Heavy tails in last-passage percolation

Ben Hambly **James B. Martin**

University of Oxford

March 21, 2006

Abstract

We consider last-passage percolation models in two dimensions, in which the underlying weight distribution has a heavy tail of index $\alpha < 2$. We prove scaling laws and asymptotic distributions, both for the passage times and for the shape of optimal paths; these are expressed in terms of a family (indexed by α) of "continuous last-passage percolation" models in the unit square. In the extreme case $\alpha = 0$ (corresponding to a distribution with slowly varying tail) the asymptotic distribution of the optimal path can be represented by a random self-similar measure on $[0, 1]$, whose multifractal spectrum we compute. By extending the continuous last-passage percolation model to $\mathbb{R}^2$ we obtain a heavy-tailed analogue of the Airy process, representing the limit of appropriately scaled vectors of passage times to different points in the plane. We give corresponding results for a directed percolation problem based on α-stable Lévy processes, and indicate extensions of the results to higher dimensions.

1 Introduction

Directed last-passage percolation in two dimensions has received much attention in recent years. In certain specific cases, for example where the weights at each site are i.i.d. with exponential or geometric distribution, very precise scaling laws and asymptotic distributions are now known, both for the passage times and for the shape of optimal paths (see for example [16, 17, 4]). Such cases are closely related to the longest increasing subsequence problem, and to Markovian interacting particle systems such as the totally asymmetric exclusion process; there are also very close links to random matrix theory, for example to the behaviour of the largest eigenvalue of a large matrix drawn from the Gaussian Unitary Ensemble (see for example [23] for a survey).

It is believed that the behaviour proved for the exponential and geometric cases should be *universal*, in that the same scaling laws and asymptotic distributions should occur in the last-passage percolation model whose underlying weight distribution is from a much more general class (certainly including any distribution with an exponentially decaying tail, and maybe also those with sufficiently light polynomial tails). The growth models corresponding

MSC 2000 subject classifications: Primary 60K35; Secondary 82B41. *Keywords and phrases:* Last-passage percolation, heavy tails, Airy process, regular variation, stable process, multifractal spectrum.

to these last-passage percolation problems should belong to the *Kardar-Parisi-Zhang (KPZ) universality class* (see for example [19]). However, only very limited universality results have been proved: for example, conditions under which laws of large numbers for the passage times (or "shape theorems") hold, and asymptotics for passage-times close to the boundary of the quadrant [22, 9, 5].

In this paper, we study cases in which the tail of the weight distribution is sufficiently heavy that such shape theorems fail, and which certainly fall outside the universality class described above. Specifically, we assume that the tail of the weight distribution is regularly varying with index $\alpha < 2$. We describe a family of "continuous last-passage percolation" models (indexed by α), and use them to provide scaling laws and asymptotic distributions for the discrete models, both for the passage times and for the shape of the optimal paths. Thus we have a universality result for these heavy-tailed models as the only information required to determine the scaling limits is the parameter α.

One example of an application where such a heavy-tailed assumption is very natural is in the use of last-passage models to represent networks of *queues in tandem.* The vertex weights in the percolation models correspond to service times in the queueing systems, and passage times in the percolation models correspond to the total time spent in the queueing system by particular customers; see for example [12, 3, 20].

In Section 2, we define the discrete last-passage percolation model precisely; we then describe the continuous last-passage model and state our main convergence results. We also derive from the continuous model a stationary process which can be seen as a heavy-tailed analogue of the *Airy process* (which was developed by Prähofer and Spohn [25] and Johansson [18]), and which gives a process limit for vectors of passage times to different points, appropriately scaled.

The proofs of the convergence results for passage-times are given in Section 3, and those for the optimal paths are given in Section 4. The results on the heavy-tailed Airy process are proved in section 7.

In Section 6 we explore the case where the tail of the weight distribution is *slowly varying* (i.e. $\alpha = 0$). It is no longer possible to provide a non-degenerate limiting distribution for the passage times; however, asymptotics for the form of the optimal paths are still possible, and in fact the distribution of the limiting path that arises can be described in a particularly simple and algorithmic way. As the path is increasing it can be thought of as the distribution of a random measure on $[0, 1]$; this measure is self-similar and we compute its multifractal spectrum.

The *Brownian directed percolation* model has recently been much studied in various contexts (see for example [6, 13, 14, 24]). In Section 5 we discuss a related model in which Brownian motion is replaced by an α-stable Lévy process, and we again prove distributional convergence to the continuous last passage percolation model.

The bulk of the paper concerns the case of two-dimensional last-passage percolation. However, almost all of the results extend easily to d dimensions, $d \geq 3$, and now apply for $\alpha < d$. We indicate these extensions in Section 8.

Simulations of trees of optimal paths, of the limiting path for $\alpha = 0$, and of heavy-tailed Airy processes are given in Sections 2, 6 and 7.

2 Main results

2.1 Definition of the discrete problem

Let F be a distribution function. We will assume that the tail of the distribution F is *regularly varying* with index $\alpha \in (0, 2)$; that is, for all $t > 0$,

$$\frac{1 - F(tx)}{1 - F(x)} \to t^{-\alpha} \text{ as } x \to \infty.$$

We will also assume (merely for convenience) that F is a continuous distribution and $F(0) = 0$.

The discrete last-passage percolation model with underlying weight distribution F is usually defined as follows.

Let $X(i, j), i, j \in \mathbb{N}$ be i.i.d. with common distribution F. The quantity $X(i, j)$ represents the *weight* at the site $(i, j) \in \mathbb{Z}_+^2$.

For $n \in \mathbb{N}$, we will define the quantity $T^{(n)}$, the *last-passage time* between $(1, 1)$ and (n, n). Let Π_n be the set of *directed paths* between $(1, 1)$ and (n, n). Each such path begins at $(1, 1)$ and ends at (n, n), and each step consists of increasing one of the two coordinates by 1. That is, for any $\pi \in \Pi_n$, we can write $\pi = (v_1, v_2, \ldots, v_{2n})$, where $v_1 = (1, 1)$, $v_{2n} = (n, n)$, and, for each $i = 1, \ldots, 2n - 1$, $v_{i+1} - v_i$ is either $(1, 0)$ or $(0, 1)$.

The weight of such a path is the sum of the weights $X(i, j)$ associated with the points (i, j) in the path. Then $T^{(n)}$ is the maximal weight of a directed path between $(1, 1)$ and (n, n); that is:

$$T^{(n)} = \max_{\pi \in \Pi_n} \sum_{v \in \pi} X(v). \tag{2.1}$$

Note that $T^{(n)}$ depends only on the weights $X(v)$, $v \in \{1, \ldots, n\}^2$.

2.2 Continuous model

We start with an alternative representation of the discrete model. Let $M_1^{(n)} \geq M_2^{(n)} \geq \cdots \geq M_{n^2}^{(n)}$ be the order statistics, written in decreasing order, from an i.i.d. sample of size n^2 from the distribution F.

Consider the set $\{1/n, 2/n, \ldots, (n-1)/n, 1\}^2 \subset [0, 1]^2$, of size n^2. Let the sequence $Y_1^{(n)}, \ldots, Y_{n^2}^{(n)}$ consist of a random ordering of the points of this set, chosen uniformly from the $(n^2)!$ possibilities. We regard $Y_i^{(n)}$ as the location of the ith largest weight $M_i^{(n)}$ (and we have scaled so that all points lie in the box $[0, 1]^2$).

For two points $y, y' \in [0, 1]^2$, we say that y and y' are *compatible*, and write $y \sim y'$, if y, y' are partially ordered in that *either* $y \leq y'$ co-ordinatewise, *or* $y' \leq y$ co-ordinatewise. (Informally, one of y and y' is below and to the left of the other). An increasing path will consist of a set of points such that every pair of points in the set is compatible. We describe the collection of increasing paths by the collection $\mathcal{C}^{(n)}$, which depends on the points $Y_1^{(n)}, \ldots, Y_{n^2}^{(n)}$ alone:

$$\mathcal{C}^{(n)} = \mathcal{C}^{(n)}(Y_1^{(n)}, \ldots, Y_{n^2}^{(n)}) = \left\{ A \subseteq \{1, \ldots, n^2\} \text{ such that for all } i, j \in A, Y_i^{(n)} \sim Y_j^{(n)} \right\}. \tag{2.2}$$

Now we can give a new definition for $T^{(n)}$, equivalent (in distribution) to (2.1):

$$T^{(n)} = \max_{A \in \mathcal{C}^{(n)}} \sum_{i \in A} M_i^{(n)}. \tag{2.3}$$

We formulate the limiting continuous model by defining the distribution of a random variable T in an analogous way.

First let $Y_1, Y_2, \dots$ be an i.i.d. sequence, with each Y_i uniformly distributed on the square $[0,1]^2$. Let $W_1, W_2, \dots$ be an i.i.d. sequence of exponential random variables with mean 1 (independent of the (Y_i)). Now write, for each $k \in \mathbb{N}$, $M_k = (W_1 + \ldots + W_k)^{-1/\alpha}$. (Then with probability 1, $M_k > M_{k+1}$ for each k, and $M_k \to 0$ as $k \to \infty$). The motivation for this definition is given by equation (2.6) below. M_k is the kth largest weight, which we imagine positioned at the point $Y_k \in [0,1]^2$. (The set of locations Y_k is of course dense in $[0,1]^2$ with probability 1).

Analogously to (2.2), we represent the set of increasing paths by the collection $\mathcal{C}$:

$$\mathcal{C} = \mathcal{C}(Y_1, Y_2, \dots) = \{A \subseteq \{1, 2, \dots\} \text{ such that for all } i, j \in A, Y_i \sim Y_j\}. \tag{2.4}$$

Then define

$$T = \sup_{A \in \mathcal{C}} \sum_{i \in A} M_i. \tag{2.5}$$

Remark: Note that one could equivalently define T in (2.5) as the sup of the weight of *finite* increasing paths A, since either the sup is finite in which case the weight of any infinite path can be arbitrarily closely approximated by that of a finite path, or the sup is infinite in which case one can find a finite path with an arbitrarily large weight. In particular T can be seen as the supremum of a countable family of measurable random variables, and so is itself measurable. In Section 7 an equivalent construction of the continuous last-passage problem is given using a Poisson random measure approach rather than the sequence of i.i.d. uniform positions in the unit square described above.

2.3 Convergence results

First note that for all k,

$$\left(Y_1^{(n)}, Y_2^{(n)}, \dots Y_k^{(n)}\right) \to (Y_1, Y_2, \dots, Y_k)$$

in distribution, as $n \to \infty$.

Now define $a_N = F^{(-1)}\left(1 - \frac{1}{N}\right)$. (As an example, if the weight distribution F is Pareto(α), with $F(x) = 1 - x^{-\alpha}$, then $a_N = N^{1/\alpha}$. In general, $\lim_{N\to\infty} \log a_N / \log N = 1/\alpha$).

Recall that $M_k^{(n)}$ is the kth largest value from a sample of size n^2 from the distribution F. We write $\tilde{M}_i^{(n)} = a_{n^2}^{-1} M_i^{(n)}$. Then from classical extreme value theory we have that, for all k,

$$\left(\tilde{M}_1^{(n)}, \tilde{M}_2^{(n)}, \dots \tilde{M}_k^{(n)}\right) \to (M_1, M_2, \dots, M_k) \tag{2.6}$$

in distribution, as $n \to \infty$ (see for example Section 9.4 of [10]).

In particular, $M_i^{(n)}$ is asymptotically of the order of a_{n^2}, for any i. (For example, for the Pareto distribution $F(x) = 1 - x^{-\alpha}$ mentioned above, we have $a_{n^2} = n^{2/\alpha}$). Since certainly

$T^{(n)} \geq M_1^{(n)}$, we have that $T^{(n)}$ grows asymptotically at least on the order of a_{n^2}. In fact, we will show that this lower bound gives the right order of magnitude.

Specifically, let $\tilde{T}^{(n)} = a_{n^2}^{-1} T^{(n)} = \sup_{A \in \mathcal{C}^{(n)}} \sum_{i \in A} \tilde{M}_i^{(n)}$. Then we will show:

Theorem 2.1 *The random variable T defined at (2.5) is almost surely finite, and $\tilde{T}^{(n)} \to T$ in distribution as $n \to \infty$.*

For comparison, one can consider the case of a lighter tail. If $\int_0^\infty [1 - F(x)]^{1/2} dx < \infty$ (this condition is very slightly stronger than the existence of a finite second moment) then a law of large numbers holds: $n^{-1} T^{(n)} \to \gamma$ as $n \to \infty$ for some deterministic γ [21]. If the weights are exponential with mean 1, then $\gamma = 4$ [27], and then in fact one has the much finer convergence result that $n^{-1/3}(T^{(n)} - 4n)$ converges in distribution as $n \to \infty$, to the *GUE Tracy-Widom distribution* [16].

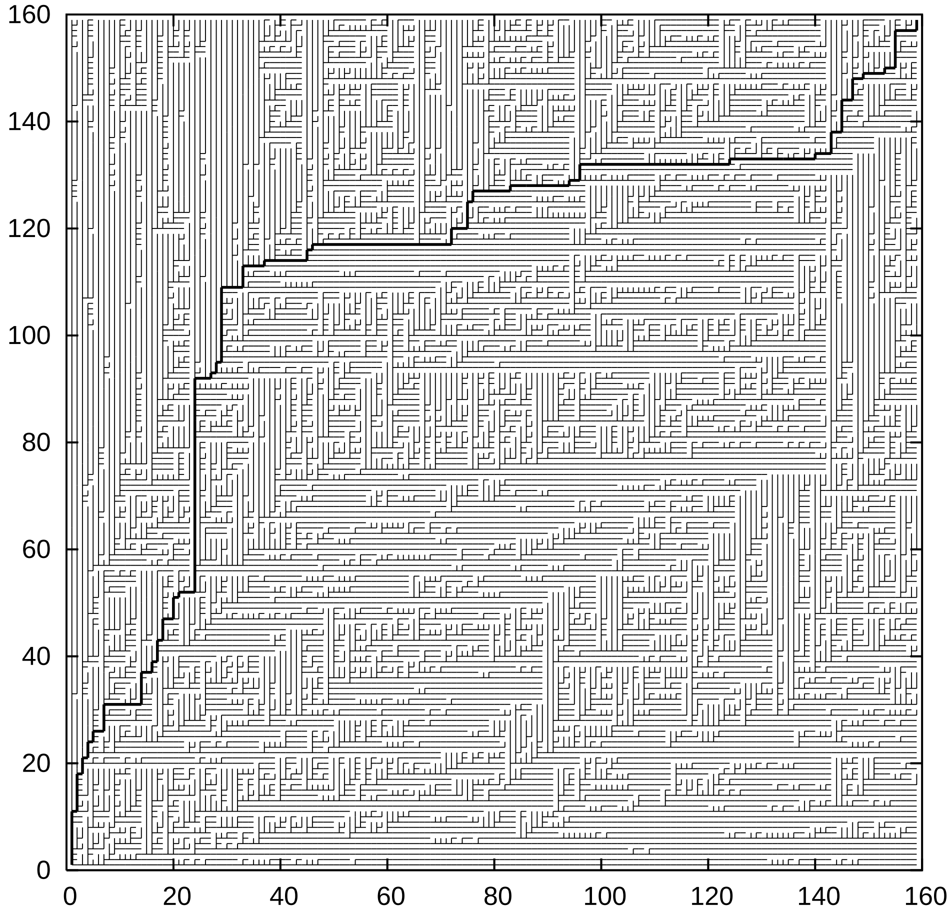

Figure 2.1: A simulation of the last-passage percolation model with F given by a Pareto distribution with index 1. The "tree" consisting of optimal paths from $(1, 1)$ to (i, j), for all $1 \leq i, j \leq 159$ is displayed; the thickened path is the optimal path from $(1, 1)$ to $(159, 159)$. This represents $P^{(n)*}$ in the language of Theorem 4.4.

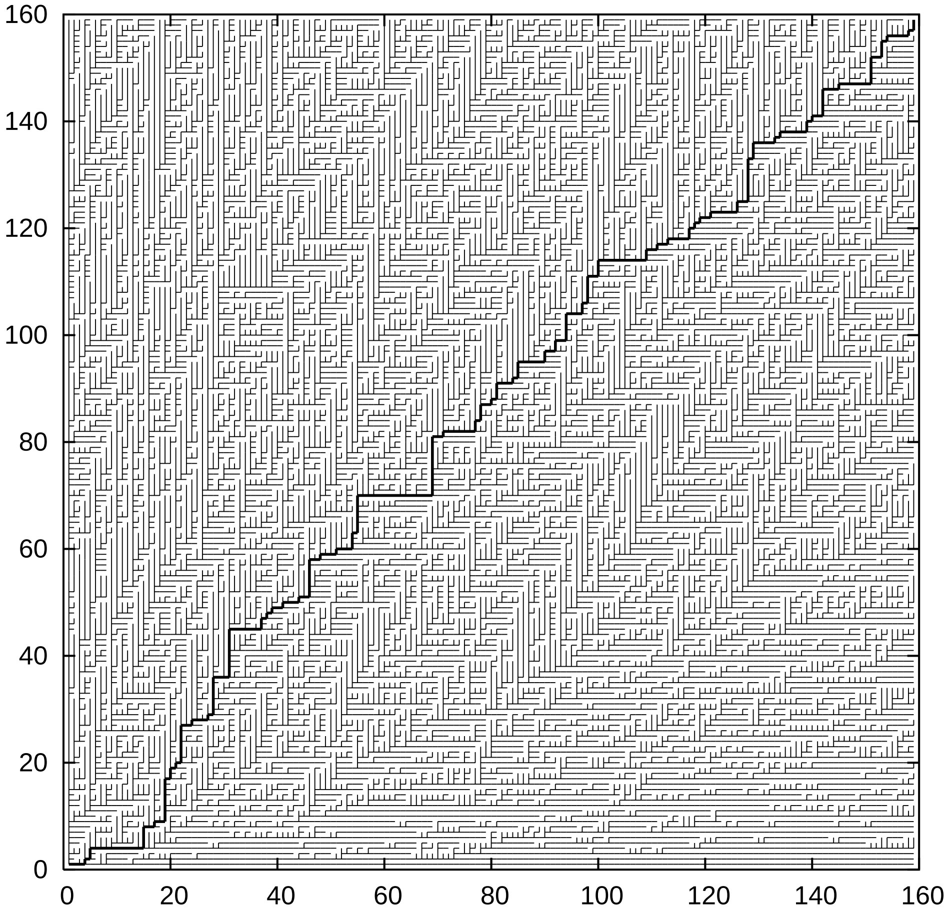

Figure 2.2: As Figure 2.1, but now with F given by an exponential distribution. As n grows, the distribution of the optimal path from $(1,1)$ to (n,n) becomes concentrated around the diagonal – the deviations of the path from a straight line are on the order of $n^{2/3}$. This contrasts with the deviations on the order of n observed in the heavy-tailed case, where the limiting path distribution is non-degenerate (as illustrated in Figure 2.1).

