive relationship between the density and the pressure of the gas [1].

Our approach to inferring different filtration laws is based on investigating solutions of (1.1) and (1.2). For any particular physical problem such as a gravity current in air on a porous membrane ($n = 3$), the motion of the liquid may be influenced by additional physical effects, such as surface tension. However, here we choose not

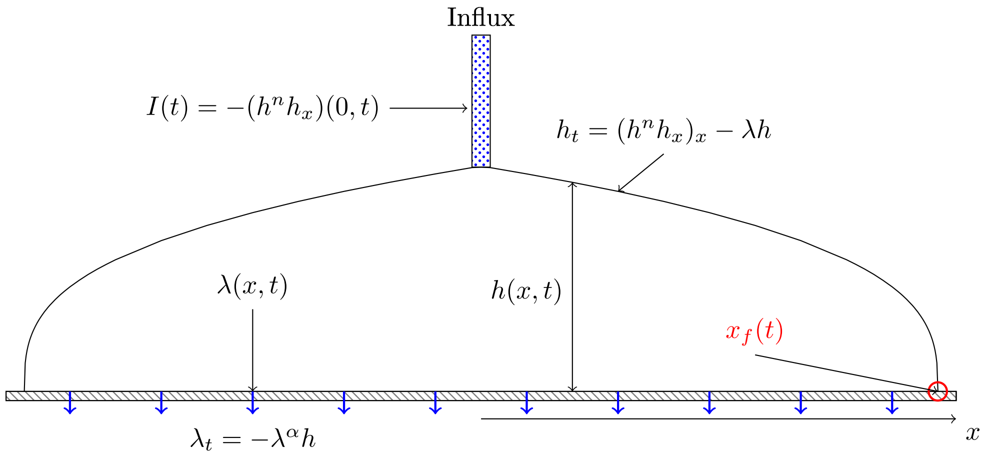


Fig. 1.1: Visualisation of a solution to (1.1) coupled with (1.2). The solution (h, λ) is symmetric in x and so we only consider the system for positive x. We are interested in inferring α from observations of $x_f(t)$.

to include these additional effects as we expect them to be negligible in the solution regimes we are interested in for inferring the filtration law.

We note that the Cauchy problem of the porous medium equation has been extensively studied [16]. We further note that versions of the porous medium equation with $n < 1$ have been used, for example to model cell population dynamics [11]. However, we require h to be concave and thus only consider $n \geq 1$. Also, while only $n = 3$ and $n = 1$ [6] correspond to gravity currents in the classical sense, we will use the terms *height* and *gravity current* for any n.

In this paper we show, based on scaling arguments and numerical experiments, how α can be determined from observations of the position of the front of the gravity current over time if $\alpha > 2 + 1/(n + 2)$ or $\alpha < 1$. In Section 2 we show that, for a constant influx $I(t) = 1$, the position of the front of the gravity current follows a power law whose exponent is unique for $\alpha > 2+1/(n+2)$. In Section 3, we show that (1.1), (1.2) and (1.3) allows for a travelling wave solution for a suitable influx $I(t)$, which depends on α. For $\alpha < 1$, there exists a second interface, the *drainage front* $x_d(t)$ where the fluid is no longer draining through the membrane, that is, $\lambda(x, t) = 0$ for $x \leq x_d(t)$ and $\lambda(x, t) > 0$ for $x > x_d(t)$. The distance between the fluid front and the drainage front position is unique for a given α and n, thus allowing for inference of α.

2. Power-law spreading for constant influx. For constant influx, $I(t) = 1$, we will first consider the two extremal cases $\alpha = -\infty, \infty$, to give upper and lower asymptotic bounds on the speed of the gravity current. Motivated by direct numerical simulations, we assume for general $\alpha \in \mathbb{R}$ that both the position of the front and the height at the origin follow a power law, that is, $x_f(t) \sim t^l$ and $h(0, t) \sim t^k$ for some parameters $l = l(\alpha)$, $k = k(\alpha)$. Based on scaling arguments and using numerical validation, we will then show that l is unique for $\alpha > 2 + 1/(n + 2)$. Thus, this allows us to infer the filtration law from observations of the position of the front over time in these instances. For $\alpha \leq 2 + 1/(n + 2)$, l will only depend on n and so inferring the filtration law will not be possible in these instances.

We assume that solutions to (1.1) and (1.2) are monotonically increasing and concave in both x and t for h, which we observe in numerical results for $n \geq 1$. This assumption is necessary for proving the inequalities for the special cases $\alpha \in \{-\infty, \infty\}$, and to estimate the volume of the current in Section 2.2. We also assume that h is monotonic in α, that is, if h_1 and h_2 correspond to solutions for α_1 and α_2 and $\alpha_1 < \alpha_2$ then $h_1 \geq h_2$. This assumption will be used in Section 2.2 to determine the largest exponent α for which the current spreads asymptotically as if there was no drainage.

