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A dual approach to multiple exercise option problems under constraints

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Abstract This paper considers the pricing of multiple exercise options in discrete time. This type of option can be exercised up to a finite number of times over the lifetime of the contract. We allow multiple exercise of the option at each time point up to a constraint, a feature relevant for pricing swing options in energy markets. It is shown that, in the case where an option can be exercised an equal number of times at each time point, the problem can be reduced to the case of a single exercise possibility at each time. In the general case there is not a solution of this type. We develop a dual representation for the problem and give an algorithm for calculating both lower and upper bounds for the prices of such multiple exercise options.

Keywords Multiple optimal stopping · Dual approach · Multiple exercise options · Swing options

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1 Introduction

There are many different types of financial derivatives with early exercise features actively traded in financial markets. The simplest being the American

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option in which the holder of the option can exercise and receive the option payoff at any time up until maturity. Even in this simple case determining the price accurately can be challenging if the underlying model for the asset price has several driving factors. Our aim here is to find representations for the price which will be useful in developing Monte Carlo algorithms for the pricing of more complex early exercise options.

The method most favored by practitioners for tackling early exercise problems with Monte Carlo is the method suggested by Longstaff and Schwartz [12] (and independently by Tsitsiklis and van Roy [15]). The method relies on approximating the value function by linear regression on a suitable space of basis functions and using this to determine the optimal stopping policy. By construction this optimal stopping policy gives a lower bound for the option price. The method is comparatively easy to implement and, for properly chosen regression functions, gives a good estimate for the price, [8].

In the work of Rogers [14] and Haugh and Kogan [9] (see also [2]), ideas for calculating an upper bound for the option price were developed. Both rely on a duality approach, expressing the problem as a minimization over a space of martingales. This provides a method to compute an upper bound, by constructing a martingale which is a good approximation to the optimal one.

More recently, there have been several articles discussing the more general multiple early exercise problem. Pricing a derivative with several early exercise opportunities is equivalent to solving a multiple optimal stopping problem. In this paper we consider such multiple optimal stopping problems where it is possible to stop more than once at the same time point. In option terminology we can exercise the option a certain number of times at a given time. The motivation for the study of the type of multiple optimal stopping problem considered here comes from the energy market contracts called swing options.

Swing options are actively traded in electricity markets. The most common contracts in electricity markets are electricity forward contracts. They are obligations to buy or sell a fixed amount of electricity at a pre-specified price (the forward price F) over a certain time period in the future. For simplicity we consider spot forwards in which the price is fixed for a certain time in the future (the expiration time T). If S_t is the electricity spot price at time t, then the payoff of the forward contract is $(S_T - F)$. The settlement price S_T is usually calculated based on the average price of electricity over the delivery period at the maturity T. Based on the delivery period during a day, electricity forwards can be categorized as forwards on on-peak electricity, off-peak electricity and 24 hour electricity.

Often the buyer is not sure about the quantity he will want to purchase due to weather changes, particular spot price expectations or other reasons. In that case the forward contracts are coupled with swing options. The simplest swing options give their holder the right to purchase each day (on- or off- peak time) for a specified period electricity at a fixed price K (strike price). Thus the payoff in this case is that of a call $(S_t - K)^+$. When exercising a swing option, the purchased quantity may vary (or swing) between a minimum and a maximum volume. An alternative version is when the swing contract allows the holder either to buy or to sell a certain quantity of electricity on a given day. The total quantity of electricity purchased for the period should be within minimum and maximum volume levels.

There are now several papers ([10], [13], [11], [5], [7], [3], [4], [6]) on pricing swing contracts. In this paper we consider an extension of the ideas developed in [13]. In that paper a stylized swing contract was considered where the number of exercise possibilities was restricted to some number $M < T$, which is an analogue of an American option that can be exercised several times during its lifetime. Our consideration of swing contracts will be slightly more general than this. We will consider options with different volume restrictions each day, as well as a total volume restriction over the lifetime of the contract, though these constraints will still be integer valued. The motivation for analysing this type of contract is the fact that the spot price of the electricity, together with its demand, have deterministic trends. For example the electricity spot price (and demand) is in general lower on weekends, which could correspond to less need for optionality in the swing contract. In short, the buyer doesn't have to pay for rights he doesn't need.

