

# Multilevel simulation of functionals of Bernoulli random variables with application to basket credit derivatives

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## Abstract

We consider  $N$  Bernoulli random variables, which are independent conditional on a common random factor determining their probability distribution. We show that certain expected functionals of the proportion  $L_N$  of variables in a given state converge at rate  $1/N$  as  $N \rightarrow \infty$ . Based on these results, we propose a multi-level simulation algorithm using a family of sequences with increasing length, to obtain estimators for these expected functionals with a mean-square error of  $\epsilon^2$  and computational complexity of order  $\epsilon^{-2}$ , independent of  $N$ . In particular, this optimal complexity order also holds for the infinite-dimensional limit. Numerical examples are presented for tranche spreads of basket credit derivatives.

**Key words:** Multilevel Monte Carlo simulation, large deviations principle, exchangeability, basket credit derivatives

## 1 Introduction

This article is concerned with the behaviour of functionals of a large number of exchangeable random variables and the efficient numerical estimation of their expectation. The objective of this work is thus two-fold: to analyse the order of convergence in  $1/N$  of expected functionals for  $N \rightarrow \infty$ , and to derive estimators for these expectations for which the computational complexity is asymptotically independent of  $N$ .

First, we analyse convergence in the case of general Lipschitz and smooth functions  $p$  of the average of  $N$  exchangeable Bernoulli random variables as  $N$  goes to infinity. We then specify  $p$  further to certain piecewise linear functions and show that the convergence order is the same as in the smooth case. These results are relevant, for instance, if one wants to approximate the result for large but finite  $N$  by its limit. A number of applications comes from the credit risk literature. In [13], Vasicek derives an expression for the limiting distribution of portfolio losses in a Normal factor model, where default of a firm is indicated by its value process being below a default barrier at maturity of the debt. In

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the large portfolio limit, the randomness comes solely from a common market factor, while a law of large numbers holds for idiosyncratic components conditionally on this factor. Bush *et al.*, in [5], extend this to a dynamic set-up where it is seen that the density of the limit empirical measure of firm values satisfies a stochastic partial differential equation (SPDE) and can be used to approximate tranche spreads of basket credit derivatives; [4] gives an extension to jump diffusion models while [9] include extensions to heterogeneity and self-exciting defaults rendering the resulting equations non-linear. Further studies focus particularly on the tail of the limiting loss distribution, see [7], [11] and the references therein.

A driving practical motivation for investigating the limiting behaviour is that the original sequence of random variables is costly to simulate, because of the large number  $N$  of underlying processes, often required over large time horizons. Moreover, often many Monte Carlo samples are necessary for sufficiently accurate estimation of, for instance, expected tranche losses of credit basket. This paper takes a different tack and develops a simulation method where the computational complexity is asymptotically independent of  $N$ . A small tweak of the algorithm can also be used to approximate the limit obtained for  $N \rightarrow \infty$ .

More concretely, it turns out that an interpretation of the multi-level Monte Carlo approach (see [10]) in the present context allows us to construct estimators based on sequences with increasing lengths and a number of samples which decreases faster than the length increases, such that the overall computational complexity is essentially no larger than for fixed small  $N$ .

A conceptually similar though distantly related approach is used in [3], where the multilevel idea is applied to a sequence of martingales to estimate a dual upper bound for the value of an early exercise option. In that setting they are able to show, as we do here, that the achievable complexity is not substantially larger than that of a non-nested simulation. The general problem of estimating conditional expectations through nested multilevel simulation is addressed in [6]. There, further extrapolation is used to reduce the bias of estimators, while here we will propose an improved estimator which reduces the variance of higher level estimators.

This article is organised as follows. In Section 2, we introduce the setting and outline the main convergence results, explaining how they can be used to construct efficient estimators. The first key result on the convergence order of expected functionals is proved in Section 3, with numerical illustrations from an example of a basket credit derivative presented in Section 4. In Section 5, we introduce in detail two multilevel simulation methods and derive bounds on their computational complexity to achieve a prescribed accuracy. Finally, in Section 6 we present numerical results illustrating the efficiency gains achieved through multilevel simulation in this context and Section 7 concludes.

## 2 Set-up and main results

In this article, we are concerned with the behaviour of “loss” variables describing the fraction of  $N$  random variables in a certain state, and expected functionals of this loss variable, as  $N$  goes to infinity. The application we have in mind, and for which we will present numerical illustrations, is that of a basket of defaultable firms, and then the loss is the fraction of firms which default over a certain period.

More precisely, on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , consider a sequence of Bernoulli random variables  $Y_i$ ,  $i \in \mathbb{N}$ , and a random variable  $L$  taking values in  $[0, 1]$ . If required we write  $\Omega = \Omega_Y \times \Omega_L$  where canonically we could take  $\Omega_Y = \{0, 1\}^{\mathbb{N}}$  and  $\Omega_L = [0, 1]$ . The probability measure  $\mathbb{P}$  is constructed as follows. The random variable  $L$  is generated according to its marginal law  $\mathbb{P}_L$  and then, conditional on  $\mathcal{F}_L$ , the  $\sigma$ -algebra generated by  $L$ , the  $Y_i$  are independent random variables with law given by

$$\mathbb{P}[Y_i = 1 | \mathcal{F}_L] = L. \quad (2.1)$$

To re-iterate, the Bernoulli random variables are conditionally independent given a common factor. Thus for each  $n \in \mathbb{N}$

$$\mathbb{P}(Y_1 = y_1, \dots, Y_n = y_n, L \in B) = \int_B l^{s_n} (1-l)^{n-s_n} \mathbb{P}_L(L \in dl), \quad \forall y_i \in \{0, 1\}, B \subset [0, 1]$$

where  $s_n = \sum_{i=1}^n y_i$ . We will often write  $\mathbb{P}_{|L} = \mathbb{P}(\cdot | \mathcal{F}_L)$  for the conditional law of the  $Y_i$  given  $\mathcal{F}_L$  and  $\mathbb{E}_{|L}$  for the associated conditional expectation. In the setting of defaultable firms,  $Y_i = 1$  iff the  $i$ -th firm defaults, and  $L$  is a global factor modelling the common tendency of firms to default. We define the loss variable to be the proportion of Bernoulli variables in state 1

$$L_N = \frac{1}{N} \sum_{i=1}^N Y_i. \quad (2.2)$$

We consider a Lipschitz function  $p$  and random variables  $P$  and  $P_N$  defined as

$$P \equiv p(L), \quad (2.3)$$

$$P_N \equiv p(L_N). \quad (2.4)$$

In particular, we will study  $p$  of the form

$$p(l) \equiv [l - K_1]^+ - [l - K_2]^+ = \begin{cases} 0 & l \leq K_1, \\ l - K_1 & K_1 \leq l \leq K_2, \\ K_2 - K_1 & l \geq K_2, \end{cases} \quad (2.5)$$

where  $[x]^+ = \max(x, 0)$  denotes the positive part and  $0 \leq K_1 < K_2 \leq 1$  are constants. In credit derivative pricing, the particular shape of the function  $p$  in (2.5) measures the losses in a certain *tranche* with attachment point  $K_1$  and detachment point  $K_2$ , and its expectation is the building block for formulae for CDO tranche spreads. A typical CDO pool consists of  $N = 125$  firms, while typical loan or mortgage books can have substantially more obligors, and it is therefore practically relevant to understand the behaviour of expected functionals for large  $N$  and to devise computationally efficient estimators.

By a conditional version of the strong law of large numbers and the continuity of  $p$

$$L_N \rightarrow L \quad \text{for } N \rightarrow \infty, \quad \mathbb{P}_{|L} - a.s., \quad (2.6)$$

$$P_N \rightarrow P \quad \text{for } N \rightarrow \infty, \quad \mathbb{P}_{|L} - a.s. \quad (2.7)$$

This convergence will also hold in  $L^2(\Omega_Y, \mathbb{P}_{|L})$  (see Lemma 3.1).

We study here the convergence rate of  $P_N - P$  and will prove the following two results. The first statement for Lipschitz and smooth functions  $p$  is a relatively straightforward consequence of (2.1) and the easily computable  $L_2$  convergence rate of  $L_N$ . The second result shows that for a specific  $p$  which is only piecewise smooth we can still obtain the same convergence order as in the smooth case and with explicitly computable bounds.

**Theorem 2.1.** *Let  $P$  and  $P_N$  be defined by (2.3) and (2.4), respectively, and assume that  $p$  is Lipschitz with constant  $c_p$ . We have that*

$$|\mathbb{E}[P_N - P]| \leq \frac{c_p}{2\sqrt{N}}, \quad (2.8)$$

$$\text{Var}[P_N - P] \leq \frac{c_p^2}{4N}. \quad (2.9)$$

If, moreover,  $p$  is differentiable and the derivative has Lipschitz constant  $C_p$ , then

$$|\mathbb{E}[P_N - P]| \leq \frac{C_p}{8N}. \quad (2.10)$$

**Theorem 2.2.** *For  $p$  defined in (2.5), if the cumulative density function (CDF)  $F_L$  of  $L$  is Lipschitz at any  $K_1 > 0$  and  $K_2 < 1$  with Lipschitz constant  $c_L$ , i.e.,*

$$|F_L(K_j) - F_L(l)| \leq c_L |K_j - l| \quad (2.11)$$

for  $j = 1, 2$  and all  $l \in [0, 1]$ , then

$$|\mathbb{E}[P_N - P]| \leq \frac{4c_L\sqrt{\pi}}{N}.$$

Note that if  $L$  has a density function which is bounded, then the CDF is certainly Lipschitz. The fact that we only need the Lipschitz property at  $K_1$  and  $K_2$  will be useful for the applications considered later.