2.1. The special cases $\alpha \in \{-\infty, \infty\}$. We identify the special case $\alpha = -\infty$ with $\lambda \equiv 0$ and the special case $\alpha = \infty$ with $\lambda \equiv 1$. This is motivated by considering the pointwise limit of (1.2). Let x, T be such that $h(x, T) = 0$ and $h(x, t) > 0$ for $t > T$. Considering the timeframe $t - T$, we can then assume w.l.o.g. that $h(x, 0) = 0$ and $h(x, t) > 0$ for $t > 0$. For the case of $1 < \alpha \to \infty$, let us fix some $t_0 > 0$ in this new timeframe. Because we assume $h(x, \cdot)$ to be monotonically increasing, we obtain

$$\begin{aligned} \lambda(x, t_0) &= \left(1 + (\alpha - 1) \int_0^{t_0} h(x, s)\ \mathrm{d}s\right)^{1/(1-\alpha)} \\ &\geq (1 + (\alpha - 1) t_0 h(x, t_0))^{1/(1-\alpha)} \\ &\geq (\max\{2, 2(\alpha - 1) t_0 h(x, t_0)\})^{1/(1-\alpha)} \to 1 \text{ as } \alpha \to \infty, \end{aligned} \tag{2.1}$$

and so we set $\lambda \equiv 1$ in the case of $\alpha = \infty$.
For $\alpha \to -\infty$, concavity of $h(x, \cdot)$ leads to $h(x, t) \geq h(x, t_0) t / t_0$ for $0 \leq t \leq t_0$ and so

$$\lambda(x, t_0) \leq \left(\max\left\{0, 1 + (\alpha - 1)\frac{t_0}{2} h(x, t_0)\right\}\right)^{1/(1-\alpha)} \to 0 \quad \text{as } \alpha \to -\infty, \tag{2.2}$$

as λ can only be nonzero while

$$t_0 \leq \frac{2}{(1 - \alpha) h(x, t_0)} \to 0 \text{ as } \alpha \to -\infty, \tag{2.3}$$

and so we set $\lambda \equiv 0$ in the case of $\alpha = -\infty$.

2.1.1. The fast-clogging limit $\alpha = -\infty$. In the fast-clogging limit, $\lambda \equiv 0$, (1.1) simplifies to

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^n \frac{\partial h}{\partial x}\right), \tag{2.4}$$

with the boundary conditions (1.3) implying conservation of volume,

$$\int_0^{x_f(t)} h\ \mathrm{d}x = t. \tag{2.5}$$

Following Huppert [7], but letting n take any value greater or equal than 1, a self-similar solution to (2.4) can be found by introducing the scalings

$$\eta = x t^{-(n+1)/(n+2)} \quad \text{and} \quad h = \eta_f^{2/n} t^{1/(n+2)} \phi(y), \tag{2.6}$$

where $\eta_f = x_f(t)t^{-(n+1)/(n+2)}$ is the position of the front and the rescaled coordinate is given by $y = \eta/\eta_f$. The scalings (2.6) transform (2.4) into

$$(\phi^n \phi')' + \frac{n+1}{n+2} y\phi' - \frac{1}{n+2}\phi = 0, \tag{2.7}$$

for $\phi(y)$, subject to the boundary condition $\phi(1) = 0$ and the volume condition (2.5). The value of η_f can be recovered from

$$\eta_f = \left(\int_0^1 \phi \; \mathrm{d}y \right)^{-n/(n+2)}, \tag{2.8}$$

and (2.7) can be solved numerically by using the expansion of ϕ around $y = 1$,

$$\phi(y) = \left(\frac{n(n+1)}{n+2} \right)^{1/n} (1-y)^{1/n} + O\left((1-y)^{(n+1)/n} \right). \tag{2.9}$$

While we are not interested in the exact value of η_f, we use the results from this subsection to verify our numerical methods and provide a reference implementation of the numerical solution to (2.7) in the supplementary material.

The key result from this subsection is that the front of the current spreads as

$$x_f(t) \propto t^{(n+1)/(n+2)} \tag{2.10}$$

for $\alpha = -\infty$. This provides us with an upper bound on the order of magnitude of the position of the front of the current for any α.

2.1.2. The slow-clogging limit $\alpha = \infty$. In the slow-clogging limit, $\lambda \equiv 1$, (1.1) becomes

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^n \frac{\partial h}{\partial x} \right) - h, \tag{2.11}$$

subject to the initial condition $h(x, 0) = 0$ and the boundary conditions (1.3) with $I(t) = 1$. We introduce

$$V(t) = \int_0^{x_f(t)} h \; \mathrm{d}x, \tag{2.12}$$

which represents the volume of fluid in the current. Making the reasonable assumption that $\mathrm{d}x_f(t)/\mathrm{d}t$ is bounded away from 0, we integrate (2.11) from 0 to $x_f(t)$ and, using the boundary conditions $h^n \partial h/\partial x = -1$ at $x = 0$ and $h^n \partial h/\partial x = 0$ at $x = x_f(t)$, we obtain the ordinary differential equation (ODE)

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 1 - V, \tag{2.13}$$

subject to the initial condition $V(0) = 0$, which has the solution

$$V(t) = 1 - \exp(-t). \tag{2.14}$$

Hence, the volume of fluid in the current converges to 1 as $t \to \infty$ and we also expect h to converge to a steady state. Using (2.11), this steady state can be found by solving

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h^n \frac{\mathrm{d}h}{\mathrm{d}x} \right) = h, \tag{2.15}$$

subject to the volume condition

$$\int_0^\infty h \, \mathrm{d}x = 1. \tag{2.16}$$

and the boundary conditions (1.3). Using the ansatz $h = a\,(x_f - x)^b$, we obtain, by a comparison of powers, that $b = 2/n$ and $a = \left(n^2/(2n+4)\right)^{1/n}$. We can then determine x_f to obtain the steady-state solution to (2.15)

$$h(x) = \left(\frac{n^2}{2n+4}\right)^{1/n} \left(\left(\frac{n}{2}\right)^{n/(n+2)} \left(\frac{2n+4}{n^2}\right)^{(n+1)/(n+2)} - x\right)^{2/n}. \tag{2.17}$$

As the solution of h only attains a finite spreading distance, we identify the case of $\alpha = \infty$ with a power-law spreading speed of 0.