We now outline the results on path convergence, which are given in full in Section 4. We will show that the optimal path for the continuous model is well defined; that is, with probability 1 there exists a unique $A^* \in \mathcal{C}$ such that $T = \sum_{i \in A^*} M_i$ (attaining the sup in (2.5)). Then there is a unique closed connected set $P^* \subset [0,1]^2$ which contains all the points $Y_i, i \in A^*$ and which itself has the directed path property (i.e. $y \sim y'$ for all $y, y' \in P^*$).

Analogously, let $A^{(n)*}$ be the optimal path for the discrete model, achieving the max in (2.3) (Since the weight distribution F is continuous, the finitely many increasing paths all have different weights a.s., and so this maximizing path is a.s. unique). Let $P^{(n)*}$ be obtained by linear interpolation between the locations $\{Y_i^{(n)} : i \in A^{(n)*}\}$ of weights used in this optimal path, taken in increasing order. Then we show that the distribution of $P^{(n)*}$ converges to that of P^* as $n \to \infty$ (under the Hausdorff metric on closed subsets of $[0,1]^2$).

In the corresponding situation for weights with exponential distribution, the optimal path converges instead to a trivial limit, the straight line from $(0,0)$ to $(1,1)$. In general, the deviations of the optimal path from $(1,1)$ to (n,n) in the discrete model are expected to be of the order of $n^{2/3}$ in cases falling into the KPZ universality class (proved rigorously in certain cases [17, 4]), rather than on the order of n as we see in the heavy-tailed case. See Figures 2.1 and 2.2 for simulations of optimal paths in the cases of weight distributions which are Pareto and exponential.

2.4 Last-passage random fields and the heavy-tailed Airy process

In Section 7 we show how the results described above can be extended to give the multivariate convergence of vectors of passage times to different points.

For $x, y > 0$, let $T^{(n)}(x,y)$ be the maximal weight of a path from $(1,1)$ to $(\lceil nx \rceil, \lceil ny \rceil)$. (So for example the quantity $T^{(n)}$ defined in (2.3) is equal to $T^{(n)}(1,1)$).

Define also $\tilde{T}^{(n)}(x,y) = a_{n^2}^{-1} T^{(n)}(x,y)$ as before.

Then we will construct a random field $\{T(x,y), x, y > 0\}$, using a Poisson random measure construction rather than the sequence of points ordered in decreasing order of weight above, in such a way that

$$\left\{\tilde{T}^{(n)}(x,y), x, y > 0\right\} \to \left\{T(x,y), x, y > 0\right\}$$

as $n \to \infty$, in the sense of convergence of finite-dimensional distributions.

From a scaling property of the distribution of the weights one has further that the random field defined by

$$\Theta(u,v) = \exp\left(-\frac{u+v}{\alpha}\right) T\left(e^u, e^v\right)$$

is *stationary* on $\mathbb{R}^2$. The convergence above can be rewritten as

$$\left\{\exp\left(-\frac{u+v}{\alpha}\right) \tilde{T}^{(n)}\left(e^u, e^v\right), u, v \in \mathbb{R}\right\} \to \left\{\Theta(u,v), u, v \in \mathbb{R}\right\}. \tag{2.7}$$

To remove the multiplicative factor on the LHS of (2.7), one can look at a line $u+v = \text{const}$; for example, the process $H_y = \Theta(y, -y)$. This process is stationary in $y \in \mathbb{R}$, and we obtain the weak convergence

$$\left\{a_{n^2}^{-1} T^{(n)}\left(e^y, e^{-y}\right), y \in \mathbb{R}\right\} \to \left\{\Theta(y,-y), y \in \mathbb{R}\right\}.$$

This gives an analogy with the "Airy process" [18] [25], which arises for example in the case where the underlying weight distribution is exponential (with mean 1, say). There one obtains a stationary process limit for the quantities

$$\left\{n^{-1/3}\left[T^{(n)}(1 + yn^{-1/3}, 1 - yn^{-1/3}) - 4n\right], y \in \mathbb{R}\right\},$$

whose marginals are given by the GUE Tracy-Widom distribution. Simulations of the "heavy-tailed Airy process" H_y are given in Figures 7.1-7.3.

We also obtain estimates on the moments and correlations of the random field T, showing for example that $\mathbb{E}\, T(x,y)^\beta < \infty$ for all $\beta < \alpha$ and giving bounds for $\mathbb{E}\,|T(x,y) - T(x',y')|^\beta$.

We note that the path convergence described above could also be extended to the multivariate setting, to describe the convergence of the distribution of trees of optimal paths to a continuous tree structure given by the set of optimal paths in the continuous last-passage percolation model. However we do not pursue this further in this paper.

3 Convergence of the last-passage time distribution

To establish the convergence in Theorem 2.1, we will work with approximations to T and $T^{(n)}$ which depend only on the k largest weights. First, define

$$\mathcal{C}_k = \mathcal{C}(Y_1, Y_2, \dots, Y_k) = \{A \subseteq \{1, 2, \dots, k\} \text{ such that for all } i, j \in A, Y_i \sim Y_j\} .$$
$$\mathcal{C}_k^{(n)} = \mathcal{C}_k^{(n)}(Y_1^{(n)}, \dots, Y_{k\wedge n^2}^{(n)}) = \left\{A \subseteq \{1, \dots, k \wedge n^2\} \text{ such that for all } i, j \in A, Y_i^{(n)} \sim Y_j^{(n)}\right\} .$$

Note that in fact $\mathcal{C}_k = \{A \in \mathcal{C} : A \subseteq \{1, \dots, k\}\} = \{A \cap \{1, \dots, k\} : A \in \mathcal{C}\}$, and similarly for $\mathcal{C}_k^{(n)}$.

Now let

$$T_k = \sup_{A\in\mathcal{C}} \sum_{i\in A, i\le k} M_i,$$

and

$$T_k^{(n)} = \sup_{A\in\mathcal{C}^{(n)}} \sum_{i\in A, i\le k} M_i^{(n)}.$$

Note that indeed T_k depends only on $(M_1, \dots, M_k)$ and $(Y_1, \dots, Y_k)$, while $T_k^{(n)}$ depends only on $(M_1^{(n)}, \dots, M_k^{(n)})$ and $(Y_1^{(n)}, \dots, Y_k^{(n)})$.

As before, define $\tilde{T}_k^{(n)} = a_{n^2}^{-1} T_k^{(n)}$.

We also define the "remainder terms" S_k, $S_k^{(n)}$ by

$$S_k = \sup_{A\in\mathcal{C}} \sum_{i\in A, i>k} M_i \quad \text{and} \quad S_k^{(n)} = \sup_{A\in\mathcal{C}^{(n)}} \sum_{i\in A, i>k} M_i^{(n)}.$$

Write also $\tilde{S}_k^{(n)} = a_{n^2}^{-1} S_k^{(n)}$.

Lemma 3.1 *With probability 1, $S_k < \infty$ for all $k \ge 0$, and $S_k \to 0$ as $k \to \infty$.*

In particular, putting $k = 0$ we will have that $T < \infty$ a.s. (Later on, we will show more, namely that $\mathbb{E}\, T^\beta < \infty$ for all $0 < \beta < \alpha$; see Proposition 7.2).

We will also have that $T_k \to T$ a.s. as $k \to \infty$, since, for all k,

$$\begin{aligned} 0 \le T - T_k &= \sup_{A\in\mathcal{C}} \sum_{i\in A} M_i - \sup_{A\in\mathcal{C}} \sum_{i\in A, i\le k} M_i \\ &\le \sup_{A\in\mathcal{C}} \sum_{i\in A, i>k} M_i \\ &= S_k. \end{aligned}$$

The convergence in Theorem 2.1 will then follow from the following two results, which provide control over $T_k - \tilde{T}_k^{(n)}$ and $\tilde{T}_k^{(n)} - \tilde{T}^{(n)}$ for appropriate k:

Proposition 3.2 *Let $\epsilon > 0$ and k be fixed. Then for all n sufficiently large, say $n \geq N_k(\epsilon)$, there is a coupling of the continuous model and the discrete model indexed by n under which*

$$\mathbb{P}\left(\sum_{i=1}^{k}\left|M_i - \tilde{M}_i^{(n)}\right| > \epsilon\right) \leq \epsilon, \tag{3.1}$$

$$\mathbb{P}\left(\sum_{i=1}^{k}\left\|Y_i - Y_i^{(n)}\right\| > \epsilon\right) \leq \epsilon, \tag{3.2}$$

$$\mathbb{P}\left(\mathcal{C}_k^{(n)} \neq \mathcal{C}_k\right) \leq \epsilon. \tag{3.3}$$

Proposition 3.3 *Let $\epsilon > 0$. Then for k sufficiently large,*

$$\mathbb{P}\left(\tilde{S}_k^{(n)} > \epsilon\right) \leq \epsilon$$

for all n.

Proof of Lemma 3.1:

First, we define $L_i = \sup_{A \in \mathcal{C}} |A \cap \{1, \ldots, i\}|$. L_i is the largest number of the points $Y_1, \ldots, Y_i$ (the locations of the i largest weights) that can be included in an increasing path. Note that the collection (L_i) is independent of the collection (M_i). L_i has the distribution of the "longest increasing subsequence" of a random permutation of length i. In particular, there is a constant c such that, for all i, $\mathbb{E}\, L_i \leq c\sqrt{i}$ and $\mathbb{E}\, L_i^2 \leq ci$; also, $L_i/\sqrt{i} \to 2$ in distribution. See for example [1] for a survey.

We will also write $U_k = \sum_{i=k+1}^{\infty} L_i(M_i - M_{i+1})$ for each $k \geq 0$. Fix $A \in \mathcal{C}$, and define $R_i = |A \cap \{1, \ldots, i\}|$. Then $I(i \in A) = R_i - R_{i-1}$, and by definition $R_i \leq L_i$.

We have

$$\begin{aligned}
\sum_{i \in A, i > k} M_i &= \lim_{n\to\infty} \sum_{i \in A, k < i \leq n} M_i \\
&= \lim_{n\to\infty} \sum_{i=k+1}^{n} M_i I(i \in A) \\
&= \lim_{n\to\infty} \sum_{i=k+1}^{n} M_i (R_i - R_{i-1}) \\
&= \lim_{n\to\infty} \left[-M_{k+1}R_k + \sum_{i=k+1}^{n-1} R_i(M_i - M_{i+1}) + M_n R_n\right] \\
&\leq \lim_{n\to\infty} \sum_{i=k+1}^{n-1} R_i(M_i - M_{i+1}) + \liminf_{n\to\infty} M_n R_n \\
&\leq \lim_{n\to\infty} \sum_{i=k+1}^{n-1} L_i(M_i - M_{i+1}) + \liminf_{n\to\infty} M_n L_n \\
&= U_k + \liminf_{n\to\infty} M_n L_n.
\end{aligned}$$

Now $\liminf_{n\to\infty} M_n L_n = 0$ a.s.; this follows, for example, since (by the law of large numbers) $M_n \sim n^{-1/\alpha}$ a.s. (with $\alpha < 2$), and since $L_n/\sqrt{n}$ converges in distribution to a constant.

Since the inequality above holds for any $A \in \mathcal{C}$, we therefore have that $S_k \le U_k$ a.s., for any k. To conclude the proof, we will show that with probability 1, $U_k < \infty$ for all k, and $U_k \to 0$ as $k \to \infty$. (In fact, as soon as $U_1 < \infty$, we necessarily have that $U_k \to 0$ as $k \to \infty$, since the quantity U_k is the "remainder" from index $k+1$ onwards in the infinite sum U_1; if the infinite sum U_1 converges, then by definition these remainders tend to 0).

Hence it's enough that $U_k < \infty$ a.s., for all k. Specifically, we'll show that $\mathbb{E}\, U_k$ is finite whenever $k > 1/\alpha$. Then certainly $U_k < \infty$ a.s. for such k, and in fact $U_r < \infty$ a.s. for all r, since if $r < k$, $U_r - U_k$ is the sum of only finitely many terms.

By independence of the collections (L_i) and (M_i),

$$\begin{aligned} \mathbb{E}\, U_k &= \sum_{i=k+1}^{\infty} \mathbb{E}\, L_i (\mathbb{E}\, M_i - \mathbb{E}\, M_{i+1}) \\ &\le \sum_{i=k+1}^{\infty} c i^{1/2} (\mathbb{E}\, M_i - \mathbb{E}\, M_{i+1}). \end{aligned} \tag{3.4}$$

Now M_r has the distribution of $(V_r)^{-1/\alpha}$, where V_r has Gamma$(r,1)$ distribution. We then obtain

$$\begin{aligned} \mathbb{E}\, M_r &= \int_0^{\infty} \frac{1}{\Gamma(r)} v^{r-1} e^{-v} v^{-1/\alpha} dv \\ &= \Gamma\big(r - 1/\alpha\big)/\Gamma(r). \end{aligned} \tag{3.5}$$

Using the identity $\Gamma(z+1) = z\Gamma(z)$ and the fact that the gamma function is log convex, one has

$$(x-1)^a \le \frac{\Gamma(x+a)}{\Gamma(x)} \le (x+a)^a \tag{3.6}$$

for $x > 1$, $a < 0$. Then

$$\begin{aligned} \mathbb{E}\, M_r - \mathbb{E}\, M_{r+1} &= \frac{\Gamma(r-1/\alpha)}{\Gamma(r)} - \frac{\Gamma(r+1-1/\alpha)}{\Gamma(r+1)} \\ &= \frac{\Gamma(r-1/\alpha)\left[1 - \frac{r-1/\alpha}{r}\right]}{\Gamma(r)} \\ &= \frac{1}{\alpha r} \frac{\Gamma(r-1/\alpha)}{\Gamma(r)} \\ &\le \frac{1}{\alpha r} \big(r - 1/\alpha - 1\big)^{-1/\alpha}. \end{aligned}$$

Returning to (3.4), we have

$$\mathbb{E}\, U_k \le \frac{c}{\alpha} \sum_{i=k+1}^{\infty} i^{-1/2} \big(i - 1/\alpha - 1\big)^{-1/\alpha},$$

which is finite for all $k > 1/\alpha$ (since $\alpha < 2$). □

Proof of Theorem 2.1: We will find a coupling of $\tilde{T}^{(n)}$ and T for each n such that $\tilde{T}^{(n)} - T \to 0$ in probability as $n \to \infty$.

For each $n \in \mathbb{N}$, define $k_n = \max\{k : n \ge N_k(1/k)\}$.

Then $k_n \to \infty$ as $n \to \infty$, and, for all n, $n \geq N_{k_n}(1/k_n)$. Hence from Propositions 3.2 and 3.3 and from Lemma 3.1, there are couplings such that, as $n \to \infty$,

$$\sum_{i=1}^{k_n} \left| M_i - \tilde{M}_i^{(n)} \right| \to 0 \tag{3.7}$$
$$\sum_{i=1}^{k_n} \left\| Y_i - Y_i^{(n)} \right\| \to 0$$
$$\tilde{S}_{k_n}^{(n)} \to 0$$
$$S_{k_n} \to 0$$

in probability, and

$$\mathbb{P}\left(\mathcal{C}_{k_n}^{(n)} \neq \mathcal{C}_{k_n} \right) \to 0.$$

Now

$$T - \tilde{T}^{(n)} = (T - T_{k_n}) + \left(T_{k_n} - \tilde{T}_{k_n}^{(n)} \right) + \left(\tilde{T}_{k_n}^{(n)} - \tilde{T}^{(n)}. \right)$$

We have $|T - T_{k_n}| \leq S_{k_n}$ and $|\tilde{T}_{k_n}^{(n)} - \tilde{T}^{(n)}| \leq \tilde{S}_{k_n}^{(n)}$, so to show that the LHS converges to 0 in probability as desired, it remains to show that $\left(T_{k_n} - \tilde{T}_{k_n}^{(n)} \right) \to 0$ in probability.

We have

$$T_{k_n} = \max_{A \in \mathcal{C}_{k_n}} \sum_{i \in A} M_i \qquad \text{and} \qquad \tilde{T}_{k_n}^{(n)} = \max_{A \in \mathcal{C}_{k_n}^{(n)}} \sum_{i \in A} \tilde{M}_i^{(n)},$$

so if $\mathcal{C}_{k_n} = \mathcal{C}_{k_n}^{(n)}$, then

$$\left| T_{k_n} - \tilde{T}_{k_n}^{(n)} \right| \leq \sum_{i=1}^{k_n} \left| M_i - \tilde{M}_i^{(n)} \right|.$$

Since $\mathbb{P}\left(\mathcal{C}_{k_n} \neq \mathcal{C}_{k_n}^{(n)} \right) \to 0$ and $\sum_{i=1}^{k_n} \left| M_i - \tilde{M}_i^{(n)} \right| \to 0$ in probability, we are done. $\square$

To complete the proof it remains to prove Propositions 3.2 and 3.3.

3.1 Convergence of $\tilde{T}_k^{(n)}$ to T_k

Proof of Proposition 3.2:

We have $(\tilde{M}_1^{(n)}, \dots, \tilde{M}_k^{(n)}, Y_1^{(n)}, \dots, Y_k^{(n)}) \to (M_1, \dots, M_k, Y_1, \dots, Y_k)$ in distribution as $n \to \infty$. By the Skorohod Representation Theorem, we can define all the variables on the same space in such a way that the convergence occurs almost surely. Then indeed (3.1) and (3.2) must hold for large enough n.

Note that since the variables Y_i are i.i.d. uniform on $[0,1]^2$, there are almost surely no two i and j such that $Y_i(d) = Y_j(d)$ for $d = 1$ or 2.

Thus if we perturb the point $(Y_1, \dots, Y_k)$ by a small enough amount, the orderings of all the coordinates remain the same, and the set $\mathcal{C}_k$ of increasing paths is unchanged. In fact, if

$$\max_{1 \leq i \leq k} \|Y_i - Y_i^{(n)}\| \leq \frac{1}{2} \min_{1 \leq i,j \leq k, i \neq j} \min_{d=1,2} |Y_i(d) - Y_j(d)|,$$

then $\mathcal{C}_k^{(n)} = \mathcal{C}_k$. Since we have $Y_i^{(n)} \to Y_i$ a.s. on the joint probability space for all $1 \leq i \leq k$, we then have $\mathcal{C}_k^{(n)} = \mathcal{C}_k$ eventually, with probability 1. Thus (3.3) must also hold for all large enough n, as desired. $\square$

3.2 Convergence of $\tilde{T}^{(n)} - \tilde{T}_k^{(n)}$ to 0

Our aim in this section is to prove Proposition 3.3.