Taking the two Theorems together, order 1 for the convergence of expectations also follows for piecewise smooth  $p$  which are Lipschitz overall, provided  $F_L$  is Lipschitz.

These Theorems show that expected functionals for large or infinite  $N$  can be successively approximated by those with smaller  $N$ . Combining this with a control variate idea leads to multilevel simulation with a substantial variance reduction for large  $N$ . Specifically, the above results imply that for Lipschitz  $p$  we have  $|\mathbb{E}[P_N - P_{MN}]| \leq c_1/\sqrt{N}$  and  $\text{Var}[P_N - P_{MN}] \leq c_2/N$  for any positive integer  $M$  with some constants  $c_1$  and  $c_2$ . We can consider a sequence  $N_l = M^l$ ,  $l \in \mathbb{N}$ , with corresponding  $L^{(l)} = L_{N_l}$  and  $P^{(l)} = P_{N_l}$ . Translating the central idea in [10] to this setting, we estimate in the decomposition

$$\mathbb{E}[P^{(l)}] = \mathbb{E}[P^{(0)}] + \sum_{k=1}^l \mathbb{E}[P^{(k)} - P^{(k-1)}] \quad (2.12)$$

every summand separately with independent samples of different sizes. These can be chosen to obtain an optimal allocation of computational cost for a given overall mean-square error (MSE). The general construction in [10] immediately gives the following result.

**Proposition 2.1** (cf. [10], Theorem 3.1). *Let  $P$ ,  $P^{(l)}$  as above. If there exist independent estimators  $Z_l$  based on  $n_l$  Monte Carlo samples, and positive constants  $\alpha, \beta, c_1, c_2, c_3$  such that  $\alpha \geq \frac{1}{2}$  and*

$$i) \quad \left| \mathbb{E}[P^{(l)} - P] \right| \leq c_1 M^{-\alpha l}$$

$$ii) \mathbb{E}[Z_l] = \begin{cases} \mathbb{E}[P^{(0)}], & l = 0 \\ \mathbb{E}[P^{(l)} - P^{(l-1)}], & l > 0 \end{cases}$$

$$iii) \mathbb{V}[Z_l] \leq c_2 n_l^{-1} M^{-\beta l}$$

$$iv) C_l \leq c_3 n_l N_l, \text{ where } C_l \text{ is the computational complexity of } Z_l$$

then there exists a positive constant  $c_4$  such that for any  $\varepsilon < e^{-1}$  there are values  $K$  and  $n_l$  for which the multilevel estimator

$$G_K = \sum_{l=0}^K Z_l, \quad (2.13)$$

has a mean-square-error with bound

$$MSE \equiv \mathbb{E}[(G_K - E[P])^2] < \varepsilon^2$$

with a computational complexity  $C$  with bound

$$C \leq \begin{cases} c_4 \varepsilon^{-2}, & \beta > 1, \\ c_4 \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ c_4 \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

**Corollary 2.1.** Let  $P_N$  and  $P$  be as in (2.3) and (2.4), and assume  $p$  is Lipschitz.

1. There is a multilevel estimator for  $\mathbb{E}[P]$  with  $MSE \epsilon^2$  with computational complexity  $C \leq c_4 (\log \epsilon)^2 \epsilon^{-2}$ .
2. For all  $N$ , there is a multilevel estimator for  $\mathbb{E}[P_N]$  with  $MSE \epsilon^2$  with computational complexity  $C \leq c_4 (\log \epsilon)^2 \epsilon^{-2}$ , where  $c_4$  is independent of  $N$ .

Note that only order 1/2 is required for the convergence of expectations in Proposition 2.1, i), and that the complexity is then dictated by  $\beta$ , the case  $\beta = 1$  implied by Theorem 2.1 for all Lipschitz payoffs being a boundary case.

We point out that for fixed  $N$  the standard (i.e., single level) Monte Carlo estimator has a complexity  $C \leq c \epsilon^{-2}$ , however,  $c$  increases linearly in  $N$ . The significance of the multilevel estimator is that the constant  $c_4$  is independent of  $N$ .

For the specific  $p$  as in (2.5), we can exploit the piecewise linearity of  $p$  to construct multilevel estimators with even better complexity, by making the following observations: The summands in (2.12) are unchanged if we replace  $P^{(k-1)} = p(L^{(k-1)})$  with any of  $p(L_m^{(k-1)})$  for  $m = 1, \dots, M$ , where

$$L_m^{(k-1)} \equiv \frac{1}{N_{k-1}} \sum_{i=1}^{N_{k-1}} Y_{i+(m-1)N_{k-1}}. \quad (2.14)$$

This is a direct consequence of the exchangeability. Now,

$$L^{(k)} = \frac{1}{M} \sum_{m=1}^M L_m^{(k-1)} \quad (2.15)$$

and, if all  $L_m^{(k-1)}$  lie in the same interval  $[0, K_1]$ ,  $(K_1, K_2]$  or  $(K_2, 1]$ , also  $P^{(k)} = \bar{P}^{(k-1)}$ , where

$$\bar{P}^{(k-1)} \equiv \frac{1}{M} \sum_{m=1}^M P_m^{(k-1)} = \frac{1}{M} \sum_{m=1}^M p(L_m^{(k-1)}), \quad (2.16)$$

since  $p$  is linear in these intervals. Because of  $\mathbb{E}[P^{(k-1)}] = \mathbb{E}[\bar{P}^{(k-1)}]$ , we can now write

$$\mathbb{E}[P^{(l)}] = \mathbb{E}[P^{(0)}] + \sum_{k=1}^l \mathbb{E}[P^{(k)} - \bar{P}^{(k-1)}], \quad (2.17)$$

and if we estimate the individual terms in the sum independently in the multilevel spirit, there is only a variance contribution from a specific sample of the  $k$ -th term if at least two  $P_m^{(k-1)}$  lie in different intervals. For large  $k$ , the probability of this is small, and we will be able to show the following result.

**Theorem 2.3.** *For  $p$  as in (2.5), let  $P^{(l)}$  as in Proposition 2.1 and  $\bar{P}^{(l-1)}$  as in (2.16). If the CDF  $F_L$  of  $L$  is Lipschitz with Lipschitz constant  $c_L$ , then*

$$\text{Var}[P^{(l)} - \bar{P}^{(l-1)}] \leq \frac{c_2}{N_l^{3/2}}, \quad (2.18)$$

where  $c_2 = c_L 4\sqrt{M\pi}(\sqrt{2} + \sqrt{M})\sqrt{\frac{7}{8}(M^2 + 6M + 1)}$ .

Here and throughout the paper we give explicit expressions for the constants. These should not be regarded as optimal in any sense.

**Corollary 2.2.** *For Lipschitz  $F_L$  and  $p$  as in (2.5), there is a constant  $c_5$  and multilevel estimators for  $\mathbb{E}[P]$  and  $\mathbb{E}[P_N]$  with MSE  $\epsilon^2$  with computational complexity  $C \leq c_5 \epsilon^{-2}$ .*

The point here is that we managed to remove the logarithmic factor present in Corollary 2.1 and that  $c_5$  does not depend on  $N$ .

### 3 Proof of convergence rates

We first prove Theorem 2.1 which contains statements in the general and smooth case. The rest of this section is devoted to the proof of Theorem 2.2 dealing with a specific non-smooth payoff relevant to our application.

**Lemma 3.1.** *Let  $P_N$  and  $P$  be as in (2.3) and (2.4), and assume  $p$  is Lipschitz with constant  $c_p$ . Then*

$$\mathbb{E}_L[(P_N - P)^2] \leq \frac{c_p^2}{4N}.$$

*Proof.* Since the function  $p$  in (2.3) is assumed Lipschitz and  $\mathbb{E}_L[L_N] = L$ , we have

$$\mathbb{E}_L[(P_N - P)^2] \leq c_p^2 \mathbb{E}_L[(L_N - L)^2] = c_p^2 \text{Var}[L_N | \mathcal{F}_L] = \frac{c_p^2}{N} \text{Var}[Y_i | \mathcal{F}_L] = \frac{c_p^2}{N} L(1 - L).$$

For  $L \in [0, 1]$ ,  $L(1 - L) \leq \frac{1}{4}$ , which gives the result.  $\square$

*Proof of Theorem 2.1.* Equation (2.9) follows directly from Lemma 3.1, and then, by Hölder's inequality,

$$|\mathbb{E}[P - P_N]| \leq \sqrt{\mathbb{E}[\mathbb{E}_{|L}[(P_N - P)^2]]} \leq \frac{c_p}{2\sqrt{N}}.$$

For differentiable  $p$ , we can write

$$\mathbb{E}[p(L) - p(L_N)] = \mathbb{E}[p'(L)(L - L_N)] + \mathbb{E}[r(L, L_N)],$$

with some remainder  $r$ , where the first term on the left-hand side is

$$\mathbb{E}[\mathbb{E}_{|L}[p'(L)(L - L_N)]] = 0.$$

For a Lipschitz derivative of  $p$ ,

$$|p(x) - p(y) - p'(x)(x - y)| \leq \frac{1}{2}C_p(x - y)^2$$

for all  $0 \leq x, y \leq 1$  and the remainder term satisfies

$$|\mathbb{E}[r(L, L_N)]| \leq \frac{C_p}{8N},$$

from which (2.10) follows.  $\square$

Now, we turn to the proof of Theorem 2.2 and show a few Lemmas first. We divide the ranges of  $L$  and  $L_N$  into the three intervals  $I_1 = [0, K_1]$ ,  $I_2 = (K_2, 1]$  and  $I_3 = (K_1, K_2]$ , in each of which the function  $p$  from (2.3) is linear; the point being that the probability of  $L$  and  $L_N$  lying in different intervals is small for large  $N$ , and the expected difference of  $P - P_N$  is small if they are in the same interval. The following Lemmas quantify this.