Our approach is to develop the duality ideas from [13] further to incorporate these extra constraints. We will show in Theorem 1 there is a dual representation for the marginal value of one more exercise right of the same form as in [13] with appropriate extensions of definitions. A version of this result was first obtained in [6]. The approach in [6] is to work with the Snell envelope to obtain the dual formulation and a representation for the optimal martingale. Our set up is slightly more general, in that the payoffs are non-increasing function of the number of exercises, as this allows the dual formulation to be used in other settings [1]. We work directly with the marginal value function, establishing various properties and working with its Doob decomposition to find the representation for the optimal martingale.

In a number of other papers e.g. [3], [4] the volume constraint is a continuous parameter between specified bounds. We do not consider that case for ease of developing our dual representation. We also avoid any discussion of hedging or other serious financial issues in pricing and risk management of swing contracts and just assume that there is a risk neutral measure and that our aim is to compute the price of a multiple exercise option in this measure.

The paper is organized as follows. In Section 2 we begin by giving the mathematical formulation of the problem for finding the price of the multiple exercise options considered here. We formulate both the optimal stopping problem and the dynamic programming equations. In Section 3 we prove some properties of the option price process. We prove that the marginal value, that is the value of an additional exercise opportunity, is decreasing as a function of the number of exercise opportunities left. In Section 4 we discuss the case where the option can be exercised up to the same number of times k for all time points. This is then the easy case, in that we prove it is essentially equivalent to k options for which it is possible to exercise only once at each time point. In Section 5 we obtain the dual representation for the multiple exercise option. In the last two sections some implementational issues and a numerical example are considered based on the Longstaff-Schwartz regression ideas and the dual representation.

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We thank Christian Bender for sending us his preprint [6]. We originally had an alternative dual formulation but being aware that there was a simpler form we were able to extend our approach to obtain the expression given in [6].

2 Definitions, problem formulation and the main result

We consider an economy in discrete time defined up to a finite time horizon T. We assume a financial market described by the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,\dots,T}, \mathbb{P})$ with $(X_t)_{t=0,1,\dots,T}$ an $\mathbb{R}^d$-valued discrete time $\mathcal{F}_t$-Markov chain describing the state of the economy - the price of the underlying assets and any other variables that affect the dynamics of the underlyings. Throughout we will assume that $\mathbb{P}$ is a risk neutral pricing measure and write $\mathbb{E}_t(X) = \mathbb{E}(X|\mathcal{F}_t)$ for any random variable X on our probability space. We will consider multiple exercise options in this market. The holder of such a multiple exercise option has the opportunity of up to k_t exercises, at time t, where k_t will be an $\mathbb{N}$-valued random variable measurable with respect to $\mathcal{F}_t$. We write $h_t^i(X_t)$ for the payoff from the $i-$th exercise of the option $i = 1, 2, ..., k_t$ at time t when the asset price is X_t (we often suppress the argument of h_t^i). We assume that the payoffs are non-negative $h_t^i(x) \geq 0$ for all $x \in \mathbb{R}^d, t = 0, \dots, T$ and $i = 1, 2, ..., k_t$ and decreasing $h_t^i(x) \geq h_t^{i+1}(x)$ for all $x \in \mathbb{R}^d, t = 0, \dots, T$ and $i = 1, 2, ..., k_t - 1$. The total payoff at time t we denote by H_t^k, that is if the option is exercised k times at time t. By definition

$$H_t^k = \sum_{i=1}^{k} h_t^i. \tag{1}$$

We will use the notation $V_t^{*,m}$ for the price (value function) at time t of an option, which can be exercised up to $\mathbf{k} = \{k_0, k_1, k_2, ..., k_T\}$ times at the corresponding time points and has m exercise opportunities left. For simplicity of notation we will assume the risk free rate is zero and exclude discounting factors from our expressions.