Define the "good event" $\mathcal{B}_k^{(n)}$:

$$\mathcal{B}_k^{(n)} = \left\{ F^{-1}\left(1 - \frac{2r}{n^2}\right) \le M_r^{(n)} \le F^{-1}\left(1 - \frac{1}{n^2}\right) \text{ for all } k < r \le n^2 \right\}. \tag{3.8}$$

Lemma 3.4 $\mathbb{P}\left(\mathcal{B}_k^{(n)}\right) \to 1$ *as* $k \to \infty$*, uniformly in* n*.*

Proof:

$$\begin{aligned}
\mathbb{P}\left(\mathcal{B}_k^{(n)} \text{ fails }\right) &\le \mathbb{P}\left(M_{k+1}^{(n)} > F^{-1}\left(1 - \frac{1}{n^2}\right)\right) + \sum_{r=k+1}^{\lfloor n^2/2 \rfloor} \mathbb{P}\left(M_r^{(n)} < F^{-1}\left(1 - \frac{2r}{n^2}\right)\right) \\
&= \mathbb{P}\left(\text{Binomial}(n^2, 1/n^2) \ge k+1\right) + \sum_{r=k+1}^{\lfloor n^2/2 \rfloor} \mathbb{P}\left(\text{Binomial}(n^2, 2r/n^2) < r\right) \\
&\le \frac{1}{k+1} + \sum_{r=k+1}^{\lfloor n^2/2 \rfloor} 2\exp\left(-\frac{r}{12}\right),
\end{aligned}$$

using Markov's inequality for the first term and an estimate from Corollary 2.3 of [15] for the second. The RHS tends to 0 as $k \to \infty$, uniformly in n, as required. $\square$

Now, we will prove a bound on the expectation of $\tilde{S}_k^{(n)}$ in terms of the "order statistics" $M_r^{(n)}$. Define

$$L_i^{(n)} = \max_{A \in \mathcal{C}^{(n)}} |A \cap \{1, 2, \ldots, i\}|.$$

Recall that $Y_r^{(n)} \in \{1, 2, \ldots, n\}^2$ is the location of the rth largest weight, $M_r^{(n)}$. Thus, $L_i^{(n)}$ is the maximum number of the points $Y_1^{(n)}, \ldots, Y_i^{(n)}$ that can be included in an increasing path.

Note that the collection $(L_r^{(n)})_{1 \le r \le n^2}$ is a function of the values $Y_r^{(n)}$ alone; in particular it is independent of the weights $M_r^{(n)}$ and of the events $B_r^{(n)}$.

Lemma 3.5 *There is a constant* c *independent of* m *and* n *such that* $\mathbb{E}\, L_m^{(n)} \le c\sqrt{m}$*, whenever* $1 \le m \le n^2$*.*

Proof: The distribution of $\{Y_1^{(n)}, \ldots, Y_m^{(n)}\}$ is uniform over the subsets of $\{1, \ldots, n\}^2$ of size m, and $L_r^{(n)}$ is the maximum number of the points $Y_1^{(n)}, \ldots, Y_r^{(n)}$ that can be included in an increasing path.

We compare this with the last-passage percolation problem in $\{1, \ldots, n\}^2$ with i.i.d. Bernoulli(p) weights.

We have the following representation for the expectation of the passage time for such a problem:

$$\mathbb{E}_{\,\text{Ber}(p)} T(n, n) = \sum_r \mathbb{P}(B_{p,n^2} = r) \mathbb{E}\, L_r^{(n)}, \tag{3.9}$$

where B_{p,n^2} has Binomial(p, n^2) distribution, since, conditional on the event that exactly r of the n^2 weights have value 1, the set of positions of the weights with value 1 is uniformly distributed among all the subsets of $\{1, \ldots, n\}^2$ of size r.

Proposition 2.2 of [21] shows that there exists a constant c_1 such that

$$\mathbb{E}_{\mathrm{Ber}(p)} T(n,n) \le c_1 p^{1/2} n \tag{3.10}$$

for all n, p.

Given $1 \le m \le n^2$, set $p = \min(2m/n^2, 1)$. Using (3.9), (3.10) and the fact that $\mathbb{E}\, L_r^{(n)}$ is increasing in r, we obtain

$$\begin{aligned}
\mathbb{P}(B_{p,n^2} \ge m)\mathbb{E}\, L_m^{(n)} &\le \sum_r \mathbb{P}(B_{p,n^2} = r)\mathbb{E}\, L_r^{(n)} \\
&\le c_1 \left(\min(2m/n^2, 1)\right)^{1/2} n \\
&\le c_2 \sqrt{m}.
\end{aligned}$$

To complete the proof, it then suffices to bound $\mathbb{P}(B_{p,n^2} \ge m)$ away from 0 uniformly in $1 \le m \le n^2$.

If $m \ge n^2/2$ then $p = 1$ and $\mathbb{P}(B_{p,n^2} \ge m) = 1$.

For $1 \le m \le n^2/2$, we have $pn^2/2 = m$, and we use the estimate

$$\mathbb{P}(B_{p,n^2} < pn^2/2) \le \exp(-pn^2/8),$$

(see for example Theorem 2.1 of [15]), to give $\mathbb{P}(B_{p,n^2} \ge m) \ge 1 - \exp(-1/8)$ uniformly in $1 \le m \le n^2/2$ as desired. □

Lemma 3.6

$$\mathbb{E}\left(\tilde{S}_k^{(n)}; \mathcal{B}_k^{(n)}\right) \le c(k+1)^{1/2} \mathbb{E}\left(\tilde{M}_{k+1}^{(n)}; \mathcal{B}_k^{(n)}\right) + c \sum_{r=k+2}^{n^2} r^{-1/2} \mathbb{E}\left(\tilde{M}_r^{(n)}; \mathcal{B}_k^{(n)}\right).$$

Proof: The argument is similar to the proof of Lemma 3.1.

Let $\tilde{A}$ achieve the max in the definition of $\tilde{S}_k^{(n)}$, so that $\tilde{S}_k^{(n)} = \sum_{i\in\tilde{A}, i>k} \tilde{M}_i^{(n)}$.

Define $R_i = |\tilde{A} \cap \{1, 2, \ldots, i\}|$ for each i. Then $R_i - R_{i-1} = I(i \in \tilde{A})$, and by definition $R_i \le L_i^{(n)}$ for each i. We then have

$$\begin{aligned}
\tilde{S}_k^{(n)} &= \sum_{i\in\tilde{A}, i>k} \tilde{M}_i^{(n)} \\
&= \sum_{i=k+1}^{n^2} \tilde{M}_i^{(n)} (R_i - R_{i-1}) \\
&= -R_k \tilde{M}_{k+1}^{(n)} + \sum_{i=k+1}^{n^2-1} R_i \left(\tilde{M}_i^{(n)} - \tilde{M}_{i+1}^{(n)}\right) + R_{n^2} \tilde{M}_{n^2}^{(n)} \\
&\le \sum_{i=k+1}^{n^2-1} L_i^{(n)} \left(\tilde{M}_i^{(n)} - \tilde{M}_{i+1}^{(n)}\right) + L_{n^2}^{(n)} \tilde{M}_{n^2}^{(n)},
\end{aligned}$$

since $R_i^{(n)} \le L_i^{(n)}$ and $\tilde{M}_i^{(n)} \ge \tilde{M}_{i+1}^{(n)}$.

We now take expectations, restricted to the event $\mathcal{B}_k^{(n)}$, using Lemma 3.5 and the independence of the $L_r^{(n)}$ from the $\tilde{M}_r^{(n)}$:

$$\mathbb{E}\left(\tilde{S}_k^{(n)}; \mathcal{B}_k^{(n)}\right) \le \sum_{i=k+1}^{n^2-1} \mathbb{E}\, L_i^{(n)} \left[\mathbb{E}\left(\tilde{M}_i^{(n)}; \mathcal{B}_k^{(n)}\right) - \mathbb{E}\left(\tilde{M}_{i+1}^{(n)}; \mathcal{B}_k^{(n)}\right)\right] + \mathbb{E}\, L_{n^2}^{(n)} \mathbb{E}\left(\tilde{M}_{n^2}^{(n)}; \mathcal{B}_k^{(n)}\right)$$

$$
\begin{aligned}
&\leq \sum_{i=k+1}^{n^2-1} c\sqrt{i}\left[\mathbb{E}\left(\tilde{M}_i^{(n)};\mathcal{B}_k^{(n)}\right)-\mathbb{E}\left(\tilde{M}_{i+1}^{(n)};\mathcal{B}_k^{(n)}\right)\right]+cn\mathbb{E}\,L_{n^2}^{(n)}\mathbb{E}\left(\tilde{M}_{n^2}^{(n)};\mathcal{B}_k^{(n)}\right)\\
&= c\sqrt{k+1}\mathbb{E}\left(\tilde{M}_{k+1}^{(n)};\mathcal{B}_k^{(n)}\right)+c\sum_{i=k+2}^{n^2}\left(\sqrt{i}-\sqrt{i-1}\right)\mathbb{E}\left(\tilde{M}_i^{(n)};\mathcal{B}_k^{(n)}\right)\\
&\leq c\sqrt{k+1}\mathbb{E}\left(\tilde{M}_{k+1}^{(n)};\mathcal{B}_k^{(n)}\right)+c\sum_{i=k+2}^{n^2} i^{-1/2}\mathbb{E}\left(\tilde{M}_i^{(n)};\mathcal{B}_k^{(n)}\right),
\end{aligned}
$$

since $i^{1/2}-(i-1)^{1/2}\leq i^{-1/2}$. This is the required result. □

The next lemma gives an estimate on the tail behaviour of the weight distribution using the regular variation condition. Note that if the weight distribution were Pareto(α), then $F^{-1}(u)=(1-u)^{-1/\alpha}$, and one then has exactly $F^{-1}(u_1)=F^{-1}(u_0)\left[(1-u_1)/(1-u_0)\right]^{-1/\alpha}$ for any u_0, u_1.

Lemma 3.7 *For any $\delta>0$, there exists $U(\delta)<1$ such that for all u_0,u_1 with $U(\delta)\leq u_1\leq u_0$,*

$$
F^{-1}(u_1)\leq 2F^{-1}(u_0)\left(\frac{1-u_1}{1-u_0}\right)^{-\frac{1}{\alpha}+\delta}.
$$

Proof: Fix $s>1$ sufficiently small that $s^{1/\alpha-\delta}<2$.

From the fact that the tail of F is regularly varying with index α, the following property holds: if u_1 is sufficiently close to 1 (at least $U(\delta)$, say), then for all $(1-u_0)<(1-u_1)/s$,

$$
\frac{F^{-1}(u_1)}{F^{-1}(u_0)}<s^{-\frac{1}{\alpha}+\delta}.
$$

Iterating, one obtains that if $U(\delta)\leq u_1\leq u_0$, then

$$
\begin{aligned}
\frac{F^{-1}(u_1)}{F^{-1}(u_0)} &< \left(s^{-\frac{1}{\alpha}+\delta}\right)^{\lfloor\log_s((1-u_1)/(1-u_0))\rfloor}\\
&= \exp\left[\left(-\frac{1}{\alpha}+\delta\right)(\log s)\left\lfloor\frac{\log((1-u_1)/(1-u_0))}{\log s}\right\rfloor\right]\\
&\leq \exp\left[\left(-\frac{1}{\alpha}+\delta\right)\left(\log\frac{1-u_1}{1-u_0}-\log s\right)\right]\\
&= \left(\frac{1-u_1}{1-u_0}\right)^{-\frac{1}{\alpha}+\delta}s^{\frac{1}{\alpha}-\delta}\\
&\leq 2\left(\frac{1-u_1}{1-u_0}\right)^{-\frac{1}{\alpha}+\delta}
\end{aligned}
$$

as required. □

Finally we use this tail estimate to control the expectation of the variables $\tilde{M}_r^{(n)}$, restricted to the "good set" $\mathcal{B}_k^{(n)}$.

Lemma 3.8 *Let $\delta>0$. Then there exist c_0, c_1 and $c_2>0$ such that*

$$
\mathbb{E}\left(\tilde{M}_r^{(n)};\mathcal{B}_k^{(n)}\right)\leq c_0r^{-\frac{1}{\alpha}+\delta}+c_1a_{n^2}^{-1}I(r\geq c_2n^2)
$$

for all n, k, r satisfying $2(1+1/\alpha)<k<r\leq n^2$.

Proof: Let $U(\delta)$ be as in Lemma 3.7, and set $c_2 = (1-U(\delta))/2$. Then $r < c_2 n^2 \Leftrightarrow 1-2r/n^2 > U(\delta)$.

Note that if

$$\max\left\{U(\delta), 1-\frac{2r}{n^2}\right\} \leq u \leq 1-\frac{1}{n^2},$$

then, by Lemma 3.7,

$$\begin{aligned}
F^{-1}(u) &\leq 2F^{-1}\left(1-\frac{1}{n^2}\right)\left[n^2(1-u)\right]^{-\frac{1}{\alpha}+\delta} \\
&= 2a_{n^2} n^{-2/\alpha}(1-u)^{-1/\alpha}\left[n^2(1-u)\right]^{\delta} \\
&\leq 2a_{n^2} n^{-2/\alpha}(1-u)^{-1/\alpha}(2r)^{\delta}. \qquad (3.11)
\end{aligned}$$

Since $r > k$, we have that

$$\mathcal{B}_k^{(n)} \subseteq \left\{F^{-1}\left(1-\frac{2r}{n^2}\right) \leq M_r^{(n)} \leq F^{-1}\left(1-\frac{1}{n^2}\right)\right\}.$$

Hence

$$\begin{aligned}
&\mathbb{E}\left(\tilde{M}_r^{(n)};\mathcal{B}_k^{(n)}\right) \\
&\quad\leq a_{n^2}^{-1}\mathbb{E}\left(M_r^{(n)}; F^{-1}\left(1-\frac{2r}{n^2}\right) \leq M_r^{(n)} \leq F^{-1}\left(1-\frac{1}{n^2}\right)\right) \qquad (3.12) \\
&\quad= a_{n^2}^{-1}\mathbb{E}\left(F^{-1}\left(U_r^{(n)}\right); 1-\frac{2r}{n^2} \leq U_r^{(n)} \leq 1-\frac{1}{n^2}\right) \\
&\quad= a_{n^2}^{-1}\int_{1-2r/n^2}^{1-1/n^2} F^{-1}(u) f_{r;n^2}(u)du \\
&\quad= a_{n^2}^{-1} I\left\{1-\frac{2r}{n^2} \leq U(\delta)\right\} F^{-1}(U(\delta)) + a_{n^2}^{-1}\int_{\max\{U(\delta),1-2r/n^2\}}^{1-1/n^2} F^{-1}(u) f_{r;n^2}(u)du \\
&\quad\leq c_1 a_{n^2}^{-1} I\left(r \geq c_2 n^2\right) + \int_{\max\{U(\delta),1-2r/n^2\}}^{1-1/n^2} 2n^{-2/\alpha}(1-u)^{-1/\alpha}(2r)^{\delta} f_{r;n^2}(u)du \\
&\quad\leq c_1 I\left(r \geq c_2 n^2\right) + c_3 a_{n^2} n^{-2/\alpha} r^{\delta}\int_0^1 (1-u)^{-1/\alpha} f_{r;n^2}(u)du, \qquad (3.13)
\end{aligned}$$

where $c_1 = F^{-1}(U(\delta))$ and $c_3 = 2^{1+\delta}$, and where $f_{r;n^2}$ is the density function of the rth largest from an i.i.d. sample of size n^2 from the uniform distribution on $[0,1]$.

Now

$$f_{r;n^2}(u) = \frac{\Gamma(n^2+1)}{\Gamma(n^2-r+1)\Gamma(r)}(1-u)^{r-1}u^{n^2-r},$$

and, since $r-1-1/\alpha > 0$, one then has

$$\begin{aligned}
\int_0^1 (1-u)^{-1/\alpha} f_{r;n^2}(u)du &= \frac{\Gamma(n^2+1)}{\Gamma(n^2-r+1)\Gamma(r)}\int_0^1 (1-u)^{r-1-1/\alpha}u^{n^2-r}du \\
&= \frac{\Gamma(n^2+1)}{\Gamma(n^2-r+1)\Gamma(r)}\left(\frac{\Gamma(n^2+1-1/\alpha)}{\Gamma(n^2-r+1)\Gamma(r-1/\alpha)}\right)^{-1} \\
&= \frac{\Gamma(n^2+1)}{\Gamma(n^2+1-1/\alpha)}\frac{\Gamma(r-1/\alpha)}{\Gamma(r)}. \qquad (3.14)
\end{aligned}$$

Using (3.6), the RHS of (3.14) is bounded above by

$$\frac{(n^2+1)^{1/\alpha}}{(r-1-1/\alpha)^{1/\alpha}}.$$

Since $r > 2(1+1/\alpha)$, this is in turn no greater than $\left(4n^2/r\right)^{1/\alpha}$. Inserting this into (3.13) gives the desired result. $\square$

Proof of Proposition 3.3: We may assume that $k \le n^2$, since if $k > n^2$ then $\tilde{S}_k^{(n)} = 0$.

Fix $\epsilon > 0$, and fix some $\delta < \frac{1}{\alpha} - \frac{1}{2}$. Using Markov's inequality, we have

$$\mathbb{P}\left(\tilde{S}_k^{(n)} > \epsilon\right) \le \mathbb{P}(\mathcal{B}_k^{(n)} \text{ fails }) + \epsilon^{-1}\mathbb{E}\left(\tilde{S}_k^{(n)}; \mathcal{B}_k^{(n)}\right).$$

By Lemma 3.4, the first term tends to 0 as $k \to \infty$, uniformly in n. For the second term, Lemmas 3.6 and 3.8 combine to give

$$\mathbb{E}\left(\tilde{S}_k^{(n)}; \mathcal{B}_k^{(n)}\right) \le cc_0(k+1)^{-\frac{1}{\alpha}+\frac{1}{2}+\delta} + cc_0 \sum_{r=k+2}^{n^2} r^{-\frac{1}{\alpha}-\frac{1}{2}+\delta} + cc_1(k+1)^{\frac{1}{2}} a_{n^2}^{-1} + cc_1 a_{n^2}^{-1} \sum_{c_2 n^2 \le r \le n^2} r^{-\frac{1}{2}}.$$

From the choice of δ and the fact that $a_{n^2}/n \to \infty$ as $n \to \infty$, one obtains that all four terms on the RHS tend to 0 as $k \to \infty$ uniformly in n such that $k \le n^2$.