**Lemma 3.2.** *For  $j = 1, 2$ , we have*

$$\mathbb{P}_{|L}(L \in I_j, L_N \in I_j^c) \leq \mathbb{1}_{L \in I_j} e^{-N(L-K_j)^2}, \quad (3.1)$$

$$\mathbb{P}_{|L}(L \in I_j^c, L_N \in I_j) \leq \mathbb{1}_{L \in I_j^c} e^{-N(L-K_j)^2}. \quad (3.2)$$

*Proof.* This is a standard large deviations result. By Theorem 2.2.3 in [8], p. 27, and Remark (c) thereafter, for  $\mathcal{F}_L$ -independent and identically distributed random variables  $(Y_i)_{1 \leq i \leq N}$  with  $\mathbb{E}[Y_i | \mathcal{F}_L] = L$ , we obtain that if  $0 < L \leq K_j$ ,

$$\mathbb{P}_{|L}(L_N > K_j) \leq e^{-Ng(L, K_j)},$$

and if  $K_j < L < 1$ ,

$$\mathbb{P}_{|L}(L_N \leq K_j) \leq e^{-Ng(L, K_j)},$$

where the rate function  $g(L, K_j)$  is given on p. 35 in [8] as

$$g(L, K_j) = K_j \log\left(\frac{K_j}{L}\right) + (1 - K_j) \log\left(\frac{1 - K_j}{1 - L}\right),$$

since  $Y_i$  are Bernoulli distributed random variables with  $P(Y_i = 1 | \mathcal{F}_L) = L$ . It is straightforward to check that for all  $L \in (0, 1)$

$$g(L, K_j) \geq (K_j - L)^2. \quad (3.3)$$

Hence, by (3.3), for  $0 < L \leq K_j$

$$\mathbb{P}_{|L}(L_N > K_j) \leq e^{-Ng(L, K_j)} \leq e^{-N(K_j - L)^2}, \quad (3.4)$$

and similarly for  $K_j < L < 1$ . These estimates are clearly true for the degenerate cases  $L = 0$  and  $L = 1$ . From this the result follows.  $\square$

**Lemma 3.3.** *Let  $p$  be as in (2.5). If  $A_N$  is the event that  $L_N$  and  $L$  are in the same interval and  $A_N^c$  its complement, then*

$$\mathbb{E}[(P_N - P)1_{A_N}] = -\mathbb{E}[(L_N - L)1_{A_N^c}1_{\{L \in I_3\}}]. \quad (3.5)$$

*Proof.* By splitting the range of  $L$  into the different intervals,

$$\begin{aligned} \mathbb{E}[(P_N - P)1_{A_N}] &= \sum_{j=1}^3 \mathbb{E}[(P_N - P)1_{A_N}1_{\{L \in I_j\}}] \\ &= \mathbb{E}[(L_N - L)1_{A_N}1_{\{L \in I_3\}}] \\ &= -\mathbb{E}[(L_N - L)1_{A_N^c}1_{\{L \in I_3\}}], \end{aligned}$$

where we have used in the second line that  $P_N = P$  if both  $L_N$  and  $L$  lie in either  $I_1$  or  $I_2$  and that  $P_N - P = L_N - L$  in  $I_3$ ; in the last line that  $\mathbb{E}_{|L}[L_N - L] = 0$  and  $1_{A_N} + 1_{A_N^c} = 1$ .  $\square$

**Lemma 3.4.** *Let  $A_N^c$  be as in Lemma 3.3. If the CDF  $F_L$  of  $L$  is Lipschitz at  $K_j$ ,  $j = 1, 2$ , with constant  $c_L$ , then*

$$\mathbb{E} \left[ \left( \mathbb{P}_{|L}[A_N^c] \right)^{\frac{1}{2}} \right] \leq \frac{c_L 4\sqrt{\pi}}{\sqrt{N}}. \quad (3.6)$$

*Proof.* Let  $I_1^c = (K_1, 1]$  and  $I_2^c = [0, K_2]$  be the complements in  $[0, 1]$  of  $I_1$  and  $I_2$ , then

$$A_N^c \subseteq \{L \in I_1, L_N \in I_1^c\} \cup \{L \in I_2, L_N \in I_2^c\} \cup \{L \in I_1^c, L_N \in I_1\} \cup \{L \in I_2^c, L_N \in I_2\}$$

and therefore

$$\begin{aligned} \mathbb{P}_{|L}[A_N^c] &\leq \mathbb{P}_{|L}[L \in I_1, L_N \in I_1^c] + \mathbb{P}_{|L}[L \in I_2, L_N \in I_2^c] + \mathbb{P}_{|L}[L \in I_1^c, L_N \in I_1] \\ &\quad + \mathbb{P}_{|L}[L \in I_2^c, L_N \in I_2]. \end{aligned}$$

By (3.1), (3.2) we have

$$\mathbb{P}_{|L}[A_N^c] \leq 2 \left( e^{-N(L-K_1)^2} + e^{-N(L-K_2)^2} \right),$$

and we obtain

$$\mathbb{E} \left[ \left( \mathbb{P}_{|L}[A_N^c] \right)^{\frac{1}{2}} \right] \leq 2^{\frac{1}{2}} \left( \mathbb{E} \left[ e^{-N \frac{(L-K_1)^2}{2}} \right] + \mathbb{E} \left[ e^{-N \frac{(L-K_2)^2}{2}} \right] \right). \quad (3.7)$$



If we extended  $F_L$  by 0 and 1 from  $[0, 1]$  to  $\mathbb{R}$  then, for  $j = 1, 2$ , we have

$$\mathbb{E} \left[ e^{-N \frac{(L-K_j)^2}{2}} \right] = \int_{-\infty}^{\infty} e^{-N \frac{(l-K_j)^2}{2}} dF_L(l) \quad (3.8)$$

$$\begin{aligned} &= N \int_{-\infty}^{\infty} (l - K_j) e^{-N \frac{(l-K_j)^2}{2}} F_L(l) dl \\ &\leq N F_L(K_j) \int_{-\infty}^{\infty} (l - K_j) e^{-N \frac{(l-K_j)^2}{2}} dl + c_L N \int_{-\infty}^{\infty} (l - K_j)^2 e^{-N \frac{(l-K_j)^2}{2}} dl \\ &= \frac{c_L \sqrt{2\pi}}{\sqrt{N}}, \end{aligned} \quad (3.9)$$

where we used the Lipschitz property of the CDF after (3.9) and then integrated exactly. The result follows directly by insertion in (3.7).  $\square$

*Proof of Theorem 2.2.* By the tower property of conditional expectations and Jensen's inequality, we have

$$|\mathbb{E}[(P_N - P) 1_{A_N^c}]| \leq \mathbb{E}[|\mathbb{E}_{|L}[(P_N - P) 1_{A_N^c}]|]. \quad (3.10)$$

Then Cauchy-Schwarz gives

$$|\mathbb{E}[(P_N - P) 1_{A_N^c}]| \leq \mathbb{E}_{|L} \left[ \left( \mathbb{E}[(P_N - P)^2] \right)^{\frac{1}{2}} \left( \mathbb{P}_{|L}[A_N^c] \right)^{\frac{1}{2}} \right]. \quad (3.11)$$

By Lemmas 3.1 and 3.4, we obtain

$$|\mathbb{E}[(P_N - P) 1_{A_N^c}]| \leq \frac{c_L 2\sqrt{\pi}}{N}. \quad (3.12)$$

Similarly, using Lemma 3.3 and the same argument as above,

$$|\mathbb{E}[(P_N - P) 1_{A_N}]| \leq \frac{c_L 2\sqrt{\pi}}{N},$$

from which the statement follows.  $\square$

## 4 An application and numerical results

To illustrate the theoretical rate of convergence, we study numerical results for expected tranche losses of a synthetic CDO for an increasing size  $N$  of the underlying CDS pool.