2.2. The general case $\alpha \in \mathbb{R}$. We shall now consider the full system (1.1)–(1.3) with $I(t) = 1$. Numerical solutions of this system were obtained using the method outlined in Appendix C. The results for $n = 3$, shown in figures 2.1a and 2.1b, indicate that both the position of the front x_f and the height at the origin $h(0,t)$ follow a power law for large t. We therefore assume $x_f(t) \sim t^l$ and $h(0,t) \sim t^k$. To find a relation between the power law of the long time height at the origin, k, and the power law of the long time front position, l, we consider the fast-clogging limit, for which we have $k = 1/(n+2)$ and $l = (n+1)/(n+2)$, see (2.6). This suggests

$$l = (n+1)k, \tag{2.18}$$

which also holds true for the slow-clogging limit $\alpha = \infty$, where $l = k = 0$. We verify the assumption (2.18) numerically for different values of $n = 2, 3, 5$ and different values of α; the results are shown in figure 2.2a.

To make progress with establishing a relationship between the power-law spreading speed l of the current and exponent of the filtration law, α, we estimate the magnitude of the volume of the current and the total drainage asymptotically. Integrating (1.1) with respect to x and t, we obtain

$$\underbrace{\int_0^{x_f(t)} h \, \mathrm{d}x}_{=:V(t)} + \int_0^t \underbrace{\int_0^{x_f(s)} \lambda h \, \mathrm{d}x}_{=:D(s)} \, \mathrm{d}s = t. \tag{2.19}$$

This result states that the total influx is composed of the *volume* $V(t)$ and the integral of the instantaneous *drainage* $D(s), 0 \leq s \leq t$. Hence, to balance the total influx t, either the volume term or the integral of the drainage (or both) must be of magnitude t asymptotically, as $t \to \infty$.

As we are assuming the solution h to (1.1) to be concave, h encloses the triangle formed by the origin, the position of the front, and the position of the height at the origin. Thus, we can bound the volume $V(t)$ of the current for sufficiently large t by

$$t \geq V(t) \geq O(t^{(n+2)k}), \tag{2.20}$$

hence $k \leq 1/(n+2)$, and we can conclude that the no-drainage solution to (2.7) serves asymptotically as an upper bound to any solution h to (1.1), hence

$$h(x,t) \leq h(0,t) \leq O(t^{1/(n+2)}) \quad \text{and} \quad x_f(t) \leq O(t^{(n+1)/(n+2)}) \quad \forall\, \alpha \in \mathbb{R},\ n \geq 1. \tag{2.21}$$

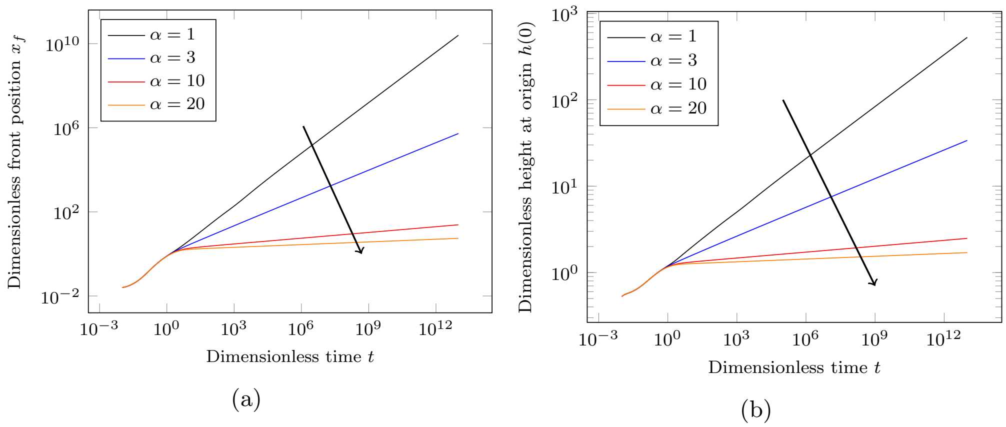


Fig. 2.1: Loglog plots of (a) the position of the front and (b) the height at the origin for $n = 3$ and different values of $\alpha = 1, 3, 10, 20$, with the arrow used to indicate increasing values of α. The results indicate that both the height at the origin and the position of the front follow a power law.

Thus, if the volume term is of magnitude t, the current has to spread asymptotically as $t^{(n+1)/(n+2)}$ and grow asymptotically at the origin as $t^{1/(n+2)}$ due to the bounds in (2.21).

The spreading distance can asymptotically only be smaller than $t^{(n+1)/(n+2)}$ if the total drainage is of order t and the volume is of strictly smaller order than t. Taking the derivative of (2.19) with respect to t, we therefore obtain the necessary condition

$$\lim_{t\to\infty} D(t) = 1 \tag{2.22}$$

for the spreading distance to be asymptotically smaller than $O(t^{(n+1)/(n+2)})$. To estimate the instantaneous drainage $D(t)$, we use the transform $\eta = x/x_f(t)$ and obtain for λ