Hence indeed $\mathbb{P}\left(\tilde{S}_k^{(n)} > \epsilon\right) < \epsilon$ for all large enough k, uniformly in n such that $k \le n^2$, as required. $\square$

4 Path convergence

In this section we will state and prove results describing the convergence of the distribution of the optimal paths for the discrete models to that for the limiting continuous model.

First we note that the optimal path for the continuous model is well-defined:

Proposition 4.1 *With probability 1, there exists a unique $A^* \in \mathcal{C}$ such that $T = \sum_{i \in A^*} M_i$.*

We will also define $A^{(n)*}$ as the set that achieves the maximum in

$$T^{(n)} = \max_{A \in \mathcal{C}^{(n)}} \sum_{i \in A} M_i^{(n)}.$$

(or in the equivalent expression for $\tilde{T}^{(n)}$). (Since the weight distribution is assumed to be continuous, this optimal set is almost surely unique).

It will be useful to extend the sequences $(Y_i^{(n)})_i$ and $(\tilde{M}_i^{(n)})_i$ to all $i \in \mathbb{N}$ (rather than only $i \le n^2$); for example, we can put $\tilde{M}_i^{(n)} = 0$, $Y_i^{(n)} = (0,0)$ for all $i \ge n^2$. We will consider always the product topology when looking at convergence of such infinite sequences.

We also use the product topology on $\mathcal{S}$, the set of subsets of $\mathbb{N}$. Thus, given A and a sequence A_k in $\mathcal{S}$, we have $A_k \to A$ if, for every m, $A_k \cap \{1, \dots, m\}$ is equal to $A \cap \{1, \dots, m\}$ for all large enough k. One has easily that any sequence A_k has at least one limit point, and also that if $A_k \in \mathcal{C}$ for each k then every limit point is also in $\mathcal{C}$ (since if the limit point contains i and j, then i and j are in A_k for some k, and hence $Y_i \sim Y_j$).

The following theorem is our first path convergence result. Later (in Theorem 4.4) we will use it to prove a more direct convergence result concerning the optimal paths viewed as random subsets of $[0,1]^2$.

Theorem 4.2 $\left(\left(Y_i^{(n)}\right)_{i\in\mathbb{N}}, A^{(n)*}\right) \to \left(\left(Y_i\right)_{i\in\mathbb{N}}, A^*\right)$ *in distribution as* $n \to \infty$.

Before proving Proposition 4.1 and Theorem 4.2, we need the following fact:

Lemma 4.3 *With probability 1, the following holds: if* A_j *is a sequence in* $\mathcal{C}$ *converging to a limit* A, *then* $\lim_{j\to\infty} \sum_{i\in A_j} M_i = \sum_{i\in A} M_i$.

[N.B. this result is not true in general for sequences A_j in $\mathcal{S}$ (unless $\alpha < 1$ so that $\sum M_i < \infty$ a.s.)]

Proof: If $A_j \cap \{1,\ldots,m\} = A \cap \{1,\ldots,m\}$ then

$$\left|\sum_{i\in A_j} M_i - \sum_{i\in A} M_i\right| \le \sup_{\tilde{A}\in\mathcal{C}} \sum_{i\in\tilde{A}, i>m} M_i = S_m.$$

But $S_m \to 0$ a.s. as $m \to \infty$ (from Lemma 3.1), and $A_j \cap \{1,\ldots,m\} = A \cap \{1,\ldots,m\}$ eventually for all m, so we are done. $\square$

Proof of Proposition 4.1: Recall that

$$T_k = \sup_{A\in\mathcal{C}} \sum_{i\in A, i\le k} M_i.$$

Since the sum on the RHS depends only on the intersection of A with $\{1,\ldots,k\}$, we only need to consider the max over finitely many $A \subseteq \{1,\ldots,k\}$. Thus there exists some A_k^* which achieves the sup.

We consider the sequence A_k^*. As observed above, this sequence has at least one limit point $A^* \in \mathcal{C}$, and by Lemma 4.3, $\sum_{i\in A^*} M_i = \lim_{k\to\infty} \sum_{i\in A_k^*} M_i = \lim_{k\to\infty} T_k = T$.

Now we wish to show that in fact a unique A^* achieves the sum T.

Suppose instead that there are two such optimising sets in $\mathcal{C}$. Then there is some k that is contained in one but not the other.

In that case,

$$\begin{aligned}\sup_{A\in\mathcal{C}, k\notin A} \sum_{i\in A} M_i &= \sup_{A\in\mathcal{C}, k\in A} \sum_{i\in A} M_i \\ &= M_k + \sup_{A\in\mathcal{C}, k\in A} \sum_{i\in A, i\ne k} M_i,\end{aligned}$$

which gives

$$M_k = \sup_{A\in\mathcal{C}, k\notin A} \sum_{i\in A} M_i - \sup_{A\in\mathcal{C}, k\in A} \sum_{i\in A, i\ne k} M_i. \tag{4.1}$$

We will show that this event has probability 0 for each k; then, by countable additivity, we are done.

The RHS of (4.1) does not depend on M_k; in fact, it is a function of the collection $\left((M_i)_{i\in\mathbb{N}\setminus\{k\}},(Y_i)_{i\in\mathbb{N}}\right)$. We condition on the value of this collection. Then the RHS is a constant, while the random variable M_k on the LHS has a continuous distribution. (Specifically, the distribution of $M_k^{-\alpha}$ conditional on this collection is uniform on the interval $\left(M_{k-1}^{-\alpha},M_{k+1}^{-\alpha}\right)$, since the sequence $\left(M_i^{-\alpha}\right)$ forms the points of a Poisson process). Hence the event (4.1) has probability 0, as required. □

Proof of Theorem 4.2: We will use the same couplings as in the proof of Theorem 2.1. As at (3.7), we then have that $\mathbb{P}\left(\mathcal{C}_{k_n}^{(n)}\neq\mathcal{C}_{k_n}\right)\to 0$ as $n\to\infty$, and that all of the quantities $\sum_{i=1}^{k_n}\left|M_i-\tilde{M}_i^{(n)}\right|$, $\sum_{i=1}^{k_n}\left\|Y_i-Y_i^{(n)}\right\|$, $\tilde{S}_{k_n}^{(n)}$ and S_{k_n} converge to 0 in probability as $n\to\infty$.

To prove Theorem 4.2, it will then suffice to show in addition that for any m,

$$\mathbb{P}\left(A^{(n)*}\cap\{1,\dots,m\}\neq A^*\cap\{1,\dots,m\}\right)\to 0$$

as $n\to\infty$. For this, it's in turn enough to show that for all r,

$$\mathbb{P}\left(r\in A^*, r\notin A^{(n)*}\right)\to 0 \tag{4.2}$$

$$\mathbb{P}\left(r\notin A^*, r\in A^{(n)*}\right)\to 0 \tag{4.3}$$

as $n\to\infty$. We will show (4.2); an analogous argument gives (4.3).

Define $T_{(-r)}=\sup_{A\in\mathcal{C},r\notin A}\sum_{i\in A}M_i$.

Suppose $r\in A^*$. Then $T_{(-r)}<T$ strictly. (Otherwise, there is a sequence of members of $\mathcal{C}$, none of which contain r, whose weight converges to T; then (by Lemma 4.3) this sequence has some limit point, itself a member of $\mathcal{C}$ not containing r, which attains the weight T. But this contradicts the uniqueness of A^* established in Proposition 4.1).

If $\mathcal{C}_{k_n}^{(n)}=\mathcal{C}_{k_n}$, then

$$\begin{aligned}\max_{A\in\mathcal{C}^{(n)},r\notin A}\sum_{i\in A}\tilde{M}_i^{(n)} &\le \max_{A\in\mathcal{C}_{k_n}^{(n)},r\notin A}\sum_{i\in A}\tilde{M}_i^{(n)}+\tilde{S}_{k_n}^{(n)}\\ &\le \max_{A\in\mathcal{C}_{k_n},r\notin A}\sum_{i\in A}\tilde{M}_i^{(n)}+\sum_{i=1}^{k_n}\left[\tilde{M}_i^{(n)}-M_i\right]_+ +\tilde{S}_{k_n}^{(n)}\\ &\le T_{(-r)}+\sum_{i=1}^{k_n}\left[\tilde{M}_i^{(n)}-M_i\right]_+ +\tilde{S}_{k_n}^{(n)},\end{aligned} \tag{4.4}$$

and similarly

$$\max_{A\in\mathcal{C}^{(n)},r\in A}\sum_{i\in A}\tilde{M}_i^{(n)}\ge T-\sum_{i=1}^{k_n}\left[M_i-\tilde{M}_i^{(n)}\right]_+ -S_{k_n}. \tag{4.5}$$

If also $r\notin A^{(n)*}$, then

$$\max_{A\in\mathcal{C}^{(n)},r\notin A}\sum_{i\in A}\tilde{M}_i^{(n)}\ge\max_{A\in\mathcal{C}^{(n)},r\in A}\sum_{i\in A}\tilde{M}_i^{(n)},$$

and using (4.4) and (4.5) we get

$$T-T_{(-r)}\le\sum_{i=1}^{k_n}\left|M_i-\tilde{M}_i^{(n)}\right|+S_{k_n}+\tilde{S}_{k_n}^{(n)}$$

So altogether we obtain

$$\mathbb{P}\left(r \in A^*, r \notin A^{(n)*}\right)$$
$$\leq \mathbb{P}\left(\mathcal{C}_k^{(n)} \neq \mathcal{C}_k\right) + \mathbb{P}\left(0 < T - T_{(-r)} \leq \sum_{i=1}^{k_n} \left|M_i - \tilde{M}_i^{(n)}\right| + S_{k_n} + \tilde{S}_{k_n}^{(n)}\right).$$

We have already observed above that the first probability on the RHS tends to 0 as $n \to \infty$. The same is true for the second probability on the RHS, since all the terms on the right of the inequality converge to 0 in probability as $n \to \infty$, while the term in the middle of the inequality does not depend on n. Hence $\mathbb{P}\left(r \in A^*, r \notin A^{(n)*}\right) \to 0$ as $n \to \infty$, as required. $\square$

We now turn to the convergence of the paths regarded as subsets of $[0,1]^2$.

First let $U^* = \bigcup_{i \in A^*} Y_i \cup \{(0,0),(1,1)\}$, and take its closure $\bar{U}^*$.

We expect that $\bar{U}^*$ is connected with probability 1, but we don't have a proof. To work around this, we will use the fact that, at least, there is a.s. a unique way to extend $\bar{U}^*$ to a connected set while preserving the increasing path property. (Here the increasing path property of a set means that if y and y' are two elements of the set then $y \sim y'$).

To see this, first note that if $\bar{U}^*$ does "contain jumps", then none of these jumps can span a rectangle of non-zero area. That is, with probability 1 there is no rectangle R of non-zero area such that $y \sim y'$ for all $y \in \bar{U}^*$ and $y' \in R$.

For if there were, then R would certainly contain some points Y_j; such j could be added to A^*, increasing the weight of the path by M_j; this contradicts the maximality of A^*.

So any jumps in $\bar{U}^*$ consist only of horizontal or vertical line segments. These segments can all be added to $\bar{U}^*$ while still preserving the increasing path property, and this gives a connected set. Conversely, any connected increasing set containing $\bar{U}^*$ must "fill in" these jumps.

Thus, define P^* by setting $y \in P^*$ if:

(i) $y \in \bar{U}^*$, or

(ii) there exists $y', y'' \in \bar{U}^*$ with either

  (a) $y'(1) = y(1) = y''(1)$, $y'(2) < y(2) < y''(2)$, or

  (b) $y'(2) = y(2) = y''(2)$, $y'(1) < y(1) < y''(1)$.

This set P^* (which we conjecture to be equal to the closure of $\bigcup_{i \in A^*} Y_i$ w. p. 1) provides the distributional limit we need.

For each n, we define the object representing the optimal path in the discrete problem indexed by n as follows: order the points $\{Y_i^{(n)}, i \in A^{(n)*}\}$ in increasing order and join successive points by a straight line (horizontal or vertical, of length $1/n$). Call the resulting path $P^{(n)*}$.

P^* and $P^{(n)*}$ are regarded as subsets of $[0,1]^2$ and we use the Hausdorff metric:

$$d_H(P_1, P_2) = \sup_{x \in P_1} \inf_{y \in P_2} |x-y| + \sup_{x \in P_2} \inf_{y \in P_1} |x-y|.$$

Theorem 4.4 *$P^{(n)*} \to P^*$ in distribution as $n \to \infty$.*

Proof: Choose a probability space on which the convergence in Theorem 4.2 occurs almost surely. We will show that in this case $P^{(n)*} \to P^*$ a.s. also.

First consider any point $y \in P^*$. We will show that for all sufficiently large n there is a point of $P^{(n)*}$ within distance $\epsilon/2$ of y.

There are two cases to consider.

First, suppose that y is a limit of some sequence of points $Y_i \in A^*$. Choose some Y_i which is with distance $\epsilon/4$. For large enough n, we have $Y_i^{(n)} \in A^{(n)*}$ and $|Y_i^{(n)} - Y_i| < \epsilon/4$ and we are done.

Otherwise, y is on a vertical or horizontal line between two points that *are* limits of sequences $Y_i \in A^*$. Call these endpoints y^- and y^+. For large enough n, there are i^- and i^+ such that $Y_{i^-}, Y_{i^+} \in P^{(n)*}$ with $|Y_{i^-}^{(n)} - y^-| < \epsilon/2$ and $|Y_{i^+}^{(n)} - y^+| < \epsilon/2$, as above. Then by the increasing path property, the subpath of $P^{(n)*}$ joining Y_{i^-} and Y_{i^+} passes within $\epsilon/2$ of every point on the line segment joining y^- to y^+, and hence in particular within $\epsilon/2$ of y as required.

Now for some $m \in \mathbb{N}$ set $\epsilon = 1/m$ and consider an increasing sequence of points $(0,0) = y^{(0)}, y^{(1)}, \ldots, y^{(2m)} = (1,1) \in P^*$ such that the L_1 distance $d_1(y^{(j)}, y^{(j+1)})$ between successive points is exactly ϵ for all j. (This is possible since P^* is an increasing path and connected).

Now for large enough n there is a sequence $\tilde{y}^{(0)}, \tilde{y}^{(1)}, \ldots, \tilde{y}^{(2m)}$ of points of $P^{(n)*}$ such that $d_1(\tilde{y}^{(j)}, y^{(j)}) < \epsilon/2$ for all j. Then necessarily $\tilde{y}^{(0)}, \tilde{y}^{(1)}, \ldots, \tilde{y}^{(2m)}$ is itself an increasing sequence and $d_1(\tilde{y}^{(j)}, \tilde{y}^{(j+1)}) < 2\epsilon$ for all j.

Using the increasing path property for P^*, we have that every point of P^* is within L_1 distance $\epsilon/2$ of one of the $y^{(j)}$. Then since each $y^{(j)}$ is within L_1 distance $\epsilon/2$ of a point of $P^{(n)*}$, we have that every point of P^* is within ϵ of $P^{(n)*}$.

Similarly, the increasing path property for $P^{(n)*}$ gives that every point of $P^{(n)*}$ is within L_1 distance ϵ of one of the $\tilde{y}^{(j)}$, and thus in turn within distance $3\epsilon/2$ of some point of P^*.

Hence $d_H(P^*, P^{(n)*}) < 5\epsilon/2$, for all large enough n. This works for any $\epsilon = 1/m$, so $P^{(n)*} \to P^*$ as required. □

5 Stable process directed percolation

In this section we consider a directed last-passage percolation model based on stable Lévy processes. This is the stable version of the Brownian directed percolation problem considered in [24, 14].

For $n \in \mathbb{N}$, $t > 0$, consider the random variable

$$L(n,t) = \sup_{0=t_0 \le t_1 \le \ldots \le t_n = t} \sum_{i=1}^{n} S^i_{t_{i-1} t_i},$$

where $S^i_{st} = S^i_t - S^i_s$, and S^i are i.i.d. α-stable processes for some $\alpha \in (0,2)$. The Brownian version of this problem, in which the stable processes are replaced by Brownian motions, gives a representation for the largest eigenvalue process in "Hermitian Brownian motion" (a matrix-valued process whose marginal at any fixed time has the GUE distribution) and has been much studied in various contexts (see for example [6, 13, 14, 24]). We do not have a random matrix interpretation of this stable process version; however, an interesting connection could be to the case of Wigner random matrices with heavy-tailed entries considered by Soshnikov in [28], where a scaling is obtained for the largest eigenvalues which corresponds to the one we have observed for the heavy-tailed last-passage percolation problem.

We will show that the asymptotic behaviour of the distribution of $L(n,t)$, as n becomes large, is again described by our continuous heavy-tailed last-passage directed percolation problem. Note that by scaling $L(n,tn) = t^{1/\alpha}L(n,n)$ in distribution and hence we can just consider $L(n,n)$.

The processes S^i have jump measure $c_+x^{-\alpha-1}I_{x>0} + c_-|x|^{-\alpha-1}I_{x<0}$, for some $c_+ > 0$ and $c_- \geq 0$. The jumps play the role of weights for the percolation problem.

Theorem 5.1

$$\left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha}L(n,n) \to T$$

in distribution as $n \to \infty$, where T is the last-passage time in the continuous last-passage percolation model with index α, defined at (2.5).

A short argument is available in the case $\alpha < 1$ (making use of the fact that the sum of all positive weights is finite), and we give this first.

5.1 Case $\alpha < 1$

Let $M_1^{(n)}, M_2^{(n)}, \dots$ be the set of positive jumps of the processes $S^1, S^2, \dots, S^n$ on the interval $[0,n]$, written in descending order.

From the form of the jump measure, we can regard the ordered sequence of jumps as a Poisson random measure. Thus by a suitable transformation we can write the sequence of jumps in terms of a Poisson process and have that for any n,

$$\left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha}\Big(M_1^{(n)}, M_2^{(n)}, \dots, M_k^{(n)}, \dots\Big)$$
$$\stackrel{d}{=} \Big(W_1^{-1/\alpha}, (W_1+W_2)^{-1/\alpha}, \dots, (W_1+\dots+W_k)^{-1/\alpha}, \dots\Big),$$

where W_i are i.i.d. exponential random variables with mean 1.