We consider a structural factor model (see, e.g., [12, 4]), where the *distance-to-default* of the  $i$ -th firm,  $i = 1, \dots, N$ , evolves according to

$$X_t^i = X_0^i + \beta t + \sqrt{1 - \rho} W_t^i + \sqrt{\rho} B_t + J_t, \quad t > 0, \quad (4.1)$$

where  $\rho \in [0, 1)$ ,  $\beta$  given. Here,  $B$  is assumed to be a standard Brownian motion and  $J_t = \sum_{k=1}^{CP_t} \Pi_k$ , where  $CP_t$  is a Poisson process with intensity  $\lambda$  and  $\Pi_k$  are independent Normals with mean  $\mu_\Pi$  and variance  $\sigma_\Pi^2$ , while all  $W^i$  are independent standard Brownian motions and independent of  $B$  and  $J$ . Thus  $B$  and  $J$  model factors affecting the whole market, whereas  $W^i$  are idiosyncratic effects.

The  $i$ -th firm is considered to be in default if its distance-to-default is below 0 at any one of the observation times  $T_j = jq$ ,  $q = 0.25$  (quarterly), up to  $T_{20} = T = 5$ , the assumed maturity of the debt here. We introduce the default time  $\tau_i$  and Bernoulli random variable  $Y_i$  indicating default of the  $i$ -th firm before  $T$ , by

$$\begin{aligned}\tau_i &= \inf \left( \{t \in \{T_1, \dots, T_M\} : X_t^i \leq 0\} \cup \{\infty\} \right), \\ Y_i &= 1_{\{\tau_i \leq T\}}.\end{aligned}$$

For the numerical experiments, the initial values  $X_0^i$  are drawn independently from a Normal distribution,

$$X_0^i \sim N(\mu_{X_0}, \sigma_{X_0}^2),$$

where the mean  $\mu_{X_0} = 4.6$  and standard deviation  $\sigma_{X_0} = 0.8$  are obtained from a calibration to iTraxx data as detailed in [4], as are  $\rho = 0.13$ ,  $\lambda = 0.04$ ,  $\mu_\Pi = -0.5$  and  $\sigma_\Pi^2 = 0.17$ .

We illustrate the CDF  $F_L$  of  $L$  for different parameters in Figure 1. To this end, we generated samples of  $L$  by simulating the SPDE satisfied by the limit empirical measure of distances-to-default for  $N \rightarrow \infty$ , see [4] for details. It appears that  $F_L$  is Lipschitz in  $(0, 1)$  but that the derivative at 0 and 1 can become very large in certain parameter ranges for  $\mu_0 = \mu_{X_0}$  and overall instantaneous correlation

$$\rho_A = (\rho + \zeta)/(1 + \zeta), \quad \zeta = \lambda(\mu_\Pi^2 + \sigma_\Pi^2), \quad (4.2)$$

between  $X_t^i$  and  $X_t^j$  (see [4]).

For large values of  $\mu_0$ , the probability of defaults becomes very small and the density of  $L$  is concentrated around 0. For  $\rho_A$  approaching 1, all  $Y_i$  become identical and therefore either all or none of the firms default, such that here the density of  $L$  is concentrated at 0 and 1. In the degenerate case  $\rho_A = 0$  (i.e.,  $\rho = \lambda = 0$ ),  $L$  is deterministic, the measure is atomic and  $F_L$  a step function.

The empirical evidence thus suggests that  $F_L$  is Lipschitz in the range  $(0, 1)$ . Given that Theorem 2.2 only requires the Lipschitz property at interior values  $K_j$ , the conditions appear to be satisfied and the Theorem to apply in this setting. Even in situations where  $F_L$  has a bounded derivative at 0 and 1, the fact that only the Lipschitz constants from  $K_1$  and  $K_2$  enter into the estimates gives us substantially smaller bounds.

We now move on to present numerical results for the payoff function  $p$  from (2.5) illustrating the convergence as the number of firms  $N$  goes to infinity. We consider portfolios consisting of  $N_k = M^k = 5^k$  companies for  $k = 1, \dots, 7$ .

To include a recovery value of defaulted firms in the model, we rescale  $L_N$  by  $(1 - R)$ , where  $R = 0.4$  is the recovery rate. Equivalently, we pick  $(K_1, K_2) = (1 - R)^{-1}(a, d)$  in (2.3) and  $(a, d) \in \{(0, 0.03), (0.03, 0.06), (0.06, 0.09), (0.09, 0.12), (0.12, 0.22), (0.22, 1)\}$  as the attachment and detachment points for iTraxx tranches, and then study  $(1 - R)p(L_N)$ .

A straightforward Monte Carlo estimator for expected tranche losses  $\mathbb{E}[P^{(k)}]$  is then given by

$$\widehat{G}_k = \frac{1}{n} \sum_{j=1}^n (1 - R)p(L^{(k,j)}), \quad (4.3)$$

$$L^{(k,j)} = \frac{1}{N_k} \sum_{i=1}^{N_k} Y_i^{(j)}, \quad (4.4)$$

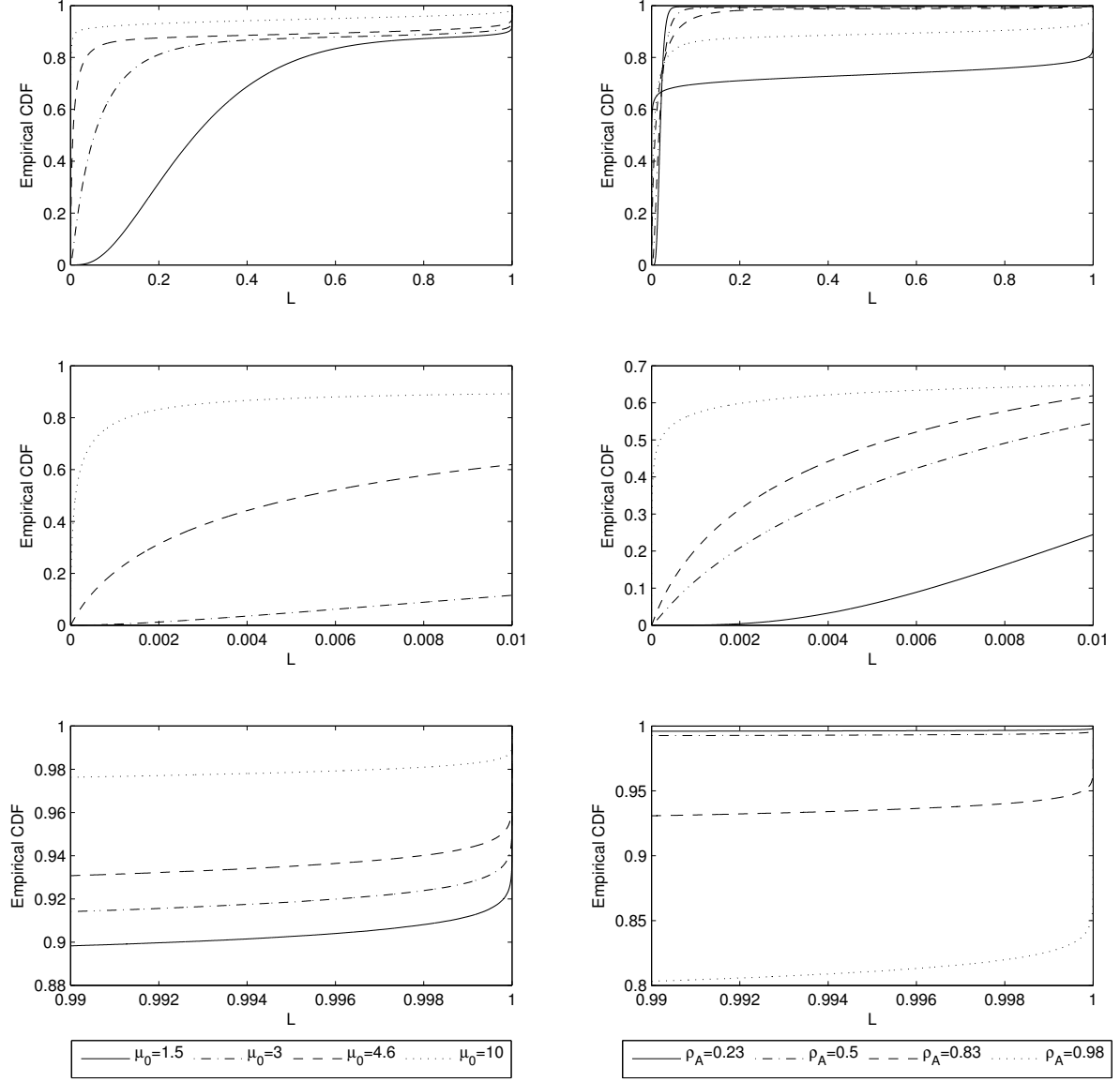


Figure 1: Top row: Empirical CDF  $F_L$  for different values of  $\mu_0 = \mu_{X_0}$  (left) and different  $\rho_A$  (right). The values of  $\rho_A$  are arrived at by (4.2) from  $(\rho, \lambda) \in \{(0.03, 0.001), (0.1, 0.002), (0.35, 0.0035), (0.35, 0.0351), (0.8, 0.1)\}$ . All other parameters are fixed as given in the text. The plots in the second and third rows are zoomed into the ranges of  $L$  close to 0 and 1, respectively.

where  $(Y_i^{(j)})$  are independent samples of  $Y_i$ , i.e., corresponding to independent paths for  $B$ ,  $W$  and  $J$ . There is no time discretisation error as (4.1) can be sampled directly. However, it turns out to be computationally prohibitively expensive to choose  $n$ , the number of samples, large enough to produce estimators with sufficiently small RMSE to allow us to distinguish between  $\widehat{G}_k$  and  $\widehat{G}_{k+1}$  for large  $k$ .