In order to give a formal definition for the value function $V_t^{*,m}$ we require a little notation. For a set of stopping times $\{\tau_i\}_{i=1}^m$ with $\tau_m \leq \tau_{m-1} \leq \dots \leq \tau_1$ taking values in $\{0, 1, \dots, T\}$ we write $N_t^m(\tau_m, \dots, \tau_1) = \#\{j : \tau_j = t\}$ for the number of stopping times taking the value t. We will also write $\bar{N}_t^m(\tau_m, \dots, \tau_1) = \#\{j : \tau_j \leq t\}$, for the number of stopping times in the set taking the value at most t. We will omit the set of stopping times when it is clear. We define an exercise policy $\pi_{\mathbf{k}}$ to be a set of stopping times $\{\tau_i\}_{i=1}^m$ with $\tau_m \leq \tau_{m-1} \leq \dots \leq \tau_1$ such that $N_t^m \leq k_t$. We will often drop the subscript and just write π for ease of notation. Then the value of the policy $\pi_{\mathbf{k}}$ at time t is given by

$$V_t^{\pi_{\mathbf{k}},m} = \mathbb{E}_t(\sum_{s=t}^{T} H_s^{N_s^m}(X_s)).$$

Definition 1 The value function is defined to be

$$V_t^{*,m} = \sup_{\pi_{\mathbf{k}}} V_t^{\pi_{\mathbf{k}},m} = \sup_{\pi_{\mathbf{k}}} \mathbb{E}_t(\sum_{s=t}^{T} H_s^{N_s^m}(X_s)).$$

We denote the corresponding optimal policy $\pi^* = \{\tau_m^*, \tau_{m-1}^*, ..., \tau_1^*\}$.

For our purposes it will be more convenient to work with two alternative formulations. Firstly using dynamic programming it is straightforward to see that the value function can be written as follows.

Lemma 1 *(Multiple exercise option price - Dynamic programming formulation). The price $V_t^{*,m}$ at time t of an option with payoff function $\{h_s^i, t \leq s \leq T, 1 \leq i \leq k_s\}$ which could be exercised up to k_s times per single exercise time $s \in \{t, \ldots, T\}$ with m exercise opportunities in total is given by*

$$\begin{aligned} V_T^{*,m} =& H_T^{\min\{k_T,m\}}, \\ V_t^{*,m} =& \max\{H_t^{\min\{k_t,m\}} + \mathbb{E}_t[V_{t+1}^{*,m-\min\{k_t,m\}}], \\ & H_t^{\min\{k_t,m\}-1} + \mathbb{E}_t[V_{t+1}^{*,m-(\min\{k_t,m\}-1)}], \\ & ..., H_t^1 + \mathbb{E}_t[V_{t+1}^{*,m-1}], \mathbb{E}_t[V_{t+1}^{*,m}]\}. \end{aligned}$$

Note that for $0 \leq i \leq k_t$ the quantity

$$H_t^{\min\{k_t,m\}-i} + \mathbb{E}_t[V_{t+1}^{*,m-\min\{k_t,m\}+i}]$$

is the payoff under the exercise of the m-th, $m-1$-th, ..., $m-\min\{k_t,m\}+i+1$-th exercise opportunities at time t plus the expected future payoff from the remaining $m-\min\{k_t,m\}+i$ exercise opportunities. There is also an optimal stopping problem formulation which is again straightforward to derive.

Lemma 2 *(Multiple exercise option price - Optimal stopping problem formulation). The price $V_t^{*,m}$ of an option, which could be exercised up to k_t times per single exercise time t with m exercise opportunities in total is given by*

$$\begin{aligned} V_t^{*,m} = & \sup_{t \leq \tau \leq T} \mathbb{E}_t\big[\max\{H_\tau^{\min\{k_\tau,m\}} + \mathbb{E}_\tau[V_{\tau+1}^{*,m-\min\{k_\tau,m\}}], H_\tau^{\min\{k_\tau,m\}-1} \\ & + \mathbb{E}_\tau[V_{\tau+1}^{*,m-(\min\{k_\tau,m\}-1)}], ..., H_\tau^1 + \mathbb{E}_\tau[V_{\tau+1}^{*,m-1}]\}\big]. \end{aligned}$$

A useful quantity in the pricing of American options in general is the continuation value C_t^*, which is the expected cashflow at the next time-step if the option has not been exercised. In our context we define the continuation value in a similar way.