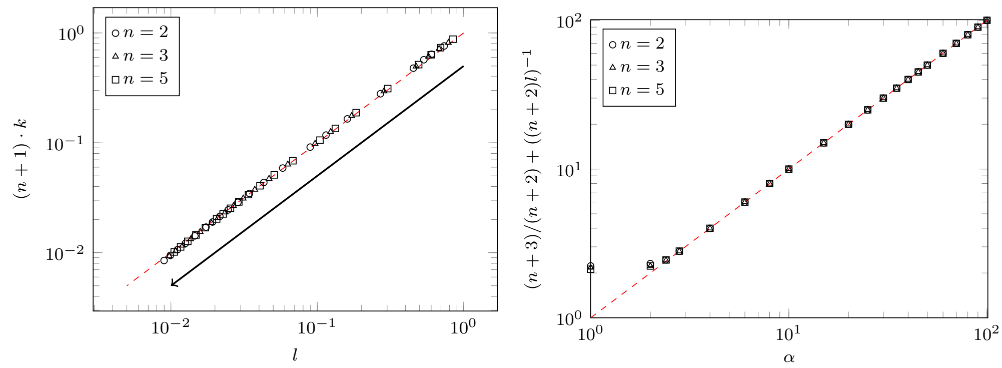
$$\frac{\partial \lambda}{\partial t} - \frac{\dot{x}_f}{x_f}\eta\frac{\partial \lambda}{\partial \eta} = -\lambda^\alpha h, \tag{2.23}$$

where a dot ($\dot{}$) denotes differentiation with respect to t. The power-law assumption $x_f(t) \sim t^l$ yields $\dot{x}_f/x_f \sim t^{-1}$ and for each η, $\partial\lambda/\partial\eta$ is bounded as $\lambda(\eta, 0) = 0$, λ is increasing in η, and $\lambda(\eta, t) \to 0$ as $t \to \infty$ for $\eta < 1$. We first consider the case where in (2.23) the term $\lambda^\alpha h$ is asymptotically larger than $\frac{\dot{x}_f}{x_f}\eta\frac{\partial\lambda}{\partial\eta}$ and later show that this is consistent with our solution. Due to the assumption of concavity, we can provide a lower bound on h as

$$h(\eta, t) \geq (1 - \eta)h(0, t) \sim (1 - \eta)t^k, \tag{2.24}$$

and we then estimate λ asymptotically by solving

$$\frac{\partial \lambda}{\partial t} = -(1 - \eta)\lambda^\alpha t^k, \tag{2.25}$$



(a) (b)

Fig. 2.2: (a) Loglog plot of the transformation $l = (n+1)k$ versus l for the power-law exponent l of the front and power-law exponent k of the height at the origin for different values of α in the range of 1 to 100, where the arrow indicates the increase in α. The dashed red line is the indicator line (l, l), and so we obtain excellent agreement between the theoretical prediction (2.18) and numerical result. (b) Loglog plot of the numerical results following the relation between α and k (2.30) for different values of α in the range of 1 to 100. We obtain excellent agreement between the theory and the numerical results, and see a divergence from the dashed red indicator line (α, α) for $\alpha \lesssim 3$ as expected.

where $k \in [0, 1/(n+2)]$ is the power law of the height h at $\eta = 0$. The solution to (2.25) subject to $\lambda(\eta, 0) = 1$ for $\alpha \neq 1$ is given by

$$\lambda(\eta, t) = \left(1 + \frac{(\alpha-1)(1-\eta)}{k+1} t^{k+1}\right)^{1/(1-\alpha)}, \tag{2.26}$$

and, for $\alpha = 1$,

$$\lambda(\eta, t) = \exp\left(-\frac{1-\eta}{k+1} t^{k+1}\right). \tag{2.27}$$

For $\alpha < 1$ and $\eta < 1$, the membrane will be blocked within a finite amount of time $T(\eta)$, i.e. $\lambda(\eta, t) = 0 \ \forall \ t \geq T(\eta)$. For $\alpha > 1$, we can use (2.26) to obtain the asymptotic behaviour of λ as

$$\lambda(\eta, t) \sim t^{-(k+1)/(\alpha-1)}, \quad \eta < 1. \tag{2.28}$$

To estimate the drainage, we use the upper bound on the height $h(\eta, t) \leq O(t^k)$ and the estimate of λ from (2.28), which yields

$$D(t) = \int_0^{x_f(t)} \lambda h \, \mathrm{d}x \sim t^{-(k+1)/(\alpha-1)} t^{(n+2)k}. \tag{2.29}$$

Using the condition (2.22) that the drainage $D(t)$ must converge to 1, we can solve (2.29) for $l = (n+1)k$ or α, obtaining

$$l = \frac{n+1}{(n+2)(\alpha-1)-1} \quad \text{or} \quad \alpha = \frac{n+3}{n+2} + \frac{n+1}{(n+2)l}. \tag{2.30}$$

Due to the bound on the asymptotic order of the height $k = 1/(n+2)$ and the bound on the asymptotic order of the position of the front $l = (n+1)/(n+2)$ (see (2.6)), the relations in (2.30) are only valid for $\alpha \geq 2 + 1/(n+2)$. For $\alpha < 2 + 1/(n+2)$, the volume $V(t)$ will be $O(t)$ and we predict the spreading position to follow the same power law asymptotically for $t \to \infty$ as in the case where there is no drainage.

We use the relation for α in (2.30) to transform the numerically obtained values for l for different n and α. The results are shown in figure 2.2b, illustrating excellent agreement between the theoretical prediction (2.30) and the numerical results.

To conclude this section, we have shown, on the basis of scaling arguments and numerical results, that the spreading position of a particle-laden gravity under a constant influx current follows a power law. This allows us to uniquely identify the filtration law if the exponent α of the filtration law satisfies $\alpha > 2 + 1/(n+2)$. However, for $\alpha \leq 2+1/(n+2)$, the position of the front will be proportional to $t^{(n+1)/(n+2)}$ asymptotically, and so it is not possible to infer α in this case. In the next section, we will use a travelling wave solution to show how to infer the filtration law when $\alpha < 1$.