We therefore use the multilevel simulation approach outlined in Section 2 and detailed further in Section 5. The point is that the differences  $G_{k+1} - G_k$  are simulated directly in the multilevel approach. Therefore, we approximate  $|G - G_k|$ , where  $G \equiv \lim_{k \rightarrow \infty} G_k$ , by

$$S_k = |G_k - G_K| = \left| \sum_{l=k+1}^K Z_l \right| \quad (4.5)$$

for  $k < K$ , where  $Z_l$  is an estimator for  $\mathbb{E}[P^{(l)} - P^{(l-1)}]$  as used in the construction of  $G_k$  in (2.13) (precisely, we used the estimator  $Z_l$  defined later in (5.1)). The difference between  $S_k$  and  $|G - G_k|$  for  $k = K - 1$  is given by  $G_{K-1} - G_K \approx (G_{K-1} - G)(1 - 1/M)$  and for  $k = K - 2$  by  $G_{K-2} - G_K \approx (G_{K-2} - G)(1 - 1/M^2)$ . Given  $M = 5$  in our examples, the error due to this approximation will be seen to be smaller than the estimation error.

The results are shown in Figure 2. We plot the logarithm of  $S_k$  to base  $M$ , together with the sample standard deviation of the the multilevel estimators  $G_k$  (see (2.13)) and

$$y_k = -k + y_0, \quad (4.6)$$

where  $y_0$  is a suitably chosen constant, to verify the predicted convergence order empirically. The data points appear to be in good agreement with first order convergence.

## 5 A multilevel method and its numerical analysis

In this section, we describe and analyse a multilevel simulation approach for the estimation of expected functionals of the form (2.3) and (2.4), the latter with a particular emphasis on the case of large  $N$ .

The multilevel Monte Carlo method proposed by Giles in [10] estimates the expected value of a functional of the solution to a stochastic differential equation obtained by a timestepping scheme. It performs computations on different refinement levels  $l$  with time steps  $h_l = h_0 M^{-l}$  for  $M > 1$ , such as to minimise the overall computational time of the Monte Carlo estimator for prescribed mean square error (MSE). Since the MSE consists of a Monte Carlo error (variance) and a discretisation error (bias), the method controls both the number of samples  $n_l$  on level  $l$ , to bound the Monte Carlo variance of order  $O(n_l^{-1})$ , and the finest  $L$  with time step  $h^{-L}$  on which to approximate the SDE, in order to reduce the bias. The multilevel method is based on two premises: Monte Carlo estimators for an increasing number of time steps converge at a certain order in  $h_l$ , and the computational cost needed to calculate an estimator increases with  $n_l h_l^{-1}$ . In this approach, estimators obtained with a smaller number of time steps are used as control variates for estimators with a larger number of time steps, which significantly decreases the computation time.

To obtain a complexity result for an estimator of  $\mathbb{E}[P]$  with  $P$  from (2.3), we substitute  $h_l$  by  $N_l^{-1}$  in Theorem 3.1 of [10] and immediately obtain Proposition 2.1 from Section

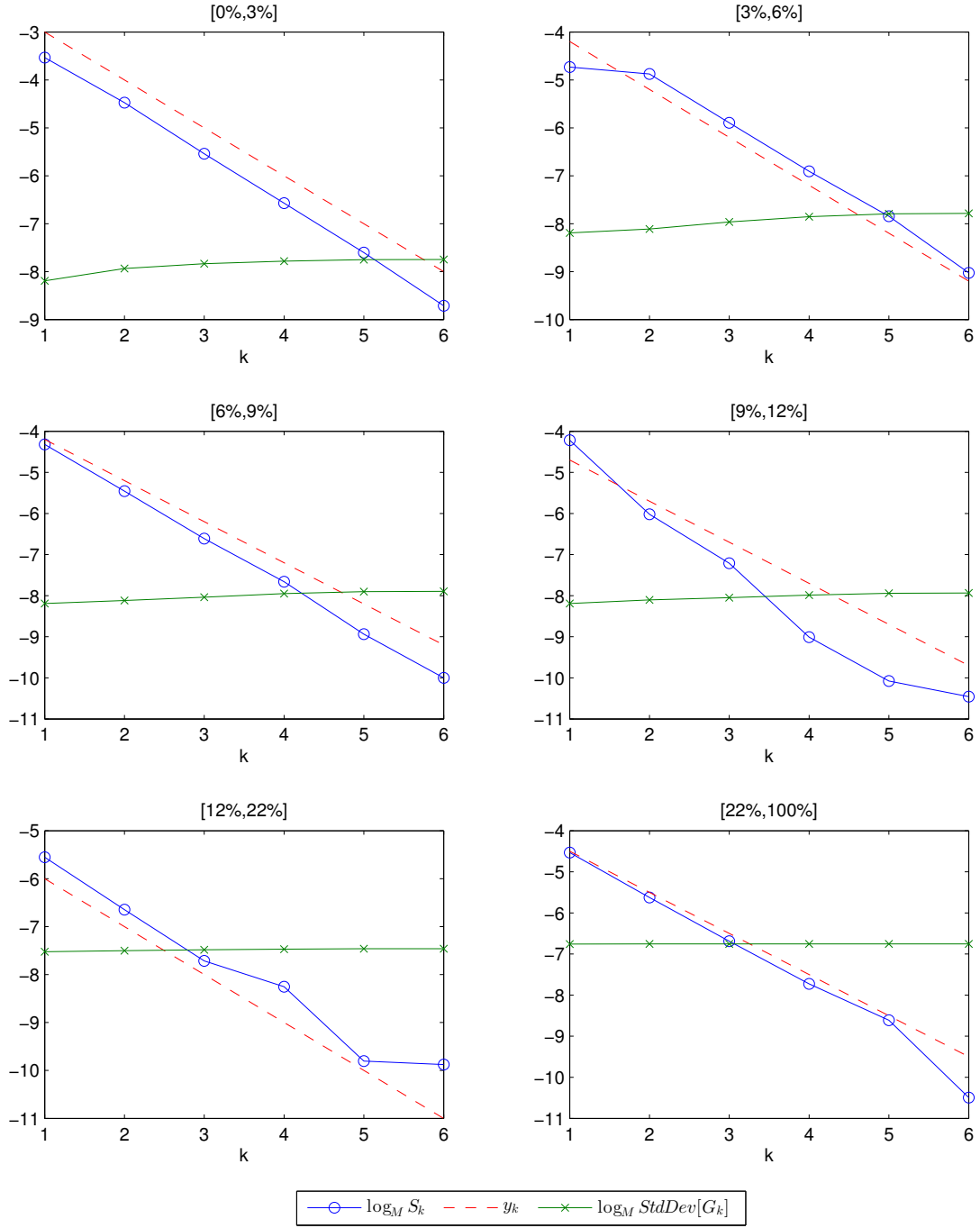


Figure 2: Shown here is  $\log_M S_k$ , where  $S_k$  given by (4.5) is an estimator for  $|\mathbb{E}[P^{(k)} - P]|$ . The various plots are for tranches ranging from [0%-3%] to [22%-100%], of a CDO basket consisting of  $N_k = M^k = 5^k$  companies, where  $k = 1, \dots, 6$ . The comparison with the predicted trend  $y_k$  from (4.6) confirms the first order convergence. Included is also the standard deviation of the estimated tranche loss  $G_k$ .

2. We hence define estimators  $Z_l$  by

$$Z_l \equiv n_l^{-1} \sum_{j=1}^{n_l} \left( P^{(l,j)} - P_c^{(l,j)} \right), \quad (5.1)$$

where ‘c’ denotes a ‘coarse’ estimator on level  $l$ , i.e., using only  $N_{l-1}$  instead of  $N_l$  Bernoulli random variables, precisely,

$$P^{(l,j)} = p(L^{(l,j)}), \quad \text{where } L^{(l,j)} = N_l^{-1} \sum_{i=1}^{N_l} Y_i^{(l,j)}, \quad (5.2)$$

$$P_c^{(l,j)} = p(L_c^{(l,j)}), \quad \text{where } L_c^{(l,j)} = N_{l-1}^{-1} \sum_{i=1}^{N_{l-1}} Y_i^{(l,j)}, \quad (5.3)$$

where  $Y_i^{(l,j)}$ ,  $j = 1, \dots, n_l$ , are independent samples of  $Y_i$  for fixed level  $l$  and independent across levels. They are constructed from a loss factor  $L^{(l,j)}$  (with the same distribution as  $L$ , independent across  $l$  and  $j$ ) in the same way that  $Y_i$  is constructed from  $L$ .

By direct inspection, for this construction of  $Z_l$ , Assumption ii) holds in Proposition 2.1. From Theorem 2.1, we know that i) holds with  $\alpha = 1/2$  for general Lipschitz  $p$ . Clearly, the computational effort to compute  $Z_l$  is proportional to  $n_l N_l$  as required in iv). Finally, iii) holds by the following simple application of Lemma 3.1.