Definition 2 The continuation value $C_t^{*,m}$ at time t of an option, which could be exercised up to k_t times per single exercise time t with m exercise opportunities in total, is given by

$$C_t^{*,m} = \mathbb{E}_t[V_{t+1}^{*,m}].$$

Now the dynamic programming equations can be written in terms of the continuation value.

$$\begin{aligned} C_T^{*,m} &= 0, \\ C_t^{*,m} &= \mathbb{E}_t\big[\max\{H_{t+1}^{\min\{k_{t+1},m\}} + C_{t+1}^{*,m-\min\{k_{t+1},m\}}, \\ &\quad H_{t+1}^{\min\{k_{t+1},m\}-1} + C_{t+1}^{*,m-(\min\{k_{t+1},m\}-1)}, \\ &\quad ..., H_{t+1}^{1} + C_{t+1}^{*,m-1}, C_{t+1}^{*,m}\}\big]. \end{aligned}$$

When valuing multiple exercise options an important quantity is the value of an additional exercise opportunity.

Definition 3 The marginal value of one additional exercise opportunity is denoted by $\Delta V_t^{*,m}$ for $m \geq 1$:

$$\Delta V_t^{*,m} = V_t^{*,m} - V_t^{*,m-1}.$$

The marginal value for $m = 1$ is just the option value for one exercise opportunity

$$\Delta V_t^{*,1} = V_t^{*,1}.$$

Of course the marginal continuation values can be given by

$$\Delta C_t^{*,m} = C_t^{*,m} - C_t^{*,m-1}.$$

We can now give our dual representation for the marginal value function.

Theorem 1 *(Marginal value - Dual Representation). The marginal value* $\Delta V_0^{*,m}$ *is equal to*

$$\Delta V_0^{*,m} = \inf_{\pi} \inf_{\mathcal{M} \in \boldsymbol{M}_0} \mathbb{E}_0\big[\max_{\substack{u=0,1,...,T \\ N_u^{m-1}(\tau_{m-1},...,\tau_1) < k_u}} (h_u^{N_u^{m-1}+1} - \mathcal{M}_u)\big],$$

The first inf *is taken over all exercise policies with* $m-1$ *rights, which are denoted by* $\pi = (\tau_{m-1}, ..., \tau_1)$*. The second* inf *is taken over* $\boldsymbol{M}_0$*, the set of integrable martingales which are null at 0. The* max *is taken over the exercise times that have not been already used in* π*. Furthermore the* inf *is attained for the optimal policy with* $m-1$ *rights* $\pi^* = (\tau_{m-1}^*, ..., \tau_1^*)$ *and martingales* $\mathcal{M}_t^*$ *defined by*

$$\mathcal{M}_{t+1}^* - \mathcal{M}_t^* = \sum_{l=0}^{m-1} (\Delta M_{t+1}^{*,m-l} - \Delta M_t^{*,m-l}) \mathbf{1}_{\bar{N}_t^{m-1}(\tau_{m-1}^*,...,\tau_1^*)=l}.$$

The proof of this result will be given in Section 5. We note that if the payoffs h^i are fixed, independent of i, we recover the dual result given in [6].

3 Properties of the Marginal Values

Here we will state and prove several properties of the marginal values, which we will use later.