Now let $L_k^{(n)+}$ be the maximal weight of a path, if one ignores all the weights except the k largest positive weights. Just as in the discrete case, one can show that

$$\left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha}L_k^{(n)+} \to T_k$$

in distribution as $n \to \infty$, and one also has $T_k \to T$ in distribution as $k \to \infty$, where T_k and T are the last passage times for the continuous problem as defined before.

Now let $L^{(n)+}$ be the maximal weight of a path, if one considers all the positive weights but ignores all the negative ones. Then

$$L^{(n)+} - L_k^{(n)+} \leq \sum_{r=k+1}^{\infty} M_r^{(n)}.$$

Now the distribution of $n^{-2/\alpha}\sum_{r=k+1}^{\infty} M_r^{(n)}$ does not depend on n, and converges to 0 in distribution as $k \to \infty$ (since the sum of all the positive weights is a.s. finite). So we have

$$\left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha} L^{(n)+} \to T$$

in distribution.

Now consider the optimal path attaining $L^{(n)+}$. Consider the sum of (the absolute values of) all the negative weights along the path; call it $S^{(n)-}$. Since the positive and negative weights occur independently, $S^{(n)-}$ has just the same distribution as the sum of the negative weights for a single stable process between times 0 and n. This is finite and on the scale $n^{1/\alpha}$ (in fact, the distribution of $n^{-1/\alpha}S^{(n)-}$ is independent of n). So certainly $n^{-2/\alpha}S^{(n)-} \to 0$ in distribution as $n \to \infty$. Since $L^{(n)+} - S^{(n)-} \le L(n,n) \le L^{(n)+}$, we obtain

$$\left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha} L(n,n) \to T$$

in distribution as $n \to \infty$, as required.

5.2 Case $1 \le \alpha < 2$

Lower bound:

As before,

$$\left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha} L_k^{(n)+} \to T_k$$

in distribution as $n \to \infty$.

Now consider a path realising $L_k^{(n)+}$ in this way; (for definiteness, say the first such path in the lexicographic order).

Let $\tilde{T}_k^{(n)}$ be the total weight of this path, including all weights, and let $\tilde{S} = \tilde{T}_k^{(n)} - L_k^{(n)+}$.

The distribution of $\tilde{S}$ can be described as follows. Generate the n independent processes $S^1, \dots, S^n$ from time 0 to time n. Remove the k largest positive jumps that occur in the n processes in time $[0,n]$. Then $\tilde{S}$ has the distribution of the altered value of S_n^1, after the k largest jumps from the set of processes have been removed.

However, as $n \to \infty$, the probability that any of the k largest jumps occur in the process S_1 tends to 0 (it is no larger than k/n); so with high probability, the procedure in the previous paragraph does not alter the value of S_n^1. Thus the limit in distribution of $n^{-1/\alpha}\tilde{S}$ is the distribution of $n^{-1/\alpha}S_n^1$ (which is independent of n). In particular, $n^{-2/\alpha}\tilde{S} \to 0$ in probability. Thus, for any k,

$$\begin{aligned}\left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha}\tilde{T}_k^{(n)} &= \left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha}\tilde{S} + \left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha} L_k^{(n)+} \\ &\to T_k\end{aligned}$$

in distribution, as $n \to \infty$. Since $L(n,n) \ge \tilde{T}_k^{(n)}$ and $T_k \to T$ as $k \to \infty$, this establishes that T is a lower bound for the limit in distribution of $\left(\frac{\alpha}{c_+}\right)^{1/\alpha} n^{-2/\alpha} L(n,n)$.

Upper bound:

We first need a lemma on the tail behaviour of the difference between the supremum and infimum of the stable process on an interval:

Lemma 5.2 *Let S be a stable process with index α and jump measure $c_+ x^{-\alpha-1} I_{x>0} + c_-|x|^{-\alpha-1} I_{x<0}$, where $c_+ > 0$ and $c_- \geq 0$. Then*

$$\mathbb{P}\Big(\sup_{0 \leq s \leq t \leq 1} \{S_t - S_s\} > x\Big) \sim \frac{c_+}{\alpha} x^{-\alpha}$$

as $x \to \infty$. That is, the quantity $\sup_{0 \leq s \leq t \leq 1}\{S_t - S_s\}$ has the same positive tail behaviour as the size of the largest positive jump of S in $[0, 1]$ (which is also the same as the upper tail of S_1 and of $\sup_{0 \leq t \leq 1} S_t$).

Proof: Let the running infimum and supremum processes be denoted by $I_t = \inf\{S_s : 0 \leq s \leq t\}$ and $S_t^* = \sup\{S_s : 0 \leq s \leq t\}$ respectively. Consider the reflected process $X_t = S_t - I_t$. Our aim is to determine the tail behaviour of $X_1^* = \sup\{X_t : 0 \leq t \leq 1\}$. By Bertoin [7] Prop VI.3, we know that for fixed t, the distribution of X_t is the same as that of S_t^*. By standard results (e.g. [7] Prop VIII.4) we have

$$\mathbb{P}(X_1 > x) \sim \mathbb{P}(S_1^* > x) \sim \frac{c_+}{\alpha} x^{-\alpha}, \tag{5.1}$$

as $x \to \infty$.

Thus we just need to show that X_1^* has the same tail as X_1. The proof is analogous to that of [7] Prop. VIII.4, and we reproduce the argument here. An easy consequence of (5.1) is that

$$\liminf_{x \to \infty} \mathbb{P}(X_1^* > x) x^\alpha \geq \frac{c_+}{\alpha}.$$

Now fix $\epsilon > 0$ and note that the reflected process is Markov by [7] Prop VI.1. As the stable process scales in that $S_{\lambda t} \stackrel{d}{=} \lambda^{1/\alpha} S_t$, this property will be inherited by the infimum and hence the reflected process itself, giving $X_{\lambda t} \stackrel{d}{=} \lambda^{1/\alpha} X_t$. Since $S_t - X_t = I_t$ is decreasing in t, we also have that $X_1 - X_t \geq S_1 - S_t$ for all $t < 1$. Applying these properties and denoting by τ_x the first hitting time of the interval (x, ∞) by the reflected process X, we have

$$\begin{aligned}
\mathbb{P}\big(X_1 > (1-\epsilon)x\big) &\geq \mathbb{P}\big(X_1^* > x, X_1 > (1-\epsilon)x\big) \\
&\geq \int_0^1 \mathbb{P}(\tau_x \in dt)\mathbb{P}(X_1 - X_t > -\epsilon x) \\
&\geq \int_0^1 \mathbb{P}(\tau_x \in dt)\mathbb{P}(S_1 - S_t > -\epsilon x) \\
&= \int_0^1 \mathbb{P}(\tau_x \in dt)\mathbb{P}(S_{1-t} > -\epsilon x) \\
&\geq \int_0^1 \mathbb{P}(\tau_x \in dt)\mathbb{P}(S_1 > -\epsilon x) \\
&= \mathbb{P}(X_1^* > x)\mathbb{P}(S_1 > -\epsilon x).
\end{aligned}$$

As $\mathbb{P}(S_1 > -\epsilon x) \to 1$ as $x \to \infty$ we have that

$$\limsup_{x \to \infty} P(X_1^* > x) x^\alpha \leq (1-\epsilon)^{-\alpha} c_+/\alpha,$$

and, as ϵ is arbitrary, we have the result. $\square$

Now for $1 \leq i, j \leq n$, define

$$X(i,j) = \sup_{j-1 \leq s \leq t \leq j} \{S_t^i - S_s^i\}.$$

Let $T(n,n)$ be the maximal passage time for the discrete model with weights $X(i,j)$. Then one can see by a direct sample path comparison that $L(n,n) \leq T(n,n)$.

Applying Lemma 5.2, $a_N \sim \left(\frac{\alpha}{c_+}\right)^{1/\alpha} N^{1/\alpha}$, where $a_N = \inf\{x : \mathbb{P}(X(i,j) > x \leq 1/N\}$. Thus, applying Theorem 2.1 for the discrete model,

$$\left(\frac{c_+}{\alpha}\right)^{1/\alpha} n^{-2/\alpha} T(n,n) \to T$$

in distribution as $n \to \infty$. This gives the required upper bound in distribution for the limit of $L(n,n)$.

6 The case $\alpha = 0$: convergence to the greedy path

In this section we consider the discrete last-passage percolation model in the case $\alpha = 0$. The distribution F is said to have a *slowly varying tail*: for all $t > 0$,

$$\frac{1 - F(tx)}{1 - F(x)} \to 1 \text{ as } x \to \infty;$$

equivalently, for all $s < 1$,

$$\frac{F^{-1}(1 - sv)}{F^{-1}(1 - v)} \to \infty \text{ as } v \downarrow 0. \tag{6.1}$$

Now it is no longer possible to find a non-degenerate limit in distribution for $T^{(n)}$ as we did in Theorem 2.1. Let $M_1^{(n)}$ be the maximum of an i.i.d. sample from F of size n^2 as before. Let b_n be any sequence of constants. Then any limit point in distribution of the sequence $b_n^{-1} M_1^{(n)}$ must be concentrated on the set $\{0, \infty\}$. From Proposition 6.2 below, the same is true if we replace $M_1^{(n)}$ by $T^{(n)}$.

However, convergence in distribution of the optimal paths can still be obtained. In fact, the form of the limiting distribution has a particularly simple description in terms of a "greedy algorithm". We give a multifractal analysis of this limiting object in Section 6.2.

The limiting object is defined as follows. Given the locations $Y_1, Y_2, \dots$ i.i.d. uniform on $[0,1]^2$, let $\mathcal{C}$, the set of increasing paths, be defined as at (2.4). We now define the *greedy path* $A^* = A^*(Y_1, Y_2, \dots) \in \mathcal{C}$ recursively as follows. Let $1 \in A^*$ always, and then, given $A^* \cap \{1, \dots, r\}$, let $r + 1 \in A^*$ if and only if $Y_{r+1} \sim Y_i$ for every $i \in A^*$, $i \leq r$. One can describe A^* as the first member of $\mathcal{C}$ in the lexicographic order.

The discrete problem is defined in terms of the locations $(Y_i^{(n)})$ and weights $(M_i^{(n)})$ as before. Write $A^{(n)*}$ for the optimal path as in Section 4. The following theorem gives the convergence of these optimal paths to the greedy path:

Theorem 6.1 $\left(\left(Y_i^{(n)}\right)_{i \in \mathbb{N}}, A^{(n)*}\right) \to \left(\left(Y_i\right)_{i \in \mathbb{N}}, A^*\right)$ *in distribution as $n \to \infty$.*

Exactly as in Section 4.2, one can also define P^* and $P^{(n)*}$ to represent the optimal paths regarded as subsets of $[0,1]^2$, and obtain the convergence in distribution of $P^{(n)*}$ to P^* as $n \to \infty$ (under the Hausdorff metric). In fact, the situation is considerably simpler here; one can simply define P^* to be the closure of $\bigcup_{i \in A^*} Y_i$, since by the results of Section 6.2 this set is connected w.p. 1.

Theorem 6.1 will follow from the next proposition:

Proposition 6.2 *For all r,*

$$\mathbb{P}\left(M_r^{(n)} > \sum_{i=r+1}^{n^2} M_i^{(n)} \right) \to 1 \text{ as } n \to \infty.$$

Proof of Theorem 6.1: Fix some $\epsilon > 0$. As in Proposition 3.2, for n large enough, we can find a coupling such that, with probability at least $1-\epsilon$,

$$\sum_{i=1}^{k} \left\| Y_i - Y_i^{(n)} \right\| < \epsilon$$

and

$$\mathcal{C}_k^{(n)} = \mathcal{C}_k \tag{6.2}$$

Suppose that (6.2) holds and also that, for all $r \leq k$, $M_r^{(n)} > \sum_{i=r+1}^{n^2} M_i^{(n)}$. Then indeed

$$A^{(n)*} \cap \{1, \dots, k\} = A^* \cap \{1, \dots, k\}, \tag{6.3}$$

where A^* is the "greedy path". Then using Proposition 6.2, we can find n such that (6.3) holds with probability at least $1-2\epsilon$. This gives the convergence in distribution in Theorem 6.1. $\square$

6.1 Proof of Proposition 6.2

Lemma 6.3 *Let $\epsilon > 0$ and $C > 0$. There exists $U(C,\epsilon) < 1$ such that if $U(C,\epsilon) < u_1 < (1+\epsilon)u_2 - \epsilon$, then*

$$F^{-1}(u_1) < F^{-1}(u_2)(1-u_1)^{-C}(1-u_2)^C.$$

Proof: Let $t = (1+\epsilon)^{2C}$. From (6.1) there exists $V > 0$ such that for all $v < V$,

$$\frac{F^{-1}\left(1 - \frac{v}{1+\epsilon}\right)}{F^{-1}(1-v)} > t.$$

Iterating,

$$\frac{F^{-1}\left(1 - \frac{v}{(1+\epsilon)^m}\right)}{F^{-1}(1-v)} > t^m, \text{ for all } m \in \mathbb{N},$$

and so in fact, for any $v_2 < v_1 < V$,

$$\frac{F^{-1}(1-v_2)}{F^{-1}(1-v_1)} > t^{\left\lfloor \log_{1+\epsilon} \frac{v_1}{v_2} \right\rfloor}.$$

Putting $u_1 = 1 - v_1$, $u_2 = 1 - v_2$ and $U(C, \epsilon) = 1 - V$, we have that for $U(C, \epsilon) < u_1 < u_2$,

$$F^{-1}(u_2) > t^{\left\lfloor \log_{1+\epsilon} \frac{1-u_1}{1-u_2} \right\rfloor} F^{-1}(u_1).$$

Now whenever $z \geq 1$, then $\lfloor z \rfloor \geq z/2$, so restricting to $(1 - u_1) > (1 + \epsilon)(1 - u_2)$ we get

$$F^{-1}(u_2) > t^{\frac{1}{2} \log_{1+\epsilon} \frac{1-u_1}{1-u_2}} F^{-1}(u_1),$$

which rearranges to the desired result. □

Lemma 6.4 *Fix $\epsilon > 0$ and $C > 0$. If $u_2 \in (0, 1)$ is sufficiently close to 1, then for all u_1 with $0 < u_1 < (1 + \epsilon)u_2 - \epsilon$,*

$$F^{-1}(u_1) < F^{-1}(u_2)(1 - u_1)^{-C}(1 - u_2)^C.$$

Proof: From Lemma 6.3, we already know that this is true when $u_1 > U(C, \epsilon)$. Now if $u_1 \leq U(C, \epsilon)$, then $F^{-1}(u_1) \leq F^{-1}(U(C, \epsilon))$. So it will suffice to show that for u_2 sufficiently close to 1, the RHS is always at least $F^{-1}(U(C, \epsilon))$. In fact, we will show that the RHS tends to ∞ as $u_2 \uparrow 1$, uniformly in u_1.

For any u_2, the RHS is minimised by $u_1 = 0$. So we wish to show that

$$F^{-1}(u_2)(1 - u_2)^C \to \infty \text{ as } u_2 \uparrow 1. \tag{6.4}$$

Fix $\tilde{u}_1 > U(2C, \epsilon)$. Then Lemma 6.3 gives, for u_2 sufficiently close to 1,

$$\begin{aligned} F^{-1}(u_2) &\geq F^{-1}(\tilde{u}_1)(1 - \tilde{u}_1)^{2C}(1 - u_2)^{-2C} \\ &= c(1 - u_2)^{-2C} \end{aligned}$$

for some constant c. This gives the desired convergence to ∞ in (6.4). □

Lemma 6.5 *Let the r.v. U be uniform on $(0, 1)$.*

$$\lim_{u \to 1} \frac{\mathbb{E}\left(F^{-1}(U) \middle| U \leq u\right)}{F^{-1}(u)(1 - u)} = 0.$$

Proof: Take any $C > 0$ and $\epsilon > 0$. If u is close enough to 1, then using Lemma 6.4,

$$\begin{aligned} \mathbb{E}\left(F^{-1}(U); U \leq u\right) &= \int_0^u F^{-1}(v)dv \\ &\leq \int_0^{(1+\epsilon)u-\epsilon} F^{-1}(u)(1 - v)^{-C}(1 - u)^C dv + \int_{(1+\epsilon)u-\epsilon}^u F^{-1}(u)dv \\ &= F^{-1}(u)\left\{(1 - u)^C \frac{(1 + \epsilon)(1 - u)^{-C+1}}{C - 1} + \epsilon(1 - u)\right\} \\ &= F^{-1}(u)(1 - u)\left\{\frac{(1 + \epsilon)^{-C+1}}{C - 1} + \epsilon\right\}. \end{aligned}$$

We can now choose ϵ as small as desired, and then C as large as desired, to give an upper bound on

$$\limsup_{u \to 1} \frac{\mathbb{E}\left(F^{-1}(U); U \leq u\right)}{F^{-1}(u)(1 - u)}$$

which is arbitrarily close to 0.

Finally, for $u \geq 1/2$,

$$\mathbb{E}\left(F^{-1}(U)\middle|U \leq u\right) \leq 2\mathbb{E}\left(F^{-1}(U); U \leq u\right)$$

and the result follows. $\square$

Proof of Proposition 6.2: We use the representation

$$\left(M_1^{(n)}, \ldots, M_{n^2}^{(n)}\right) = \left(F^{-1}(U_1^{(n)}, \ldots, U_{n^2}^{(n)})\right),$$

where $(U_1^{(n)}, \ldots, U_{n^2}^{(n)})$ are the order statistics of an i.i.d. sample of size n^2 from the uniform distribution on $(0,1)$, written in decreasing order.

We need to show that

$$\mathbb{P}\left(\sum_{m=r+1}^{n^2} F^{-1}(U_m^{(n)}) \geq F^{-1}(U_r^{(n)})\right) \to 0 \text{ as } n \to \infty.$$

Let $R > 0$ and suppose that $u > 1 - R/n^2$. Then

$$\begin{aligned}
\frac{n^2\mathbb{E}\left(F^{-1}(U)\middle|U \leq u\right)}{F^{-1}(u)} &= \frac{R\mathbb{E}\left(F^{-1}(U)\middle|U \leq u\right)}{F^{-1}(u)R/n^2} \\
&\leq \frac{R\mathbb{E}\left(F^{-1}(U)\middle|U \leq u\right)}{F^{-1}(u)(1-u)} \\
&\leq R \sup_{u > 1-R/n^2} \frac{\mathbb{E}\left(F^{-1}(U)\middle|U \leq u\right)}{F^{-1}(u)(1-u)}.
\end{aligned} \tag{6.5}$$

Now for (almost) all u,

$$\mathbb{E}\left(\sum_{m=r+1}^{n^2} F^{-1}(U_m^{(n)})\middle|U_r^{(n)} = u\right) = (n^2 - r)\mathbb{E}\left(F^{-1}(U)\middle|U \leq u\right),$$

so for (almost) all $u > 1 - R/n^2$ we have, from Markov's inequality,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{m=r+1}^{n^2} F^{-1}(U_m^{(n)}) \geq F^{-1}(M_r^{(n)})\middle|U_r^{(n)} = u\right) \\
&\qquad \leq \frac{1}{F^{-1}(u)}\mathbb{E}\left(\sum_{m=r+1}^{n^2} F^{-1}(U_m^{(n)})\middle|U_r^{(n)} = u\right) \\
&\qquad \leq n^2\frac{\mathbb{E}\left(F^{-1}(U)\middle|U \leq u\right)}{F^{-1}(u)} \\
&\qquad \leq R \sup_{u > 1-R/n^2} \frac{\mathbb{E}\left(F^{-1}(U)\middle|U \leq u\right)}{F^{-1}(u)(1-u)}
\end{aligned}$$

using (6.5).