**Proposition 5.1.** *Let  $P^{(l)} = P_{N_l}$  as per (2.4), where  $p$  is Lipschitz with constant  $c_p$ , then*

$$\text{Var}[P^{(l)} - P^{(l-1)}] \leq c_p^2 \frac{M+1}{2N_l}. \quad (5.4)$$

*Proof.* This follows directly from

$$\begin{aligned} \mathbb{E}_{|L}[(P^{(l)} - P^{(l-1)})^2] &= \mathbb{E}_{|L}[((P^{(l)} - P) - (P^{(l-1)} - P))^2] \\ &\leq 2 \left( \mathbb{E}_{|L}[(P^{(l)} - P)^2] + \mathbb{E}_{|L}[(P^{(l-1)} - P)^2] \right) \end{aligned} \quad (5.5)$$

by Lemma 3.1 and taking expectations over  $L$ .  $\square$

We have therefore proven the first statement of Corollary 2.1.

In practice, it is also relevant to be able to compute  $\mathbb{E}[P_N]$  efficiently for finite  $N$ . It is clear that for fixed  $N$ , the complexity is bounded by  $c \epsilon^{-2}$  for some  $c > 0$ , but for a naïve (single-level) estimator the constant  $c$  will increase with  $N$ . From the proof of Theorem 3.1 in [10] it is clear, however, that there is a multilevel estimator with *a priori* bounded upper level  $K$  which satisfies the second statement in Corollary 2.1.

We now propose a multilevel estimator with even lower variance, based on the faster decay rate  $3/2$  in Theorem 2.3, which we prove subsequently. Specifically, we define estimators  $\bar{Z}_l$  by

$$\bar{Z}_l \equiv n_l^{-1} \sum_{j=1}^{n_l} \left( P^{(l,j)} - \bar{P}^{(l,j)} \right), \quad (5.6)$$

where  $P^{(l,j)}$  is defined as in (5.2), but instead of  $P_c^{(l,j)}$  we use

$$\bar{P}^{(l,j)} = M^{-1} \sum_{m=1}^M p(L_m^{(l,j)}), \quad \text{where } L_m^{(l,j)} = N_{l-1}^{-1} \sum_{i=1}^{N_{l-1}} Y_{i+(m-1)N_{l-1}}^{(l,j)}, \quad (5.7)$$

and where the rest of the set-up is as earlier.

It is clear that  $\bar{Z}_l$  satisfies ii) in Proposition 2.1 and that the computational complexity is still bounded as required per iv). In fact, as the main computational cost is typically in sampling  $Y_i$ , the computational complexity is virtually identical to that of  $Z_l$ . In particular, if we evaluate (5.2) by using (2.15) and the already computed (5.7), the difference in evaluating  $Z_l$  and  $\bar{Z}_l$  is an  $O(M)$  cost, i.e., independent of  $N_l$ . Now, given Theorem 2.3, we have that

$$\text{Var}[\bar{Z}_l] \leq c n_l^{-1} M^{-3/2^l}, \quad (5.8)$$

for some  $c$ , such that we are in the first regime in the complexity result of Proposition 2.1, i.e., we have optimal complexity order.

The remainder of this section is devoted to the proof of Theorem 2.3.

**Lemma 5.1.** *Assume the CDF  $F_L$  of  $L$  is Lipschitz with constant  $c_L$ . Let  $B^{(l)}$  be the event that  $L^{(l)}$  lies in the same interval as  $L^{(l-1)}$ ,  $B^{(l),c}$  its complement, then*

$$\mathbb{E} \left[ \left( \mathbb{P}_{|L}[B^{(l),c}] \right)^{\frac{1}{2}} \right] \leq \frac{C}{\sqrt{N_l}},$$

where  $C = c_L 4\sqrt{\pi}(\sqrt{2} + \sqrt{M})$ .

*Proof.* Let  $A^{(l)}$  again be the event that  $L^{(l)}$  and  $L$  are in the same interval,  $A^{(l),c}$  its complement. Then from

$$B^{(l),c} \subseteq \left( A^{(l)} \cap A^{(l-1),c} \right) \cup \left( A^{(l),c} \cap A^{(l-1)} \right) \cup \left( A^{(l),c} \cap A^{(l-1),c} \right)$$

follows

$$\begin{aligned} \mathbb{P}_{|L}[B^{(l),c}] &\leq \mathbb{P}_{|L}[A^{(l)} \cap A^{(l-1),c}] + \mathbb{P}_{|L}[A^{(l),c} \cap A^{(l-1)}] + \mathbb{P}_{|L}[A^{(l),c} \cap A^{(l-1),c}] \\ &\leq 2 \mathbb{P}_{|L}[A^{(l),c}] + \mathbb{P}_{|L}[A^{(l-1),c}], \end{aligned}$$

which leads to

$$\mathbb{E} \left[ \left( \mathbb{P}_{|L}[B^{(l),c}] \right)^{\frac{1}{2}} \right] \leq \sqrt{2} \mathbb{E} \left[ \left( \mathbb{P}_{|L}[A^{(l),c}] \right)^{\frac{1}{2}} \right] + \mathbb{E} \left[ \left( \mathbb{P}_{|L}[A^{(l-1),c}] \right)^{\frac{1}{2}} \right].$$

By Lemma 3.4, we obtain the result.  $\square$

**Lemma 5.2.** *For  $P$ ,  $P^{(l)}$ ,  $P^{(l-1)}$  and  $\bar{P}^{(l-1)}$  as above,  $p$  Lipschitz with constant 1,*

$$\mathbb{E}_{|L}[(P^{(l)} - P)^4] \leq \frac{3}{16N_l^2} \left( 1 + \frac{4}{3N_l} \right) \leq \frac{7}{16N_l^2}, \quad (5.9)$$

$$\mathbb{E}_{|L}[(P^{(l)} - P^{(l-1)})^4] \leq \frac{C}{N_l^2}, \quad (5.10)$$

$$\mathbb{E}_{|L}[(P^{(l)} - \bar{P}^{(l-1)})^4] \leq \frac{C}{N_l^2}, \quad (5.11)$$

where  $C = \frac{7}{8}(M^2 + 6M + 1)$ .

*Proof.* See Appendix A. □

*Proof of Theorem 2.3.* Let  $E^{(l)}$  be the event that all  $L_m^{(l-1)}$  lie in the same interval,  $1 \leq m \leq M$ , and  $E^{(l),c}$  its complement, then

$$\mathbb{E}[(P^{(l)} - \bar{P}^{(l-1)})^2] = \mathbb{E}[(P^{(l)} - \bar{P}^{(l-1)})^2 1_{E^{(l)}}] + \mathbb{E}[(P^{(l)} - \bar{P}^{(l-1)})^2 1_{E^{(l),c}}].$$

By (2.15) and linearity of  $p$  in each interval, we have

$$\mathbb{E}[(P^{(l)} - \bar{P}^{(l-1)})^2 1_{E^{(l)}}] = 0.$$

By Cauchy-Schwartz, we have

$$\mathbb{E}_{|L}[(P^{(l)} - \bar{P}^{(l-1)})^2 1_{E^{(l),c}}] \leq \left( \mathbb{E}_{|L}[(P^{(l)} - \bar{P}^{(l-1)})^4] \right)^{\frac{1}{2}} \left( \mathbb{P}_{|L}[E^{(l),c}] \right)^{\frac{1}{2}},$$

hence,

$$\mathbb{E}[(P^{(l)} - \bar{P}^{(l-1)})^2 1_{E^{(l),c}}] \leq \mathbb{E} \left[ \left( \mathbb{E}_{|L}[(P^{(l)} - \bar{P}^{(l-1)})^4] \right)^{\frac{1}{2}} \left( \mathbb{P}_{|L}[E^{(l),c}] \right)^{\frac{1}{2}} \right].$$

By Lemma 5.2, we have that

$$\left( \mathbb{E}_{|L}[(P^{(l)} - \bar{P}^{(l-1)})^4] \right)^{\frac{1}{2}} \leq \frac{\sqrt{c_1}}{N_l}, \quad (5.12)$$

where  $c_1 = \frac{7}{8}(M^2 + 6M + 1)$ .

If we denote by  $B_m^{(l)}$  the event that  $L_m^{(l-1)}$  and  $L^{(l)}$  lie in the same interval, then

$$E^{(l),c} = \bigcup_{m=1}^M B_m^{(l),c}$$

and therefore

$$\mathbb{P}_{|L}(E^{(l),c}) \leq \sum_{m=1}^M \mathbb{P}_{|L}(B_m^{(l),c}) = M \mathbb{P}_{|L}(B^{(l),c}).$$

By Lemma 5.1, this gives

$$\mathbb{E} \left[ \left( \mathbb{P}_{|L}[E^{(l),c}] \right)^{\frac{1}{2}} \right] \leq \frac{c_2}{N_l},$$

where  $c_2 = c_L 4\sqrt{M\pi}(\sqrt{2} + \sqrt{M})$ . Together with (5.12), we obtain the result. □

## 6 Multilevel tests

In this section, we present multilevel simulation results based on the estimators from the previous section and illustrating the theoretical findings from there. We return to the example from Section 4 and estimate expected tranche losses for credit baskets with an increasing number of firms  $N_l = M^l$ .