3. A travelling wave solution. In this section, we show that there exists a travelling wave solution to the system (1.1), (1.2) and (1.3) , with constant speed 1,

$$h(x,t) = H(x-t) \quad \text{and} \quad \lambda(x,t) = \Lambda(x-t), \tag{3.1}$$

for all $\alpha \in \mathbb{R}$ when a suitable influx $I(t) = -(h^n h_x)(0,t)$ is applied.

We will show that the influx needed to obtain a travelling wave solution must follow a power law whose order is unique for $\alpha > 2 + 1/n$. As in the constant-influx case, any point on the membrane clogs within a finite time for $\alpha < 1$. This implies that for $\alpha < 1$ the width of the area where the current drains is finite, creating a second *no-drainage front*. For a travelling wave, the distance between the liquid front and the no-drainage front is constant and we will furthermore show that it is unique in α, providing a method to determine α in this instance. The existence of a no-drainage front has similarities to imbibition fronts that can be observed in the spreading of gravity currents on initially dry meshes where a capillary pressure has to be overcome (see for example Sayag and Neufeld [14]). However, in our case the drainage region at the front is followed by a no-drainage region, whereas for initially dry meshes, the situation is reversed.

Using the ansatz (3.1) in (1.1) and (1.2), we obtain the system of ODEs

$$(H^n H')' - \Lambda H + H' = 0, \tag{3.2a}$$

$$\Lambda' - \Lambda^\alpha H = 0, \tag{3.2b}$$

subject to the boundary conditions $H(0) = 0$, $\Lambda(0) = 1$, and

$$(H^{n-1} H')(0) = -1. \tag{3.3}$$

from (1.3). Given that $H(0) = 0$, we can immediately infer that H will not be Lipshitz continuous at the front, thus not guaranteeing existence and uniqueness. We can circumvent this problem by introducing the two variable transformations

$$\Phi = H^n \quad \text{and} \quad \xi = -\eta, \tag{3.4}$$

where the second transformation is employed for convenience to work with positive variables, as $H > 0$ only for $\eta < 0$. Applying the transformation (3.4) to (3.2), we obtain a new system of ODEs,

$$\frac{1}{n}(\Phi')^2 + \Phi\Phi'' - \Phi' - n\Lambda\Phi = 0, \tag{3.5a}$$

$$\Lambda' + \Lambda^\alpha \Phi^{1/n} = 0, \tag{3.5b}$$

subject to the boundary conditions $\Lambda(0) = 1$, $\Phi(0) = 0$, and $\Phi'(0) = n$, making the problem Lipshitz continuous at $\xi = 0$, guaranteeing existence and uniqueness at least up to some finite ξ. The system (3.5) can be solved numerically by initiating the solver a small distance away from $\xi = 0$ and using the approximation $\Phi(\epsilon) = n\epsilon + O(\epsilon^2)$ and $\Lambda = 1$ around $\xi = 0$ for small ϵ.

We will now reproduce the exact solution to the fast-clogging limit $\alpha \to -\infty$, that is, $\Lambda \equiv 0$, from [16], as this will provide a lower bound to influx necessary to obtain a travelling wave solution. For the slow-clogging limit $\alpha \to \infty$, that is $\Lambda \equiv -1$, we will propose a power law solution for large ξ, which then will be generalised for $\alpha > 1$.

3.1. The special cases $\alpha \in \{-\infty, \infty\}$. As in Section 2, we first consider the special cases of $\alpha \to -\infty$, that is, $\Lambda \equiv 0$, and $\alpha \to \infty$, that is, $\Lambda \equiv 1$. Instead of providing us with upper and lower bounds on the asymptotic spreading speed, these two special cases will provide us with asymptotic bounds on the influx to create a travelling wave.

3.1.1. The fast clogging limit $\alpha = -\infty$. For $\Lambda \equiv 0$, the system of equations (3.5) simplifies to

$$\frac{1}{n}(\Phi')^2 + \Phi\Phi'' - \Phi' = 0, \tag{3.6}$$

subject to the boundary conditions $\Phi(0) = 0$ and $\Phi'(0) = n$. We can verify by direct evaluation that

$$\Phi(\xi) = \max\{n\xi, 0\} \tag{3.7}$$

solves the equation (3.6), and the influx required to create this travelling wave solution is given by

$$I(\xi) = H(\xi)^n \frac{\mathrm{d}H(\xi)}{\mathrm{d}\xi} = \frac{1}{n}\Phi^{1/n}\frac{\mathrm{d}\Phi}{\mathrm{d}\xi} = (n\xi)^{1/n}. \tag{3.8}$$

This corresponds to requiring an influx of $O(t^{1/n})$ to create a travelling wave.

3.1.2. The slow clogging limit $\alpha = \infty$. For $\Lambda \equiv 1$, the system of equations (3.5) simplifies to

$$\frac{1}{n}(\Phi')^2 + \Phi\Phi'' - \Phi' - n\Phi = 0, \tag{3.9}$$

subject to the boundary conditions $\Phi(0) = 0$ and $\Phi'(0) = n$. We are not aware of a closed-form solution to (3.9), but we can make progress by looking for a solution of the form $\Phi \sim \xi^p$ for large ξ. Balancing the terms in (3.9), we obtain $p = 2$ and

$$I(\xi) \sim \frac{2}{n}\xi^{(n+2)/n}. \tag{3.10}$$

This corresponds to requiring an influx of $O(t^{(n+2)/n})$.