So in fact, integrating over $u > 1 - R/n^2$,

$$\mathbb{P}\left(\sum_{m=r+1}^{n^2} F^{-1}(U_m^{(n)}) \geq F^{-1}(M_r^{(n)}) \Big| U_r^{(n)} > 1 - R/n^2\right)$$
$$\leq R \sup_{u>1-R/n^2} \frac{\mathbb{E}\left(F^{-1}(U) \big| U \leq u\right)}{F^{-1}(u)(1-u)}.$$

Then

$$\limsup_{n\to\infty} \mathbb{P}\left(\sum_{m=r+1}^{n^2} F^{-1}(U_m^{(n)}) \geq F^{-1}(U_r^{(n)})\right)$$
$$\leq \limsup_{n\to\infty} \mathbb{P}\left(U_r^{(n)} \leq 1 - R/n^2\right) + \limsup_{n\to\infty} R \sup_{u>1-R/n^2} \frac{\mathbb{E}\left(F^{-1}(U) \big| U \leq u\right)}{F^{-1}(u)(1-u)}.$$

The second term is 0 by Lemma 6.5, for every R. The first term is the probability $\mathbb{P}(B_{n^2,R/n^2} \leq r-1)$, where $B_{n^2,R/n^2}$ is a binomial $(n^2, R/n^2)$ random variable. This converges as $n \to \infty$ to the probability $\mathbb{P}(P_R \leq r-1)$ where P_R is a Poisson mean R random variable. Thus this probability can be made as small as desired by choosing R large, and hence the limsup is in fact 0 as required. □

6.2 The properties of the greedy path

In this section we discuss the properties of the greedy path which we have obtained as a distributional limit of the optimal path for the discrete problem when $\alpha = 0$. The path can be regarded as a function $y = G(x)$ from $[0,1]$ to itself in a natural way (specifically, one could define $G(x) = \sup\{y : Y_i = (x', y)$ for some $x' \leq x$ and some $i \in A^*\}$. See Figure 6.1 for a realisation of the greedy path). This function G is monotone non-decreasing, and hence defines a measure μ on $[0,1]$. We will show that μ is a random self-similar measure which is singular with respect to Lebesgue measure, and we will be able to compute its multifractal spectrum.

We recall the definition of the multifractal spectrum for our setting. Let $B_r(x)$ denote the ball of radius r around the point $x \in \mathbb{R}$. For $a \geq 0$, we define E_a, the set of points at which the measure has local dimension a by

$$E_a = \left\{x : \lim_{r\downarrow 0} \frac{\log \mu(B_r(x))}{\log r} = a\right\}.$$

The multifractal spectrum is then defined to be

$$f(a) = \dim_H(E_a),$$

where $\dim_H(A)$ denotes the Hausdorff dimension of a set A. In our setting we have

$$E_a = \left\{x \in [0,1] : \lim_{r\downarrow 0} \frac{\log(G(x+r) - G(x-r))}{\log r} = a\right\}.$$

In particular we note that if μ had a density with respect to Lebesgue measure, then the spectrum would be the function $f(a) = 0$ for all $a \neq 1$ and $f(1) = 1$.

There are a number of papers making rigorous the *multifractal formalism*, the heuristic argument for computing the multifractal spectrum in terms of the Legendre transform of

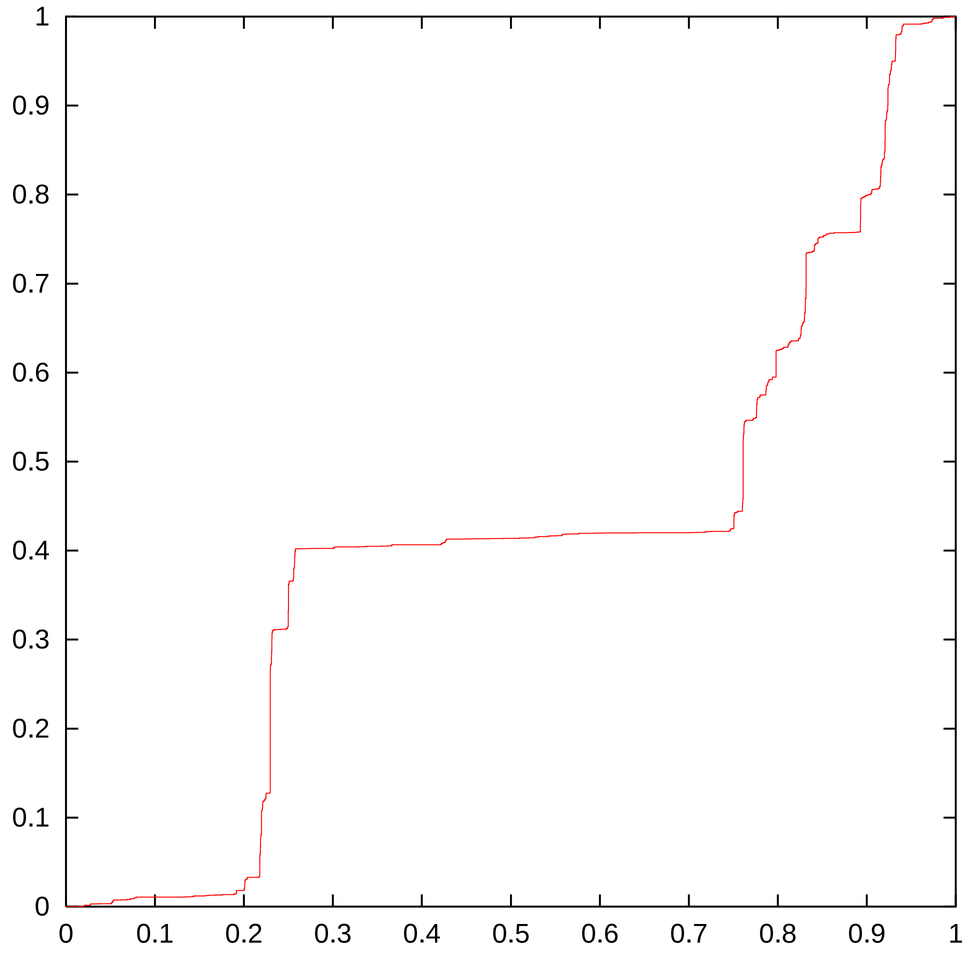

Figure 6.1: A simulation from the distribution of the "greedy path" which occurs as the limit of the distribution of optimal paths in the case $\alpha = 0$.

the moment measures, and we will be able to set our measure in a framework within which we can apply this formalism. The study of the multifractal spectrum for random self-similar measures is the topic of [11, 2, 8] where the underlying assumptions are successively weakened.

A *scaling law* on a space E consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and for each $\omega \in \Omega$ a collection of weights and maps $(\phi_1(\omega), p_1(\omega), \ldots, \phi_N(\omega), p_N(\omega))$, where $p_i \in \mathbb{R}_+$ and $\phi_i : E \to E$ is a contraction with Lipschitz constant r_i. For a given scaling law a random self-similar measure is a measure μ which satisfies the distributional equality

$$\mu(\cdot) = \sum_{i=1}^{N} p_i \mu_i(\phi_i^{-1}(\cdot)),$$

where μ_i are i.i.d. copies of μ (independent of the weights and maps). The support of the measure is typically a random self-similar set.

The multifractal formalism enables the multifractal spectrum for the random self-similar measure to be calculated in the following way. Let $m(q, \theta) = E(\sum_i p_i^q r_i^\theta)$ and $\beta(q) = \inf\{\theta :$

$m(q,\theta) \le 1\}$. Under the formalism the multifractal spectrum is the Legendre transform of $\beta(q)$,

$$f(a) = \inf_{q\in\mathbb{R}} \{aq + \beta(q)\}. \tag{6.6}$$

We will now give a more formal version.

We introduce a little notation. We write $\mathcal{T}_n = \{1,\dots,N\}^n$ for the sequences which index the sets after n applications of the scaling law, and $\mathcal{T}$ for the tree $\cup_{n=0}^{\infty}\mathcal{T}_n$. Let $(\Omega^{\otimes\mathcal{T}}, \mathcal{F}^{\otimes\mathcal{T}}, \mathbb{P}^{\otimes\mathcal{T}})$ denote the product probability space for random variables on the tree; for each node, we have an independent copy of the scaling law. Now for each $\mathbf{i} \in \mathcal{T}_n$, define

$$p_{\mathbf{i}} = p_{i_1}(\omega_\emptyset)p_{i_2}(\omega_{i_1})\dots p_{i_n}(\omega_{i_1 i_2 \dots i_{n-1}})$$

and

$$r_{\mathbf{i}} = r_{i_1}(\omega_\emptyset)r_{i_2}(\omega_{i_1})\dots r_{i_n}(\omega_{i_1 i_2 \dots i_{n-1}}).$$

The total mass of the random measure over the unit interval is given by $W = \lim_{n\to\infty} \sum_{\mathbf{i}\in\mathcal{T}_n} p_{\mathbf{i}}$. In our setting we will consider random probability measures, so that $W = 1$. The other limit random variable we need is is $W(q) = \lim_{n\to\infty} W_n(q)$ where $W_n(q) = \sum_{\mathbf{i}\in\mathcal{T}_n} p_{\mathbf{i}}^q r_{\mathbf{i}}^\beta$.

We also define the set $I_\beta \subseteq \mathbb{R}$ as follows: $q* \in I_\beta$ if, for some $a \ge 0$, the infimum $\inf_{q\in\mathbb{R}}\{aq + \beta(q)\}$ is non-negative and is achieved at $q*$.

Finally the *strong open set condition* is that there is an open set O such that, with probability 1, one has that $\phi_i(O) \cap \phi_j(O) = \emptyset$ for $i \ne j$, that $\cup_{i=1}^N \phi_i(O) \subset O$ and that $\mu(O) > 0$.

We now state a version of the main result of [8] which can be applied in our setting.

Lemma 6.6 *Let μ be a random self-similar probability measure satisfying the strong open set condition. If the following three sets of conditions are satisfied:*

1. $-\infty < \mathbb{E}\sum_i p_i \log p_i < 0$;
2. *For all* $q \in I_\beta$, $\mathbb{E}\sum_i (\log p_i) p_i^q r_i^{\beta(q)} < \infty$ *and* $\mathbb{E}\sum_i (\log r_i) p_i^q r_i^{\beta(q)} < \infty$;
3. *For all* $q \in I_\beta$, $\mathbb{E}\sum_i \left((\log p_i)^2 + (\log r_i)^2\right) p_i^q r_i^{\beta(q)} < \infty$ *and* $\mathbb{E}W(q)\log_+ W(q) < \infty$;

then the multifractal formalism holds in the following sense. Define

$$\beta^*(a) = \inf_{q\in\mathbb{R}} \{aq + \beta(q)\}.$$

Then for any $a \ge 0$,

$$f(a) = \max\{\beta^*(a), 0\},$$

with probability 1, and $E_a = \emptyset$ *with probability 1 if* $\beta^*(a) < 0$.

We now return to the measure arising from our greedy path. From the definition, we can describe the greedy path recursively as follows: choose a point $Y = (Y(1), Y(2))$ uniformly in the box $[0,1]^2$. Then the original path is the union of two independent greedy paths, one scaled to lie in $[0, Y(1)] \times [0, Y(2)]$ and the other to lie in $[Y(1), 1] \times [Y(2), 1]$. From this representation we can regard the induced measure as a random self-similar measure for a scaling law. Let V and $\tilde{V}$ be independent uniform random variables. Then the scaling law has $N = 2$ with the set of weights $(p_1, p_2) = (\tilde{V}, 1 - \tilde{V})$ and the set of contractions (ϕ_1, ϕ_2), where $\phi_1(x) = Vx$ has contraction factor $r_1 = V$ and $\phi_2(x) = 1 - (1 - V)x$ has contraction factor $r_2 = 1 - V$. Then the random self-similar measure μ satisfies $\mu(\cdot) = \sum_{i=1}^{2} p_i \mu_i(\phi_i^{-1}(\cdot))$ in distribution. From the construction it is clear that the measure is a probability measure whose support is the unit interval.

We note that our measure does not fit into the framework of [11] or [2]. The interval does not satisfy the strong separation condition as required in [11] (for a definition see [11]) but instead it satisfies the strong open set condition with $O = (0, 1)$. This ensures that the overlap between a pair of contractions applied to the unit interval occurs at one point. It also uses uniform random variables for the contraction ratios and therefore there is no strictly positive lower bound on the contraction ratios as required in [2].

Theorem 6.7 *The measure corresponding to the greedy path has for, any given $a \geq 0$, with probability 1,*

$$f(a) = \begin{cases} \sqrt{8a} - a - 1, & 3 - 2\sqrt{2} \leq a \leq 3 + 2\sqrt{2}, \\ 0, & \textit{otherwise}. \end{cases}$$

If $a \in [0, 3 - 2\sqrt{2}) \cup (3 + 2\sqrt{2}, \infty)$ then $E_a = \emptyset$ with probability 1.

Proof: We determine the multifractal spectrum using the multifractal formalism. We need to consider $m(q, \theta) = E(\tilde{V}^q V^\theta + (1 - \tilde{V})^q (1 - V)^\theta)$ and $\beta(q) = \inf\{\theta : m(q, \theta) \leq 1\}$. It is straightforward to compute these quantities and we have

$$m(q, \theta) = 2 \int_0^1 y^q dy \int_0^1 x^\theta dx = \frac{2}{(1 + q)(1 + \theta)} \text{ for } q > -1, \theta > -1.$$

Thus

$$\beta(q) = \frac{2}{q + 1} - 1,$$

and we can calculate that

$$\beta^*(a) = \sqrt{8a} - a - 1$$

for all $a \geq 0$. We have that $\beta^*(a)$ is non-negative on the interval $[3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$ and negative elsewhere. Hence, if we can establish the three conditions of Lemma 6.6, to justify the formalism, we will have proved our theorem.

Another calculation gives that $I_\beta = [1 - \sqrt{2}, 1 + \sqrt{2}] \subset (-1/2, 3)$, and that for all $q \in I_\beta$, one also has $\beta(q) \in (-1/2, 3)$.

For condition (1), we can compute $\mathbb{E} \sum_i p_i \log p_i = -1/2$.

For (2) straightforward calculations give $\mathbb{E} \sum_i (\log p_i) p_i^q r_i^\beta = 2((q + 1)^2 (1 + \beta))^{-1} < \infty$ and $\mathbb{E} \sum_i (\log r_i) p_i^q r_i^\beta = 2((q + 1)(1 + \beta)^2)^{-1} < \infty$, for all $q > -1, \beta > -1$.

Finally for the conditions (3) we have to do some work. It is easy to calculate the first condition

$$\mathbb{E} \sum_i \left((\log p_i)^2 + (\log r_i)^2 \right) p_i^q r_i^\beta = \frac{2}{(1 + q)^3 (1 + \beta)} + \frac{2}{(1 + q)(1 + \beta)^3} < \infty,$$

for $q > -1, \beta > -1$.

Thus we only have to verify the final condition.

We begin by observing that all we need is to prove that for some $\epsilon > 0$, $\mathbb{E}W(q)^{1+\epsilon} < \infty$ for each q. To do this we will show that $\mathbb{E}W_n(q)^{1+\epsilon}$ converges. First we observe that $W_n(q)$ is a martingale by the definition of $\beta(q)$ and we compute the bracket process

$$\begin{aligned} [W(q)]_n &= \sum_{i=1}^n \mathbb{E}\Big(\big(W_i(q) - W_{i-1}(q)\big)^2 | \mathcal{F}_{i-1}\Big) \\ &= \sum_{i=1}^n \mathbb{E}\left(\left(\sum_{j\in\mathcal{T}_{i-1}} p_j^q r_j^{\beta(q)}(\chi_j - 1)\right)^2 \Big| \mathcal{F}_{i-1}\right), \end{aligned}$$

where $\chi_j = \tilde{V}_j^q V_j^{\beta(q)} + (1-\tilde{V}_j)^q (1-V_j)^{\beta(q)}$ and the $(V_j, \tilde{V}_j)$ are independent of each other and independent over j. By definition of β we know that $\mathbb{E}\chi_j = 1$ and we can also compute

$$\mathbb{E}\chi_j^2 = \frac{2}{(1+2q)(1+2\beta)} + 2\frac{\Gamma(q+1)^2}{\Gamma(2q+2)}\frac{\Gamma(\beta+1)^2}{\Gamma(2\beta+2)}.$$

One can easily check that this quantity is finite over q and β in $(-1/2, 3)$, and hence for all $q \in I_\beta$. Thus we have

$$[W(q)]_n = \sum_{i=1}^n W_{i-1}(2q, 2\beta)(\mathbb{E}\chi^2 - 1),$$

where we write $W_n(a,b) = \sum_{\mathbf{i}\in\mathcal{T}_n} p_{\mathbf{i}}^a r_{\mathbf{i}}^b$.