For the estimator  $Z_l$  from (5.1), an upper bound for the variance – although not a sharp one – is analytically known from (5.4) and we could use that to determine the number  $n_l$  of samples on level  $l$  which is required to bring the variance contribution under a desired threshold. For the improved estimator  $\bar{Z}_l$  from (5.6), however, the bound in (5.8) contains the unknown Lipschitz constant of the CDF of  $F_L$  via Theorem 2.3. In order to determine the optimal allocation  $n_l^*$ , we use the following algorithm as per [10]. In contrast to there, the upper level  $K$  is fixed here which simplifies the stopping criterion somewhat.

1. Start with  $k = 1$ .
2. Estimate the variance  $V_k$  of a single sample using  $n_k = 10^4$  realisations.
3. Calculate the optimal number of samples,  $n_l^*$ , for  $l = 0, 1, \dots, k$ , using

$$n_l^* = \left\lceil \gamma^{-2} \sqrt{V_l N_l^{-1}} \left( \sum_{j=1}^k \sqrt{V_j N_j} \right) \right\rceil, \quad (6.1)$$

where  $\gamma^2$  is a chosen upper bound of  $\text{Var}[G_K]$ .

4. Draw extra samples for each level according to  $n_l^*$ .
5. If  $k < K$ , set  $k = k + 1$  and go to 2.
6. If  $k = K$ , finish.

**Remark 6.1.** *As per [10], choosing  $n_l^*$  by (6.1), guarantees that the variance  $\text{Var}[G_K]$  is bounded by  $\gamma^2$ , since*

$$\text{Var}[G_K] = \sum_{l=1}^K (n_l^*)^{-1} V_l \leq \sum_{l=1}^K \left( \gamma^{-2} \sqrt{V_l N_l^{-1}} \sum_{j=1}^K \sqrt{V_j N_j} \right)^{-1} V_l = \gamma^2.$$

A side effect is that, for  $k < K$ , the variance is smaller than for  $k = K$ , since

$$\text{Var}[G_k] = \sum_{l=1}^k (n_l^*)^{-1} V_l < \gamma^2 \frac{\sum_{l=1}^k \sqrt{V_l N_l}}{\sum_{l=1}^K \sqrt{V_l N_l}}.$$

Hence, if we compute estimators  $G_k$  for all  $k$  as a by-product of  $G_K$ , the variance is the smallest for  $G_1$  and then for  $G_k$ ,  $k = 2, \dots, K$ , gradually reaches the upper bound  $\gamma^2$ . This effect can be observed in Figure 3.D.

In Figure 3 we show results for the same parameter setting as in Section 4 and only for the equity tranche. Results from other tests were very similar and did not show any noteworthy additional effects. In order to easily see the rate of convergence in 3.A., we plot the logarithm of  $V_l$  to base  $M$ , together with

$$f_k = -\beta k + f_0 \quad (6.2)$$

for different values of  $\beta$ . The estimated slope is  $\hat{\beta} \approx 1$  for the original estimator and  $\hat{\beta} \approx 3/2$  for the improved estimator, which agrees with the theoretical findings. The order

of convergence of  $|E[P^{(l)} - P^{(l-1)}]|$  is  $\widehat{\alpha} \approx 1$ , which also agrees with the previous results. As can be observed in Figure 3.C, the number of samples ranges from 150 millions for  $k = 1$  to 34000 for  $k = 7$ . The improved estimator gives further reductions in computational time: the total number of samples ranges now from 35 millions for  $k = 1$  to only 350 for  $k = 7$ . The standard deviation of  $G_k$  is an increasing function of  $k$ , and is less than or equal to the chosen upper bound  $\gamma = 4 \times 10^{-6}$ .

## 7 Conclusions and extensions

A main focus of this paper was the construction of an efficient simulation algorithm for functionals of a large number of exchangeable random variables. For a specific set-up, we were able to demonstrate optimal complexity order by theoretical analysis and numerical illustrations.

### Discussion

The results from the previous section show that the computational savings can be significant in situations of practical relevance. As seen from Figure 3.C, already for  $N = 125$  (i.e.,  $k = 3$ ), the size of a CDO basket, the required number  $n_3$  of samples on this level is reduced by about two orders of magnitude compared to the number of samples for  $k = 1$ ,  $n_1$ . It is roughly this number which would be required for a standard (i.e., single level) estimator on level 3 for a variance comparable to the one achieved by the multilevel estimator at substantially lower cost.

### Extensions – random recovery and random factor loadings

There is ample empirical evidence that a basic factor model such as the one described in section 4 does not adequately reproduce observed market spreads of credit derivatives and other stylised facts of credit markets. Two effects that have so far been neglected are credit contagion (i.e., the default of one firm has an impact on the credit worthiness and dependence structure of others) and the dependence of recovery values on the wider credit environment. We focus here on the latter effect and follow [2] for a model that captures this dependence.

Consider thus the total loss as given by

$$\tilde{L}_N = \frac{1}{N} \sum_{i=1}^N l_i Y_i,$$

where  $l_i$  represent a random *loss-given-default* for company  $i$  and  $Y_i$  are default indicators as previously. It is sometimes convenient to write

$$l_i = l_{\max} \cdot (1 - R_i),$$

where  $l_{\max}$  is the (constant) maximum loss and  $R_i \in [0, 1]$  is the (random) recovery rate of the  $i$ -th firm. In keeping with our general framework, we assume that the  $R_i$  are identically distributed and independent conditional on  $\mathcal{F}_L$ .

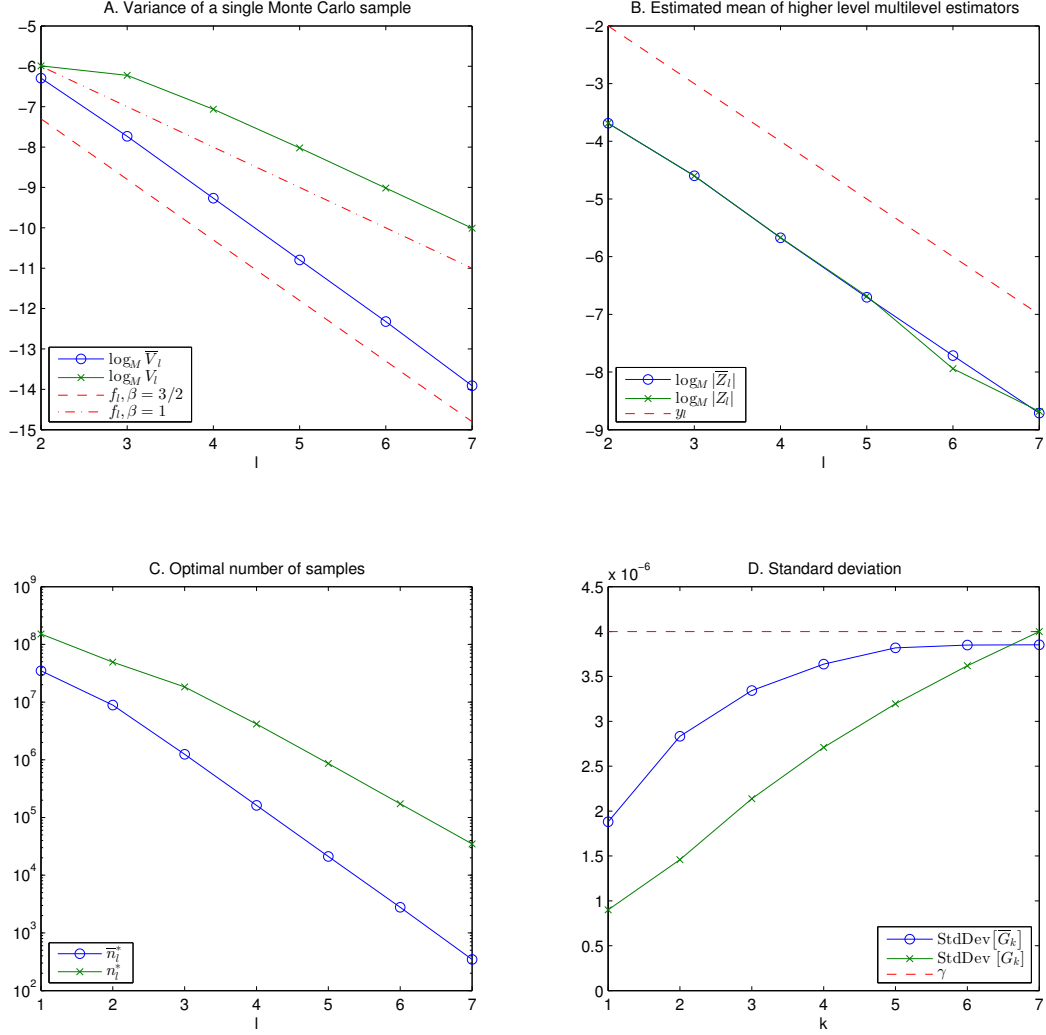


Figure 3: Multilevel results for the expected loss in the equity tranche of a CDO basket consisting of  $N_k$  companies,  $N_k = M^k = 5^k$ ,  $k = 1, \dots, 7$ . Overlined quantities refer to the estimator  $\bar{Z}_l$  from (5.6), all others to the standard estimator  $Z_l$  from (5.1). A. Variance of a single Monte Carlo sample,  $V_l$  and  $\bar{V}_l$ , together with a predicted trend,  $f_l$ , given by (6.2), where  $\beta = 1$  or  $\beta = 3/2$ . B. Mean at level  $l$ ,  $Z_l$  and  $\bar{Z}_l$ , and a trend,  $y_l$ , defined by (4.6), with slope -1. C. Optimal number of simulations in both cases,  $n_l^*$  and  $\bar{n}_l^*$ , calculated according to (6.1) for  $k = K = 7$ . D. Standard deviation of multilevel estimators  $G_k$  defined in (2.13), and similar for  $\bar{G}_k$ , with their chosen upper bound,  $\gamma$ .