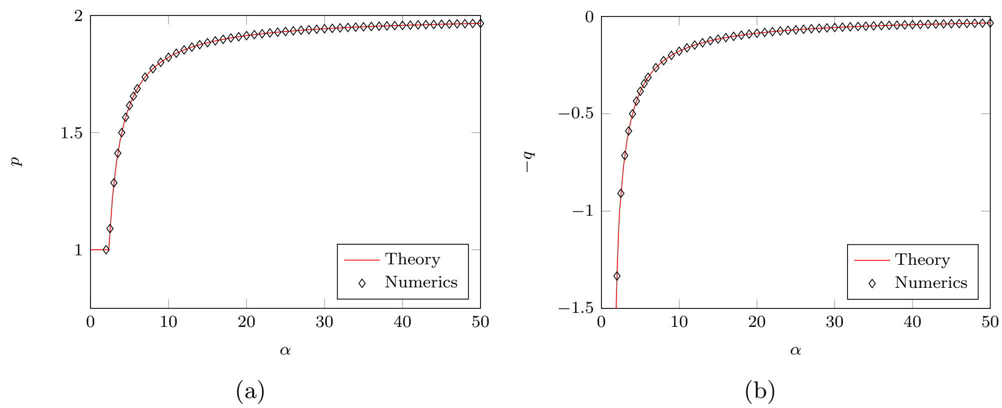


Fig. 3.1: Numerical validation of (3.13)–(3.16) for $n = 3$, (a) contains the results for Φ while (b) contains the results for λ. We obtain excellent agreement between the numerical results and the theoretical predictions. The change between the drainage-driven scaling and the volume-driven scaling occurs at $\alpha = 2 + 1/n$, both for p and q, where $p = 1$ and q is given by (3.17), although this is much more visible in (a) than in (b).

3.1.3. The general case α in $\mathbb{R}$. Motivated by the power-law approach for $\Lambda \equiv 1$, we now choose the same approach when Λ varies and assume

$$\Phi \sim \xi^{p} \quad \text{and} \quad \Lambda \sim \xi^{-q} \tag{3.11}$$

for large ξ, which corresponds to requiring an influx

$$I(\xi) \sim \xi^{((n+1)p-n)/n}. \tag{3.12}$$

For the power-law approach, it is therefore sufficient to determine the leading-order power law p of Φ to determine the leading-order magnitude of the influx I. As the influx for the fast-clogging limit is of order $1/n$ (3.8) and so for any α, the influx must be of power law order at least $1/n$, we can conclude from (3.12) that $p \geq 1$ and similarly, $p \leq 2$ by considering the slow-clogging limit. Balancing terms in (3.5), we obtain the values of p and q as

$$p = \frac{n(2\alpha - 3)}{n(\alpha - 1) + 1} \quad \text{and} \quad q = \frac{n + 2}{n(\alpha - 1) + 1}. \tag{3.13}$$

Since $1 \leq p \leq 2$, (3.13) is only valid for

$$\alpha \geq 2 + \frac{1}{n}, \tag{3.14}$$

and we automatically obtain $p = 1$ for $\alpha \leq 2 + 1/n$, as shown by the horizontal red line in figure 3.1 (a). We can validate this result, as we have for large ξ and for $\alpha \leq 2 + 1/n$,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\xi} = -\lambda^{\alpha}\xi^{1/n}, \tag{3.15}$$

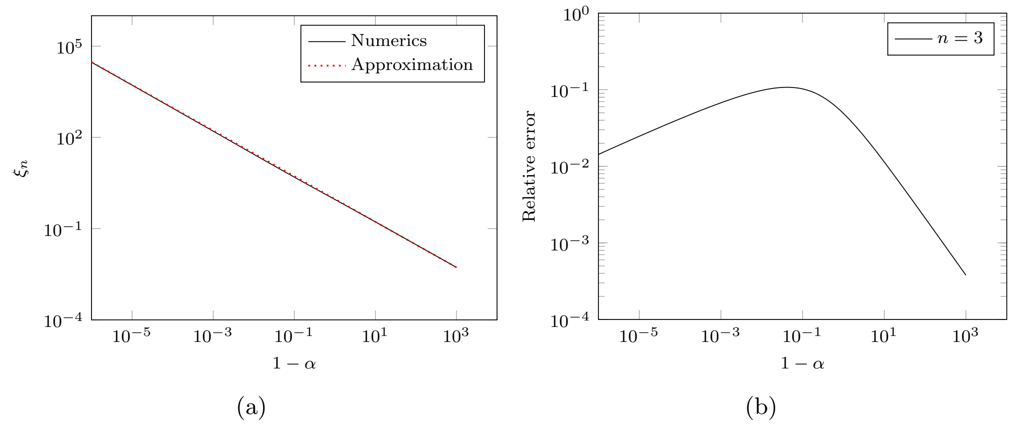


Fig. 3.2: (a) Loglog plot of the exact numerical solution to (3.18) and the approximation (3.20) for $n = 3$, we obtain excellent agreement between the two. In (b), we plot the relative error of inferring $1-\alpha$ using the approximation (3.20) and see that the approximation provides a good estimate for $1-\alpha$.

which has the solution

$$\lambda = \left(1 + \frac{\alpha - 1}{n+1}\xi^{(n+1)/n}\right)^{1/(1-\alpha)} \tag{3.16}$$

and we have $\lambda \sim \xi^{-1}$ for $\alpha = 2 + 1/n$. Hence, we know that the integral of $\lambda(\xi)$ from 0 to ∞ diverges for $\alpha = 2 + 1/n$ but converges for $\alpha < 2 + 1/n$. In the case where $1 < \alpha \leq 2 + 1/n$, we can furthermore use $p = 1$ to balance terms in (3.5) and obtain

$$q = \frac{n+1}{n(1-\alpha)}. \tag{3.17}$$

We show the excellent agreement between the numerical results and our theoretical predictions for $n = 3$ in figure 3.1. We obtain similar accuracy for $n = 1, 2, \ldots, 100$.