With the bracket process we can control the moments of the martingale as for any $\gamma \geq 1$ there is a constant c_γ such that

$$\mathbb{E}W_n(q)^\gamma \leq c_\gamma \mathbb{E}[W(q)]_n^{\gamma/2}.$$

Thus we need to compute the moments of the bracket process. As we will take $1 < \gamma < 2$ and all terms in the sum are positive, straightforward estimates give

$$\begin{aligned} \mathbb{E}[W(q)]_n^{\gamma/2} &= \mathbb{E}(\sum_{i=1}^n W_{i-1}(2q, 2\beta)(\mathbb{E}\chi^2 - 1))^{\gamma/2} \\ &\leq \mathbb{E}\sum_{i=1}^n W_{i-1}(2q,2\beta)^{\gamma/2}(\mathbb{E}\chi^2 - 1)^{\gamma/2} \\ &\leq \sum_{i=1}^n \mathbb{E}W_{i-1}(\gamma q, \gamma\beta)(\mathbb{E}\chi^2 - 1)^{\gamma/2}. \end{aligned}$$

Thus we just need to find $\mathbb{E}W_i(\gamma q, \gamma\beta) = (\mathbb{E}(\tilde{V}^{\gamma q} V^{\gamma\beta} + (1-\tilde{V})^{\gamma q}(1-V)^{\gamma\beta}))^i$. Integration gives

$$\mathbb{E}W_i(\gamma q, \gamma\beta) = \left(\frac{2}{(1+\gamma q)(1+\gamma\beta)}\right)^i = \left(\frac{2(1+q)}{(1+\gamma q)(1+q+\gamma(1-q))}\right)^i.$$

Thus our result will hold if we can establish that over the range of q, we can find a $\gamma > 1$ such that

$$\frac{2(1+q)}{(1+\gamma q)(1+q+\gamma(1-q))} < 1.$$

A numerical calculation with the quadratic formula shows that this is the case and hence we have our result. $\square$

The explicit form of the spectrum shows that $f(a) > 0$ for $3 - 2\sqrt{2} < a < 3 + 2\sqrt{2}$. We also observe that the set of points for which $a = 2$ has full dimension 1. The next result shows that the measure corresponding to the path is "singular":

Corollary 6.8 *With probability 1, the greedy path is continuous and strictly increasing, with zero derivative almost everywhere.*

Proof: These properties can be proved fairly directly from the construction of the greedy path, but here we deduce them immediately from the multifractal spectrum.

By construction, the distribution of the path is symmetric in the x and y coordinates; hence the continuity property and the property that the path is strictly increasing are equivalent.

Any point of discontinuity of the path belongs to the set E_0, by definition. But from Theorem 6.7 we have that E_0 is a.s. empty, so the path is a.s. continuous as desired.

For the derivative, note that as G is a distribution function it is almost everywhere differentiable with non-negative derivative. Let D_1 denote the set of points where the path is differentiable and has strictly positive derivative. It is straightforward to see that $D_1 \subset E_1$. Thus as $f(1) = \sqrt{8} - 2 < 1$, $\mathbb{P} - a.s.$, we have that the Lebesgue measure of D_1 is 0 with probability 1. $\square$

7 Last-passage random fields and an Airy process

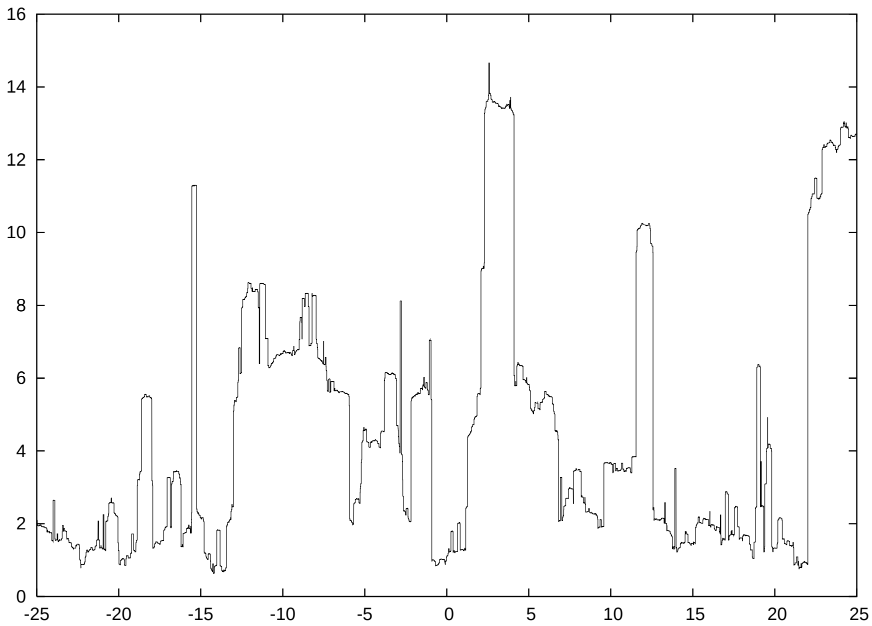

Figure 7.1: Figures 7.1-7.3 show simulations of the heavy-tailed Airy process H_t for three different values of α. Here $\alpha = 1$.

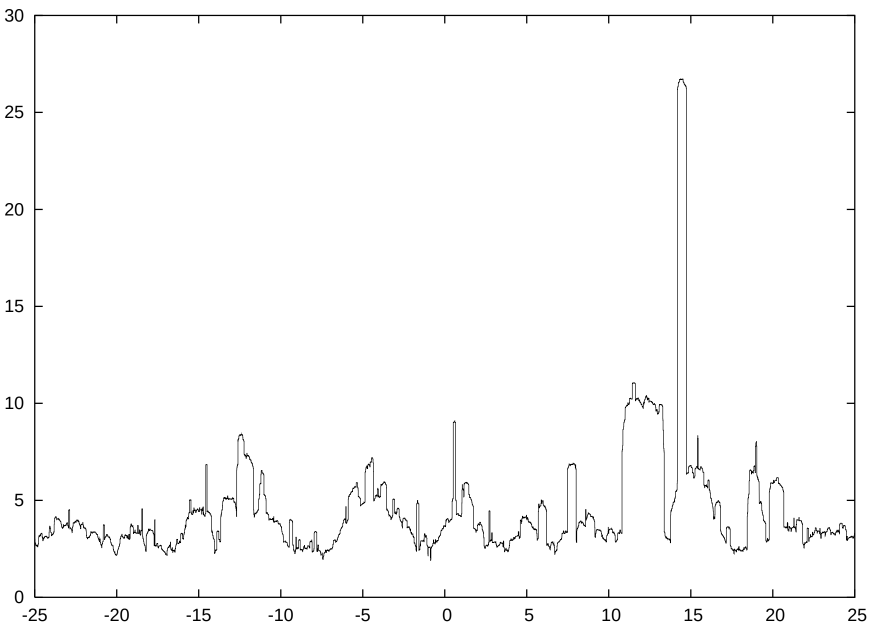

Figure 7.2: The heavy-tailed Airy process H_t in the case $\alpha = 1.5$.

In this section we consider the extension of the results to the case of a random field. To do this we give a Poisson random measure construction of the continuous limit model.

Let μ denote a Poisson random measure on $\mathbb{R}_+^3$ with intensity measure $\lambda(dx\,dy\,dz) = dx\,dy\,\alpha z^{-\alpha-1}dz$. That is, if $N(x,y,z) = \mu((0,x)\times(0,y)\times(z,\infty)) = \int_0^x\int_0^y\int_z^\infty \mu(dx\,dy\,dz)$ denotes the number of points in $(0,x)\times(0,y)\times(z,\infty)$, then this has a Poisson distribution with mean $\mathbb{E}N(x,y,z) = xyz^{-\alpha}$, and the number of points in disjoint sets are independent.

We can now extend our limiting model to this setting. To relate this to our original model we can consider the unit square in $\mathbb{R}_+^2$ and order the points of the Poisson random measure in decreasing order of their z-coordinates and we recover the sequence of weights $Z_i = M_i$ in the original model.

We will write (Y_i, Z_i) for a point of the Poisson random measure, where the points are labelled as above in that we regard Y_i as the location in $\mathbb{R}_+^2$ and Z_i as the weight in our continuous last passage percolation model. Let

$$\mathcal{C}_{xy} = \{A \subset \mathbb{N} \text{ such that for all } Y_i \sim Y_j \text{ for all } i,j \in A \text{ and } Y_i \in [0,x]\times[0,y] \text{ for all } i \in A\}$$

and define

$$T(x,y) = \sup_{A\in\mathcal{C}_{xy}} \sum_{i\in A} Z_i.$$

It is clear by Theorem 2.1 that this random variable will exist for each fixed x, y. This can be extended to show that the random field $\{T(x,y), x>0, y>0\}$ exists and arises as the limit of the last passage model. Let

$$T^{(n)}(x,y) = \max_{\pi\in\Pi_{xy}^n} \sum_{v\in\pi} X(v),$$

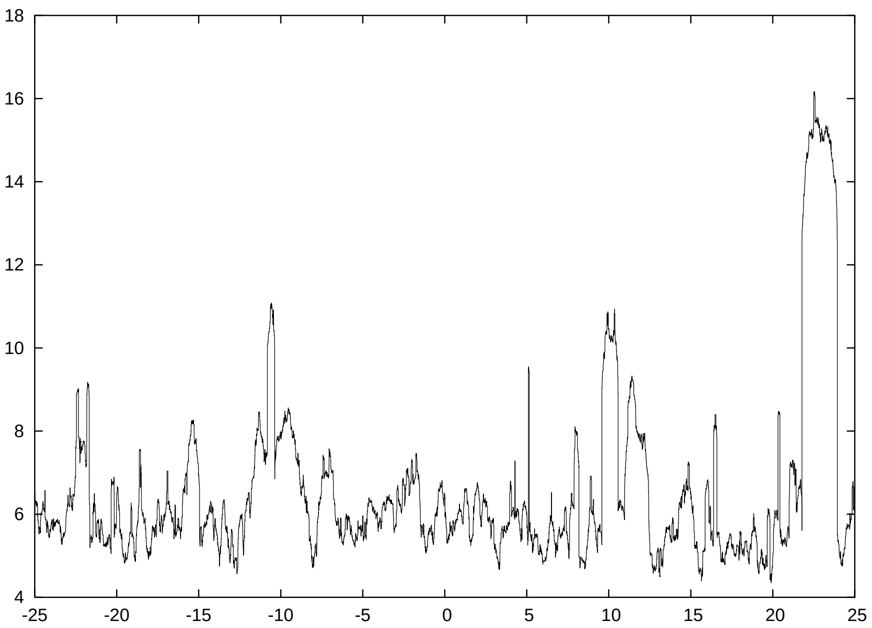

Figure 7.3: The heavy-tailed Airy process H_t in the case $\alpha = 1.99$.

where Π^n_{xy} is the set of directed paths from $(1,1)$ to $(\lceil nx \rceil, \lceil ny \rceil)$, and let

$$\tilde{T}^{(n)}(x,y) = \frac{T^{(n)}(x,y)}{a_{n^2}}.$$

Theorem 7.1 *The field* $\{T(x,y), x > 0, y > 0\}$ *exists almost surely and* $\{\tilde{T}^{(n)}(x,y), x > 0, y > 0\} \to \{T(x,y), x > 0, y > 0\}$ *in the sense of convergence of finite dimensional distributions.*

Proof: Given a realisation of the PRM, the random variable $T(x,y)$ is well defined for all x, y, and over any finite box $[0, c_x] \times [0, c_y]$ we have $T(x,y) \leq T(c_x, c_y)$ whenever $x \leq c_x, y \leq c_y$; for any fixed c_x, c_y we have $P(T(c_x, c_y) < \infty) = 1$ and hence by countable additivity

$$P(T(x,y) < \infty \text{ for all } x, y) = 1.$$

We now need to establish the convergence of finite dimensional distributions. For one particular point we already have the one-dimensional convergence result given in Theorem 2.1. Exactly the same couplings between the discrete and continuous problems that we used to prove Theorem 2.1 can be applied to extend the result to a finite collection of points within any finite box. The size of the box is arbitrary and we obtain convergence of all the finite dimensional distributions. $\square$

We now proceed to define a stationary field on the whole of $\mathbb{R}^2$. To do this we observe that by simple scaling $N(\lambda x, \nu y, (\lambda\nu)^{1/\alpha} z) = N(x, y, z)$ in distribution. In particular we have

$$T(\lambda x, \nu y) = (\lambda\nu)^{1/\alpha} T(x,y) \text{ in distribution.} \tag{7.1}$$

Now put

$$\Theta(u_1, u_2) = \exp\left(-\frac{(u_1+u_2)}{\alpha}\right) T(e^{u_1}, e^{u_2})$$

for all $u_1, u_2 \in \mathbb{R}$. Then we have that for any $v_1, v_2 \in \mathbb{R}$, the collections $\{\Theta(u_1, u_2), u_1, u_2 \in \mathbb{R}\}$ and $\{\Theta(u_1+v_1, u_2+v_2), u_1, u_2 \in \mathbb{R}\}$ have the same distribution.

The next two results concern the moments and correlations of this stationary field.

Proposition 7.2 *For all $\beta \in (0, \alpha)$ and all $\mathbf{u} \in \mathbb{R}^2$, we have $\mathbb{E}\Theta(\mathbf{u})^\beta = \mathbb{E}T(1,1)^\beta < \infty$.*

Proof: This is a continuation of the argument given to establish the existence of $T(1,1)$ in the proof of Lemma 3.1. Recalling the setting of that proof, we have $T(1,1) = S_0 \leq U_0$, where for $k \geq 0$, we defined $U_k = \sum_{i=k+1}^{\infty} L_i(M_i - M_{i+1})$. Here L_i is the largest number of the first i locations that can be included in an increasing path; there is a constant c such that $\mathbb{E}\, L_i \leq c\sqrt{i}$ and $\mathbb{E}\, L_i^2 \leq ci$ for all i, and the collections (M_i) and (L_i) are independent.

Since $T(1,1) \leq U_0$, it will be enough to show that $\mathbb{E}\, U_0^\beta < \infty$. If a finite collection of random variables each have a finite βth moment, then so does their sum. Hence, since $U_0 = L_1(M_1 - M_2) + \cdots + L_k(M_k - M_{k+1}) + U_k$, it's enough to show both of the following:

(i) for any i, $\mathbb{E}\left[L_i(M_i - M_{i+1})\right]^\beta < \infty$;

(ii) for some k, $\mathbb{E}\, U_k^\beta < \infty$.

For property (i), note that $\left[L_i(M_i - M_{i+1})\right]^\beta < (iM_1)^\beta$, so it's enough to show that $\mathbb{E}\, M_1^\beta < \infty$. But $M_1 = W_1^{-1/\alpha}$, where W_1 has exponential distribution with mean 1. Thus $\mathbb{E}\, M_1^\beta = \int_0^\infty w^{-\beta/\alpha} e^{-w} dw$, which is finite for all $\beta < \alpha$ as required.

So it remains to show (ii). Since $\beta < \alpha < 2$, it is enough to show that $\mathbb{E}\, U_k^2 < \infty$ for all large enough k, and this is what we will do.

Recall that we can write $M_i = (W_1 + \cdots + W_i)^{-1/\alpha}$ where W_j are i.i.d. exponential random variables with mean 1. We also write $V_i = W_1 + \cdots + W_i$, so that $M_i = V_i^{-1/\alpha}$. Then, using the fact that $(1+x)^{-1/\alpha} > 1 - x/\alpha$ for all $x > 0$, we have that for all i,

$$\begin{aligned}
\mathbb{E}\,(M_i - M_{i+1})^2 &= \mathbb{E}\left[V_i^{-1/\alpha}\left(1 - \left(\frac{V_{i+1}}{V_i}\right)^{-1/\alpha}\right)\right]^2 \\
&= \mathbb{E}\left[V_i^{-2/\alpha}\left(1 - \left(1 + \frac{W_{i+1}}{V_i}\right)^{-1/\alpha}\right)^2\right] \\
&\leq \mathbb{E}\left[\frac{1}{\alpha^2} V_i^{-2/\alpha} V_i^{-2} W_{i+1}^2\right] \\
&= C\mathbb{E}\, V_i^{-2-2/\alpha},
\end{aligned}$$

for some constant C, since V_i and W_{i+1} are independent. Arguing as at (3.5) and (3.6), we have that V_i has Gamma$(i,1)$ distribution, and we obtain that for some constant $\tilde{C}$ and all large enough i

$$\mathbb{E}\,(M_i - M_{i+1})^2 \leq \tilde{C} i^{-2-2/\alpha}. \tag{7.2}$$

Suppose k is large enough that (7.2) holds for all $i > k$. Then using Cauchy-Schwarz, we obtain that for all $\epsilon > 0$,

$$\begin{aligned} U_k &= \sum_{i=k+1}^{\infty} L_i(M_i - M_{i+1}) \\ &\le \left[\sum_{i=k+1}^{\infty} \left(i^{-1/2-\epsilon}\right)^2\right]^{1/2} \left[\sum_{i=k+1}^{\infty} \left(i^{1/2+\epsilon} L_i(M_i - M_{i+1})\right)^2\right]^{1/2}. \end{aligned}$$

The first sum is finite for any $\epsilon > 0$. Thus squaring and taking expectations, we have that

$$\begin{aligned} \mathbb{E}\, U_k^2 &\le c' \sum_{i=k+1}^{\infty} i^{1+\epsilon} \mathbb{E}\, L_i^2 \mathbb{E}\, (M_i - M_{i+1})^2 \\ &\le c'' \sum_{i=k+1}^{\infty} i^{1+\epsilon} i i^{-2-2/\alpha} \\ &= c'' \sum_{i=k+1}^{\infty} i^{-2/\alpha+\epsilon}, \end{aligned}$$

for some constants c', c''. Since $\alpha < 2$, this sum is finite for small enough ϵ, and so $\mathbb{E}\, U_k^2 < \infty$ as desired. $\square$

Proposition 7.3 *For all $\beta < \alpha$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^2$, we have*

$$\mathbb{E}|T(\mathbf{u}) - T(\mathbf{v})|^\beta \le 4 \max\{\|\mathbf{u}\|, \|\mathbf{v}\|\}^{\beta/\alpha} \|\mathbf{u} - \mathbf{v}\|^{\beta/\alpha} \mathbb{E} T(\mathbf{1})^\beta.$$

Proof: Recall that $T(\mathbf{u})$ is increasing in the partial order on $\mathbb{R}_+^2$. We just need to consider the two cases where $\mathbf{u} \le \mathbf{v}$ and where they are not comparable, so that, say, $u_1 < v_1$ and $u_2 > v_2$.