For continuous payoffs  $p$ , we still have

$$\tilde{L}_N \rightarrow \mathbb{E}[\tilde{L}_N] =: \tilde{L} \quad \text{for } N \rightarrow \infty, \mathbb{P}_L - a.s., \quad (7.1)$$

$$\tilde{P}_N := p(\tilde{L}_N) \rightarrow \mathbb{E}[\tilde{P}_N] =: \tilde{P} \quad \text{for } N \rightarrow \infty, \mathbb{P}_L - a.s. \quad (7.2)$$

The  $L^2$ -convergence is described in the following.

**Corollary 7.1** (to Theorem 2.1). *Let  $\tilde{P}$  and  $\tilde{P}_N$  be given by (7.2), and assume that  $p$  is Lipschitz with constant  $c_p$ . We have that*

$$|\mathbb{E}[\tilde{P}_N - \tilde{P}]| \leq \frac{c_p l_{\max}}{\sqrt{N}}, \quad (7.3)$$

$$\text{Var}[\tilde{P}_N - \tilde{P}] \leq \frac{c_p^2 l_{\max}^2}{N}. \quad (7.4)$$

*Proof.* Similar to the proof of Lemma 3.1,

$$\mathbb{E}_L[(\tilde{P}_N - \tilde{P})^2] \leq c_p^2 \mathbb{E}_L[(\tilde{L}_N - \tilde{L})^2] = c_p^2 \text{Var}[\tilde{L}_N | \mathcal{F}_L] = \frac{c_p^2}{N} \text{Var}[l_i Y_i | \mathcal{F}_L] \leq \frac{c_p^2 l_{\max}^2}{N},$$

which gives the result for the variance. The result for the expectation follows again immediately from Hölder's inequality.  $\square$

The order 1/2 for the convergence of the expectations and order 1 for the variances is sufficient to be able to apply Corollary 2.1 to establish the  $\epsilon^2 \log^2 \epsilon$  complexity for MSE  $\epsilon^2$  of the multilevel method. The following numerical tests indicate that the order 1/2 is not sharp and indeed we expect order 1 for sufficient regularity of the payoffs and/or distribution function of  $L$ . The proof of this becomes more technical than in the pure Bernoulli case because we lose the explicit form of the characteristic function. Thus, and because of the irrelevance of this for the convergence speed of the multilevel method, we do not pursue this further here.

We now consider a particular model similar to the one in [2] and give a numerical illustration. Specifically, let

$$X_t^i = X_0^i + \beta t + \sqrt{1 - \rho} W_t^i + \sqrt{\rho} B_t + J_t, \quad t > 0, \quad (7.5)$$

$$R_t^i = \Phi(\mu^R + \beta^R t + \sigma^R \sqrt{1 - \rho^R} \xi_t^i + \sigma^R \sqrt{\rho^R} B_t + J_t), \quad (7.6)$$

where the processes in the first line are defined as in (4.1), and in the second line ( $\xi^i$ ) is a standard Brownian motion independent of everything else, while  $\mu^R, \beta^R, \sigma^R > 0$  and  $0 \leq \rho^R \leq 1$  are constants.  $\Phi$  is the cumulative density of the standard normal, but could be replaced by any increasing mapping function  $\mathbb{R} \rightarrow [0, 1]$ . This has the effect that the recovery rate is positively correlated with the market factors  $B$  and  $J$ , with some idiosyncratic noise, and thus there is a negative dependence between recovery rates and default frequencies. See, for instance, [1] for an early but influential study of this empirical fact.

Since  $\mathcal{F}_L$  is identical to the filtration generated by the market factors  $B$  and  $J$ , the above convergence result holds.

Numerical results.

Another extension, also proposed in [2], are *random factor loadings* of the type

$$X_t^i = X_0^i + \beta t + \nu W_t^i + a(B_t) B_t, \quad t > 0, \quad (7.7)$$

where  $a$  is a given deterministic function and  $\nu^2 = 1 - \text{Var}(a(B_t)B_t)$ . This model can capture contagion effects where, for decreasing  $a$ , firm values are more closely correlated to the common market factor in bad times. Such an extension fits directly in the general framework developed earlier in this paper, assuming the technical conditions on the cumulative density  $F_L$  hold, where needed (see in particular Theorem 2.2).

## Outlook

We would expect there to be scope to apply the presented nested simulation approach to a wider range of settings beyond the particular application studied here. An interesting extension would be to the model from [9], where the analysis requires further tools accounting especially for the heterogeneity of the basket, resulting in non-exchangeability. While our motivation comes from credit baskets and some of the later results are specific to piecewise linear functionals encountered in the valuation of basket credit derivatives, there appears to be a wider relevance of the main approach to the simulation of certain functionals arising in large interacting particle systems and elsewhere.

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## A Moment computations

*Proof.* [of Lemma 5.2] We begin by showing (5.9) and then deduce (5.10) and (5.11). We have

$$|P^{(l)} - P| \leq |L^{(l)} - L|,$$

where

$$L^{(l)} = \frac{1}{N_l} \sum_{i=1}^{N_l} Y_i.$$

Hence, we get

$$\mathbb{E}_{|L}[(P^{(l)} - P)^4] \leq \mathbb{E}_{|L}[(L^{(l)} - L)^4] = \mathbb{E}_{|L}\left[\left(\frac{1}{N_l} \sum_{i=1}^{N_l} (Y_i - L)\right)^4\right].$$

As  $L$  is  $\mathcal{F}_L$ -measurable and the  $Y_i$  are independent and identically distributed given  $\mathcal{F}_L$  with  $\mathbb{E}_{|L}[Y_i - L] = 0$ , we have

$$\begin{aligned} \mathbb{E}_{|L}[(L^{(l)} - L)^4] &= \frac{1}{N_l^4} \mathbb{E}_{|L}\left[\sum_{i=1}^{N_l} (Y_i - L)^4 + 3 \sum_{i \neq j} (Y_i - L)^2 (Y_j - L)^2\right] \\ &= \frac{1}{N_l^3} ((1-L)^4 L + L^4 (1-L)) + \frac{3(N_l - 1)}{N_l^3} ((1-L)^2 L + L^2 (1-L))^2 \\ &= \frac{3L^2(1-L)^2}{N_l^2} + \frac{L(1-L)(1-6L(1-L))}{N_l^3}. \end{aligned}$$

Using the fact that  $0 \leq L(1-L) \leq 1/4$  we have the required bound in (5.9).

For (5.10), observe that there are many ways of estimating this fourth moment; we choose the following

$$\begin{aligned} (P^{(l)} - P^{(l-1)})^4 &= \left((P^{(l)} - P) - (P^{(l-1)} - P)\right)^4 \\ &\leq 2 \left((P^{(l)} - P)^4 + (P^{(l-1)} - P)^4\right) + 12(P^{(l)} - P)^2(P^{(l-1)} - P)^2. \end{aligned}$$

Therefore, using Cauchy-Schwarz on the last term and applying (5.9) we have

$$\begin{aligned}
\mathbb{E}_{|L}[(P^{(l)} - P^{(l-1)})^4] &\leq 2 \left( \mathbb{E}_{|L}[(P^{(l)} - P)^4] + \mathbb{E}_{|L}[(P^{(l-1)} - P)^4] \right) \\
&\quad + 12 \mathbb{E}_{|L}[(P^{(l)} - P)^4]^{1/2} \mathbb{E}_{|L}[(P^{(l-1)} - P)^4]^{1/2} \\
&\leq \frac{7}{8N_l^2} + \frac{7}{8N_{l-1}^2} + \frac{42}{8N_l N_{l-1}}
\end{aligned}$$

as required to obtain (5.10).

Finally, (5.11) follows from

$$\begin{aligned}
\mathbb{E}_{|L}[(P^{(l)} - \bar{P}^{(l-1)})^4] &= \mathbb{E}_{|L} \left[ \left( \frac{1}{M} \sum_{m=1}^M (P^{(l)} - P_m^{(l-1)}) \right)^4 \right] \\
&\leq \mathbb{E}_{|L} \left[ \frac{1}{M} \sum_{m=1}^M (P^{(l)} - P^{(l-1)})^4 \right] \\
&= \mathbb{E}_{|L}[(P^{(l)} - P^{(l-1)})^4],
\end{aligned}$$

and an application of (5.10). □