3.1.4. Computing the width of the drainage front for $\alpha < 1$. From the asymptotic solution (3.16), we can conclude that for $\alpha < 1$ we obtain $\lambda = 0$ for finite ξ. Therefore, by finding the position of the no-drainage front we could determine α from the width of the region where the current drains. To do so, we use the transformation $\Psi = \Lambda^{1-\alpha}$, under which (3.5) becomes the numerically more tractable system

$$\frac{1}{n}(\Phi')^2 + \Phi\Phi'' - \Phi' - n\Psi^{1/(1-\alpha)}\Phi = 0, \tag{3.18a}$$

$$\Psi' + (1-\alpha)\Phi^{1/n} = 0, \tag{3.18b}$$

subject to the boundary conditions $\Psi(0) = 1$, $\Phi(0) = 0$, and $\Phi'(0) = n$. We use a root-finding method to determine the position of the no-drainage front ξ_n. We can find an approximation to ξ_n by using the solution $\Phi = n\xi$ for $\alpha \to -\infty$. Then, we have that

$$\lambda = \left(1 + \frac{n^{(n+1)/n}(\alpha - 1)}{n+1}\xi^{(n+1)/n}\right)^{1/(1-\alpha)}, \tag{3.19}$$

which we can solve for $\lambda = 0$ to obtain the no-drainage front ξ_n as

$$\xi_n = \left(\frac{n+1}{n^{(n+1)/n}(1-\alpha)} \right)^{n/(n+1)}. \tag{3.20}$$

We show the excellent agreement between the approximation (3.20) and the exact solution in figure 3.2. This result allows us to approximately determine the filtration law for $\alpha < 1$ from the distance between the front and the no-drainage front.

4. Discussion and conclusions. This paper has addressed the question of determining a filtration law from observations of the position of the front of a gravity current that is spreading on a long, thin membrane. Based on scaling arguments and using numerical validation, we first showed that, for a constant influx, the position of the front follows a power law. Estimating the order of magnitude of both the instantaneous drainage and the volume of a current, we then determined the power law l of the position of the front in dependence of α. This relationship is unique for $\alpha > 2 + 1/(n+2)$, which allows us to determine the filtration law in these instances.

We also showed that the combined problem of a gravity current and clogging allows for a travelling wave solution if a suitable influx is applied. After showing that this influx has to follow, to leading order, a power law whose power r,

$$r = \frac{(n+2)(\alpha-2)}{n(\alpha-1)+1}, \tag{4.1}$$

is unique for $\alpha > 2 + 1/n$ and the same constant $r = 1/n$ otherwise, we showed that the travelling wave creates a second no-drainage front for $\alpha < 1$. Using the no-drainage solution to approximate the shape of the current, we determined an effective approximation to infer the filtration law from the distance between the front and the no-drainage front.

There are several directions in which this work could be extended. First, the obvious question remains how to determine the filtration law if $\alpha \in [1, 2 + 1/(n+2)]$. While the position of the front follows $x_f(t) \sim t^{(n+1)/(n+2)}$, numerical simulations indicate different constants of proportionality, which might be a suitable starting point.

Second, it would be interesting to explore whether the inverse problem is also tractable for a different class of filtration laws and whether there even is uniqueness for the general problem where the permeability factor λ satisfies

$$\frac{\partial \lambda}{\partial t} = -f(\lambda)h. \tag{4.2}$$

Third, the results in this paper heavily rely on numerical evidence and so it would be important to formally prove the results to gain certainty.

Finally, it would be interesting to see in which way the ideas and results in this paper can be transferred to experimental settings.

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Appendix A. Notes on the nondimensionalisation. We nondimensionalise the system of equations

$$\frac{\partial H}{\partial T} = c_n \frac{\partial}{\partial X}\left(H^n \frac{\partial H}{\partial X}\right) - c_{\mathrm{imb}} \lambda H \tag{A.1a}$$

$$\frac{\partial \lambda}{\partial T} = -c_\alpha \lambda^\alpha H, \tag{A.1b}$$

subject to the initial conditions $H(X, 0) = 0$, $\lambda(x, 0) = 1$ and boundary conditions $c_n H^n \frac{\partial H}{\partial X} = -\mathcal{I}$ at $X = 0$ as well as $H = 0$, $c_n H^n \frac{\partial H}{\partial X} = 0$ at $X = X_f$. The constants c_n and c_{imb} depend on n as well as the physical quantity that h represents (e.g. height, density), the filtration law constant c_α additionally depends on α. Choose

$$T = \frac{1}{c_{\mathrm{imb}}} t, \quad H = \frac{c_{\mathrm{imb}}}{c_\alpha} h, \quad \text{and} \quad X = \left(\frac{c_n c_{\mathrm{imb}}^{n-1}}{c_\alpha^n}\right)^{1/2} x \tag{A.2}$$

to obtain the equations (1.1) and (1.2), subject to the initial conditions $h(x, 0) = 0$, $\lambda(x, 0) = 1$ and boundary conditions $h = 0$, $h^n \frac{\partial h}{\partial x} = 0$ at $x = x_f$ and

$$h^n \frac{\partial h}{\partial x} = -I(t), \tag{A.3}$$

where

$$I(t) = \left(\frac{c_\alpha^{n+2} c_n}{c_{\mathrm{imb}}^{n+3}}\right)^{1/2} \mathcal{I}. \tag{A.4}$$

Appendix B. Derivation of filtration laws. To derive the governing equations for the four filtration laws, we model the membrane as a plate of constant depth, the pores as cylindrical holes with equal radius, and the particles as spheres of equal radius. We expect the Reynolds number of the flow to be very small and so we assume Poiseuille flow within the pores, and we also assume that there is no fluid interaction between the pores so that the overall permeability λ of the membrane is the sum of the individual permeabilities of the pores.