For $\mathbf{u} \le \mathbf{v}$, it is a simple observation that

$$T(\mathbf{v}) - T(\mathbf{u}) \le T\big((0, u_2), \mathbf{v}\big) + T\big((u_1, 0), \mathbf{v}\big), \tag{7.3}$$

where, for $\mathbf{x} \le \mathbf{y} \in \mathbb{R}_+^2$, $T(\mathbf{x}, \mathbf{y})$ denotes the maximal weight of an increasing path from $\mathbf{x}$ to $\mathbf{y}$. By the scaling in the field we have the distributional relationships

$$T\big((a, b), (c, d)\big) \stackrel{d}{=} T(c - a, d - b) \stackrel{d}{=} (c - a)^{1/\alpha} (d - b)^{1/\alpha} T(1, 1). \tag{7.4}$$

For any random variables X_1, X_2, we have $\mathbb{E}\, (X_1 + X_2)^\beta \le 2^{\max\{\beta, 1\}} \max\{\mathbb{E}\, X_1^\beta, \mathbb{E}\, X_2^\beta\}$. Thus from (7.3) and (7.4) we obtain

$$\begin{aligned} \mathbb{E}|T(\mathbf{v}) - T(\mathbf{u})|^\beta &\le 2^{\max\{\beta, 1\}} \max\left\{\mathbb{E}\, T(v_1 - u_1, v_2)^\beta, \mathbb{E}\, T(v_1, v_2 - u_2)^\beta\right\} \\ &\le 4 \mathbb{E}\, T(\mathbf{1})^\beta \max\left\{(v_1 - u_1)^{\beta/\alpha} v_2^{\beta/\alpha}, v_1^{\beta/\alpha} (v_2 - u_2)^{\beta/\alpha}\right\}. \end{aligned} \tag{7.5}$$

For the case $u_1 < v_1$, $v_2 < u_2$, we observe similarly that

$$|T(\mathbf{v}) - T(\mathbf{u})| \le T\big((u_1, 0), (v_1, u_2)\big) + T\big((0, v_2), (v_1, u_2)\big). \tag{7.6}$$

Raising to the power β and taking expectations as above, we obtain that

$$\mathbb{E}\,|T(\mathbf{v}) - T(\mathbf{u})|^\beta \le 4\mathbb{E}\,T(\mathbf{1})^\beta \max\left\{(v_1-u_1)^{\beta/\alpha}v_2^{\beta/\alpha}, (u_2-v_2)^{\beta/\alpha}u_1^{\beta/\alpha}\right\}. \tag{7.7}$$

Combining the estimates from (7.5) and (7.7) now gives the result. □

We are now ready to define our analogue of the Airy process. If we set $H_u = \Theta(u,-u) = T(e^u, e^{-u})$, we have a one-dimensional stationary process $\{H_u, u \in \mathbb{R}\}$, as the processes $(H_{u+t}, u \in \mathbb{R})$ and $(H_u, u \in \mathbb{R})$ have the same distribution for all $t \in \mathbb{R}$. We note that as $H_0 = T(1,1)$, the marginal distribution for our stationary process is the distribution of the limit random variable in our continuous model for heavy-tailed last passage percolation. By applying the estimates for the random field we have estimates on the Hölder continuity of the heavy-tailed Airy process.

Corollary 7.4 *For $\beta < \alpha$, we have:*

(i) $\mathbb{E}H_0^\beta < \infty$.

(ii) For each $\tau > 0$ and all $u, v \in [-\tau, \tau]$,

$$\mathbb{E}|H_u - H_v|^\beta \le 2e^{2\beta\tau/\alpha}|u-v|^{\beta/\alpha}\mathbb{E}H_0^\beta.$$

Proof: The first part is Proposition 7.2. The second follows from the second part of the proof of Proposition 7.3 as, assuming $u < v$ and using (7.6) with $v_1 = e^v, v_2 = e^{-v}, u_1 = e^u, u_2 = e^{-u}$,

$$\begin{aligned}\mathbb{E}|H_u - H_v|^\beta &\le (e^v(e^{-u}-e^{-v}))^{\beta/\alpha}\mathbb{E}H_0^\beta + ((e^v-e^u)e^{-u})^{\beta/\alpha}\mathbb{E}H_0^\beta \\ &= 2(e^{v-u}-1)^{\beta/\alpha}\mathbb{E}H_0^\beta \\ &\le 2(v-u)^{\beta/\alpha}e^{(v-u)\beta/\alpha}\mathbb{E}H_0^\beta,\end{aligned}$$

and the result follows. □

Finally we give a weak convergence result for our heavy-tailed Airy process.

Theorem 7.5 *The sequence $\{a_{n^2}^{-1}T^{(n)}(e^u, e^{-u})\}_{u\in[-\tau,\tau]}$ converges weakly to $\{H_u\}_{u\in[-\tau,\tau]}$ in $D[-\tau,\tau]$.*

Proof: We follow the approach outlined in [26] for such weak convergence problems, in particular the proof of [26] Proposition 3.4.

Since we restrict to $u \in [-\tau, \tau]$, we can consider only those points of the PRM whose locations fall in the box $[0, c_x] \times [0, c_y]$ where $c_x = c_y = e^\tau$. Thus we can write the points as a sequence $(Y_i, Z_i), i = 1, 2, \ldots$ such that the sequence (Z_i) is decreasing and such that $Y_i \in [0, c_x] \times [0, c_y]$ for all i.

Define $H_u^k = T_k(e^u, e^{-u})$ where

$$T_k(x,y) = \sup_{A \in \mathcal{C}_{xy}^k} \sum_{i \in A} Z_i,$$

with

$$\mathcal{C}_{xy}^k = \{A \subset \{1, \ldots, k\} : Y_i \sim Y_j \,\forall i, j \in A\}.$$

As before, we need a corresponding formulation of the discrete model. For given n, we work on the box $B^n = \{1, \ldots, \lceil ne^\tau \rceil\}^2$. The weight at a point $y \in B^n$ has distribution F, independently for different points. We represent the weights and their positions by a vector $(Y_i^n, M_i^n, 1 \le i \le \lceil ne^\tau \rceil^2)$, where M_i^n form a decreasing sequence. The interpretation is that Y_i^n is the location of the ith largest weight M_i^n in the box B^n.

We can then define $H^{(n),k} = \tilde{T}_k^{(n)}(\lceil ne^u \rceil, \lceil ne^{-u} \rceil)$ where

$$\tilde{T}_k^{(n)}(\lceil nx \rceil, \lceil ny \rceil) = \sup_{A \in \mathcal{C}_{xy}^{(n),k}} \sum_{i \in A} a_{n^2}^{-1} M_i^{(n)},$$

with

$$\mathcal{C}_{xy}^{(n),k} = \{A \subset \{1, \ldots, k \wedge n^2\} : \forall i, j \in A, Y_i^{(n)} \sim Y_j^{(n)}\}.$$

For the proof we will establish that $H^{(n),k}$ converges weakly to H^k, that $H^k \to H$ locally uniformly and that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(\rho(H^{(n),k}, H^{(n)}) > \epsilon) = 0,$$

for each $\epsilon > 0$, where ρ is the metric for the Skorohod topology.

We begin by establishing $H^k \to H$ as $k \to \infty$. This is a consequence of the construction via a PRM. For each u, H_u^k is an increasing function of k and converges to H_u. With probability 1, this holds uniformly for each $u \in [-\tau, \tau]$, since for all such u we have

$$0 \le H_u - H_u^k \le \sup_{A \in \mathcal{C}_{e^\tau, e^\tau}} \sum_{i \in A, i > k} M_i;$$

the upper bound is finite with probability 1 exactly as in Lemma 3.1.

Next we wish to show the weak convergence of $H^{(n),k}$ to H^k. This can be done by an extension of the method of Proposition 3.2. Note that with probability 1, no two points of the PRM share a vertical coordinate or a horizontal coordinate, and in addition no point of the PRM falls on the line parametrised by $(x, y) = (e^u, e^{-u})$. Then under the same couplings used in the proof of Proposition 3.2, one obtains that with probability 1, $H^{(n),k} \to H^k$ in the Skorohod space. Thus this weak convergence also holds as desired.

Finally we need to control $\rho(H^{(n),k}, H^{(n)})$. Fix an $\epsilon > 0$. Consider

$$\begin{aligned}
\mathbb{P}\left(\rho(H^{(n),k}, H^{(n)}) > \epsilon\right) &\le \mathbb{P}\left(\sup_{-\tau<u<\tau} \left|H_u^{(n),k} - H_u^{(n)}\right| > \epsilon\right) \\
&= \mathbb{P}\left(\sup_{-\tau<u<\tau} \left|\tilde{T}_k^{(n)}(e^u, e^{-u}) - \tilde{T}^{(n)}(e^u, e^{-u})\right| > \epsilon\right) \\
&= \mathbb{P}\left(\sup_{-\tau<u<\tau} \left|\sup_{A \in C_{e^u, e^{-u}}^{(n),k}} \sum_{i \in A} a_{n^2}^{-1} M_i^n - \sup_{A \in C_{e^u, e^{-u}}^{(n)}} \sum_{i \in A} a_{n^2}^{-1} M_i^n\right| > \epsilon\right) \\
&\le \mathbb{P}\left(\sup_{-\tau<u<\tau} \left|\tilde{S}_k^{(n)}(e^u, e^{-u})\right| > \epsilon\right),
\end{aligned}$$

where $\tilde{S}_k^{(n)}(e^u, e^{-u}) = \sup_{A \in C_{e^u, e^{-u}}^{(n)}} \sum_{i \in A, i > k} a_{n^2}^{-1} M_i^n$. By monotonicity we have

$$\sup_{-\tau<u<\tau} \left|\tilde{S}_k^{(n)}(e^u, e^{-u})\right| \le \tilde{S}_k^{(n)}(e^\tau, e^\tau),$$

and using the scaling

$$\mathbb{P}\Big(\rho(H^{(n),k}, H^{(n)}\Big) > \epsilon) \le \mathbb{P}\Big(\tilde{S}_k^{(n)}(1,1) > \epsilon e^{-2\tau/\alpha}\Big).$$

By Proposition 3.3 we have that $\mathbb{P}\big(\tilde{S}_k^{(n)} > \epsilon e^{-2\tau/\alpha}\big) \to 0$ as $k \to \infty$ uniformly in n. Thus indeed we have

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\big(\rho(H^{(n),k}, H^{(n)}\big) > \epsilon) = 0.$$

Putting the three pieces together we have shown the weak convergence. □

8 Higher-dimensional heavy-tailed last passage percolation

Up to this point we have considered only two-dimensional models. In this section we indicate how to extend most of the results to higher dimensions in a natural way.

For general $d \ge 2$, we consider the passage time from the point $(1, 1, \dots, 1)$ to the point $(n, n, \dots, n)$.

We now consider a sequence of locations $Y_i^{(n)}$, $i = 1, 2, \dots, n^d$ which form a uniform random permutation of the set $\{1/n, 2/n, \dots, 1\}^d \subset [0, 1]^d$, and a corresponding sequence of weights $M_i^{(n)}$, $i = 1, 2, \dots, n^d$ which are given by the order statistics, in decreasing order, of a sample of size n^d from the underlying weight distribution F. We now assume that the tail of F is regularly varying with index $\alpha < d$.

Defining $\mathcal{C}^{(n)}$ as before, we set $T^{(n)} = \sup_{A\in\mathcal{C}^{(n)}} \sum_{i\in A} M_i^{(n)}$, and $\tilde{T}^{(n)} = a_{n^d}^{-1} T^{(n)}$.

The continuous model is defined just as before; the locations Y_i are now drawn i.i.d. and uniformly at random from the box $[0, 1]^d$ rather than the square $[0, 1]^2$.

Then $\tilde{T}^{(n)} \to T$ in distribution as $n \to \infty$; the method of proof is essentially identical to that used for Theorem 2.1 in the case $d = 2$.

The multivariate extensions described in Sections 2.4 and 7 go through in an analogous way. For example, we can now obtain a process Θ which is stationary on $\mathbb{R}^d$ such that

$$\left\{ \exp\left(-\frac{u_1 + \cdots + u_d}{\alpha}\right) a_{n^d}^{-1} T^{(n)}\left(e^{u_1}, \dots, e^{u_d}\right), \mathbf{u} \in \mathbb{R}^d \right\} \to \Big\{\Theta(u_1, \dots, u_d), \mathbf{u} \in \mathbb{R}^d\Big\}$$

as $n \to \infty$, in the sense of convergence of finite-dimensional distributions; here $T^{(n)}(u_1, \dots, u_d)$ is the maximal weight of a path from $(1, \dots, 1)$ to the point $(\lceil nu_1 \rceil, \dots, \lceil nu_d \rceil)$.

We turn to the path convergence as developed in Section 4. Proposition 4.1 and Theorem 4.2 extend easily, with the same method of proof. However, extending Theorem 4.4, concerning the convergence of optimal paths viewed as random subsets of $[0, 1]^d$, is more problematic. Again we are unable to prove that the optimal path for the continuous model (i.e. the closure of $\bigcup_{i\in A^*} Y_i$) is connected (although we expect this to be true). In the case $d = 2$ this caused a little inconvenience but we could work around it by observing that any "jumps" in the path consist of horizontal or vertical line segments, and hence that at least there exists a unique connected increasing path that contains the optimal path.

For $d \geq 3$, however, a jump could, for example, cross a square of zero volume in $\mathbb{R}^d$ but with non-zero area. Then there is no longer a unique way to extend the optimal path to a connected increasing path, and thus there is an ambiguity in the limit object. If we could prove the conjecture that the optimal path itself is connected, the convergence in distribution of the discrete optimal paths $P^{(n)*}$ would follow as before.

In the case $\alpha = 0$, we can in fact prove the connectedness of the optimal path for the continuous model (defined as in Section 6 using the "greedy algorithm"). It is not clear how to extend the multifractal analysis of Section 6.2. However, by analysing a branching random walk associated with the algorithm which constructs the greedy path, one can obtain that the function from, say, $x_1 \in [0,1]$ to $(x_2, \ldots, x_d) \in [0,1]^{d-1}$ which describes the path is almost surely everywhere continuous and strictly increasing (although a.s. it also has derivative 0 almost everywhere). Thus one can show that $P^{(n)*} \to P^*$ in distribution (under the Hausdorff metric on subsets of $[0,1]^d$) for all d in the case $\alpha = 0$.

References

[1] Aldous, D. and Diaconis, P., (1999) Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem. *Bull. Amer. Math. Soc. (N.S.)* **36**, 413–432.

[2] Arbeiter, M. and Patzschke, N., (1996) Random self-similar multifractals. *Math. Nachr.* **181**, 5–42.

[3] Baccelli, F., Borovkov, A. and Mairesse, J., (2000) Asymptotic results on infinite tandem queueing networks. *Probab. Theory Related Fields* **118**, 365–405.

[4] Baik, J., Deift, P., McLaughlin, K. T.-R., Miller, P. and Zhou, X., (2001) Optimal tail estimates for directed last passage site percolation with geometric random variables. *Adv. Theor. Math. Phys.* **5**, 1207–1250.

[5] Baik, J. and Suidan, T., (2005) A GUE central limit theorem and universality of directed first and last passage site percolation. *Int. Math. Res. Not.* **2005:6**, 325–337.

[6] Baryshnikov, Y., (2001) GUEs and queues. *Probab. Theory Related Fields* **119**, 256–274.

[7] Bertoin, J., (1996) *Lévy processes*, vol. 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge.

[8] Biggins, J. D., Hambly, B. M. and Jones, O. D., (2006) Multifractal spectra for random self-similar measures via branching processes. In preparation.

[9] Bodineau, T. and Martin, J. B., (2005) A universality property for last-passage percolation paths close to the axis. *Electron. Comm. Probab.* **10**, 105–112 (electronic).

[10] David, H. A., (1981) *Order statistics*. John Wiley & Sons Inc., New York, 2nd ed. Wiley Series in Probability and Mathematical Statistics.

[11] Falconer, K. J., (1994) The multifractal spectrum of statistically self-similar measures. *J. Theoret. Probab.* **7**, 681–702.

[12] Glynn, P. W. and Whitt, W., (1991) Departures from many queues in series. *Ann. Appl. Probab.* **1**, 546–572.

[13] Gravner, J., Tracy, C. A. and Widom, H., (2001) Limit theorems for height fluctuations in a class of discrete space and time growth models. *J. Statist. Phys.* **102**, 1085–1132.

[14] Hambly, B. M., Martin, J. B. and O'Connell, N., (2002) Concentration results for a Brownian directed percolation problem. *Stochastic Process. Appl.* **102**, 207–220.

[15] Janson, S., Łuczak, T. and Rucinski, A., (2000) *Random graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York.

[16] Johansson, K., (2000) Shape fluctuations and random matrices. *Comm. Math. Phys.* **209**, 437–476.

[17] Johansson, K., (2000) Transversal fluctuations for increasing subsequences on the plane. *Probab. Theory Related Fields* **116**, 445–456.

[18] Johansson, K., (2003) Discrete polynuclear growth and determinantal processes. *Comm. Math. Phys.* **242**, 277–329.

[19] Krug, J. and Spohn, H., (1992) Kinetic roughening of growing surfaces. In C. Godrèche, ed., *Solids far from equilibrium*, Collection Aléa-Saclay: Monographs and Texts in Statistical Physics, 1, pages 479–582. Cambridge University Press, Cambridge.

[20] Martin, J. B., (2002) Large tandem queueing networks with blocking. *Queueing Syst. Theory Appl.* **41**, 45–72.

[21] Martin, J. B., (2002) Linear growth for greedy lattice animals. *Stochastic Process. Appl.* **98**, 43–66.

[22] Martin, J. B., (2004) Limiting shape for directed percolation models. *Ann. Probab.* **32**, 2908–2937.

[23] O'Connell, N., (2003) Random matrices, non-colliding particle systems and queues. In *Séminaire de Probabilités XXXVI*, no. 1801 in Lecture Notes in Mathematics, pages 165–182. Springer-Verlag.

[24] O'Connell, N. and Yor, M., (2001) Brownian analogues of Burke's theorem. *Stochastic Process. Appl.* **96**, 285–304.

[25] Prähofer, M. and Spohn, H., (2002) Scale invariance of the PNG droplet and the Airy process. *J. Statist. Phys.* **108**, 1071–1106.

[26] Resnick, S. I., (1986) Point processes, regular variation and weak convergence. *Adv. in Appl. Probab.* **18**, 66–138.

[27] Rost, H., (1981) Nonequilibrium behaviour of a many particle process: density profile and local equilibria. *Z. Wahrsch. Verw. Gebiete* **58**, 41–53.

[28] Soshnikov, A., (2004) Poisson statistics for the largest eigenvalues of Wigner random matrices with heavy tails. *Electron. Comm. Probab.* **9**, 82–91 (electronic).

Mathematical Institute,
University of Oxford,
24-29 St Giles,
Oxford OX1 3LB,
UK
hambly@maths.ox.ac.uk
http://www.maths.ox.ac.uk/~hambly

Department of Statistics,
University of Oxford,
1 South Parks Road,
Oxford OX1 3TG
UK
martin@stats.ox.ac.uk
http://www.stats.ox.ac.uk/~martin