We are interested in how the permeability λ changes in time due to particles being filtered out from the fluid through the membrane. We will derive the governing equations for the four different filtration laws – complete, intermediate, and standard blocking, as well as cake filtration – by assuming that Q is provided externally, and may thus vary in time. For the case of equation (1.2), we have assumed that, in dimensionless form, $Q = \lambda h$, where h is the height of the current. As we are only interested in the functional form of the governing equations and not in the constants, we will work with proportional arguments only; a complete derivation that includes constants can be found in [5].

For *complete blocking*, every particle arriving at the membrane blocks a pore completely. The permeability λ of the membrane is therefore proportional to the number of unblocked pores. Assuming no random fluctuations, the number of particles arriving at the membrane at any given time is proportional to the flux Q, and each of these particles will block one pore. Hence, the decrease in permeability is proportional to the reduction in number of unblocked pores, which in turn is proportional to the flux, and so

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} \propto -Q. \tag{B.1}$$

For *intermediate blocking*, particles can either block a pore completely or settle on another particle at an already blocked pore, and we assume that the arrival of a particle at a given pore, blocked or unblocked, is equally likely. The permeability λ is again proportional to the number of unblocked pores, while the number of particles arriving at the membrane at any given time, assuming no random fluctuations, is proportional to the flux Q. However, as particles can stack on already blocked pores in contrast to complete blocking, the probability of arriving at an unblocked pore is the number of unblocked pores divided by the total number of pores, which is proportional to the permeability. Hence, the change in permeability must be proportional to the number of particles arriving at the pore multiplied by the probability of arriving at an unblocked pore, and so

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} \propto -\lambda Q. \tag{B.2}$$

For *standard blocking*, the particles deposit inside the pore and their volume is assumed to be distributed equally onto the inside wall of the pore. Poiseuille flow through each pore leads to $\lambda \propto R^4$, where R is the radius of the pores. As the volume of each particle is distributed equally on the inner wall of the pore, the cross section of the pore, R^2 decreases proportionally to the volume of particles arriving at the pore, which is proportional to Q, and so we have

$$\frac{\mathrm{d}(R^2}{\mathrm{d}t} \propto \frac{\mathrm{d}(\lambda^{1/2})}{\mathrm{d}t} \propto -Q \Rightarrow \frac{\mathrm{d}\lambda}{\mathrm{d}t} \propto -\lambda^{1/2} Q. \tag{B.3}$$

For *cake filtration*, an additional resistive layer is built up on top of the membrane due to particles depositing. The increase in resistance, r, where $r = \lambda^{-1}$, of the combined membrane and filtercake is only due to the increase in thickness of the filtercake. This increase in thickness is proportional to the rate of particles depositing, which is proportional to the flux Q, and so

$$\frac{\mathrm{d}(\lambda^{-1})}{\mathrm{d}t} \propto -Q \Rightarrow \frac{\mathrm{d}\lambda}{\mathrm{d}t} \propto -\lambda^2 Q. \tag{B.4}$$

The derivations of the four filtration laws (B.1)–(B.4) are valid for any, potentially externally prescribed, flux Q. In our model $Q \propto \lambda h$, and so we can summarise equations (B.1)–(B.4) as

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} \propto -\lambda^\alpha h, \tag{B.5}$$

where $\alpha \in \{1, 3/2, 2, 3\}$. Choosing a suitable nondimensionalisation, we obtain (1.2).

Appendix C. Numerical methods. To solve the system of equations (1.1)–(1.3), we introduce the mapping $\eta = x/x_f$ and thus obtain the transformed system

$$\frac{\partial h}{\partial t} - \frac{\dot{x}_f}{x_f}\eta\frac{\partial h}{\partial \eta} = \frac{1}{x_f^2}\frac{\partial}{\partial \eta}\left(h^n \frac{\partial h}{\partial \eta}\right) - \lambda h, \tag{C.1a}$$

$$\frac{\partial \lambda}{\partial t} - \frac{\dot{x}_f}{x_f}\eta\frac{\partial \lambda}{\partial \eta} = -\lambda^\alpha h, \tag{C.1b}$$

subject to the boundary conditions $h(1,t) = 0$, $\lambda(1,t) = 1$, the transformed influx condition $(h^n h_x)(0,t) = -x_f(t)$, and the condition for $\dot{x}_f$,

$$\dot{x}_f = \lim_{\eta \to 1} h^{n-1} \frac{\partial h}{\partial \eta}. \tag{C.2}$$

The initial conditions for $x_f(0)$ and $h(\eta, 0)$ are imposed by considering the no-drainage solution to (2.4) for an early time $t = 10^{-2}$, and we set $\lambda(\eta, 0) = 1$.

We discretise (C.1) using $N = 250$ interpolation points at $\eta = 0/N, 1/N, \ldots (N-1)/N$ so that we can employ the boundary conditions $h(1,t) = 0$ and $\lambda(1,t) = 1$ implicitly. The corresponding set of 250 ODEs is then solved using MATLAB's ODE15s routine, which allows us to determine x_f for large times up to $t = 10^{13}$.

To compute the exponent of the power law $x_f(t) \sim t^l$, we fit a linear model $y = ax + b$ to $\log(x_f)$ where $x = \log(t)$ for $t \in [t_{\text{end}} \cdot 10^{-2}, t_{\text{end}}]$. The coefficient a is then the power-law exponent l.

Reference implementations for the numerical methods described in the main text and the scripts that generate the figures in the paper can be found in the supplementary material.

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