From the dynamic programming equations we can see that

$$\mathbb{E}_t[V_{t+1}^{*,m}] \leq V_t^{*,m}.$$

Thus the process $V_t^{*,m}$ is a supermartingale and has a Doob decomposition

$$V_t^{*,m} = V_0^{*,m} + M_t^{*,m} - D_t^{*,m},$$

where $M_t^{*,m}$ is a martingale vanishing at $t = 0$ and $D_t^{*,m}$ is a previsible increasing process also vanishing at $t = 0$. The increments of the process $D_t^{*,m}$ are given by the previsible parts of the option price process

$$D_{t+1}^{*,m} - D_t^{*,m} = V_t^{*,m} - \mathbb{E}_t[V_{t+1}^{*,m}].$$

The martingale difference is

$$M_{t+1}^{*,m} - M_t^{*,m} = V_{t+1}^{*,m} - \mathbb{E}_t[V_{t+1}^{*,m}].$$

Corresponding to the marginal values we can further introduce $\Delta M_t^{*,m}$ and $\Delta D_t^{*,m}$ as

$$\begin{aligned} \Delta M_t^{*,m} &= M_t^{*,m} - M_t^{*,m-1}, \\ \Delta D_t^{*,m} &= D_t^{*,m} - D_t^{*,m-1}. \end{aligned}$$

Here $\Delta M_t^{*,m}$, being a difference of martingales is again a martingale and $\Delta D_t^{*,m}$ is a previsible process. Both vanishing at $t = 0$.

For simplicity of notation we introduce

$$\begin{aligned} A_t^{*,m,l,i} = \big[h_t^i - \mathbb{E}_t[\Delta V_{t+1}^{*,m}] + [h_t^{i+1} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}] \\ + [... + [h_t^{i+l-1} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-l+1}]]_+ ...]_+]_+\big]_+, \end{aligned}$$

where we have embedded in a sum the positive parts of l terms of the kind $h_t^{i+\cdot} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-\cdot}]$. We will give a simpler expression for this quantity in Remark 3 after the proof of Proposition 3

Proposition 1 *The increment of the process $D_t^{*,m}$ can be expressed as*

$$D_{t+1}^{*,m} - D_t^{*,m} = A_t^{*,m,min\{k_t,m\},1}.$$

Proof: Using the dynamic programming formulation and assuming $k_t < m$,

$$\begin{aligned}
& D^{*,m}_{t+1} - D^{*,m}_t = V^{*,m}_t - \mathbb{E}_t[V^{*,m}_{t+1}] \\
&\quad = \max\{H^{k_t}_t + \mathbb{E}_t[V^{*,m-k_t}_{t+1}], H^{k_t-1}_t + \mathbb{E}_t[V^{*,m-(k_t-1)}_{t+1}], \\
&\quad\ \ ..., H^1_t + \mathbb{E}_t[V^{*,m-1}_{t+1}], \mathbb{E}_t[V^{*,m}_{t+1}]\} - \mathbb{E}_t[V^{*,m}_{t+1}] \\
&\quad = \big[\max\big\{H^{k_t}_t + \mathbb{E}_t[V^{*,m-k_t}_{t+1}], H^{k_t-1}_t + \mathbb{E}_t[V^{*,m-(k_t-1)}_{t+1}], \\
&\quad\ \ ..., H^1_t + \mathbb{E}_t[V^{*,m-1}_{t+1}]\big\} - \mathbb{E}_t[V^{*,m}_{t+1}]]_+ \\
&\quad = \big[h^1_t - \mathbb{E}_t[\Delta V^{*,m}_{t+1}] + [\max\{H^{k_t}_t + \mathbb{E}_t[V^{*,m-k_t}_{t+1}], H^{k_t-1}_t \\
&\quad\ \ + \mathbb{E}_t[V^{*,m-(k_t-1)}_{t+1}], ..., H^2_t + \mathbb{E}_t[V^{*,m-2}_{t+1}]\} - h^1_t - \mathbb{E}_t[V^{*,m-1}_{t+1}]]_+\big]_+ .
\end{aligned}$$

Proceeding in this way we have the result. The case where $k_t \geq m$ follows the same argument. □

Now we will write the marginal value as an optimal stopping problem.

Proposition 2 *The marginal value $\Delta V^{*,m}_t$ can be written as*

$$\Delta V^{*,m}_t = \sup_{t\leq\tau\leq T} \mathbb{E}_t\big[h^1_\tau + A^{*,m-1,min\{k_\tau-1,m-1\},2}_\tau - D^{*,m-1}_{\tau+1}\big] + D^{*,m-1}_t .$$

Proof: Using the optimal stopping formulation

$$\begin{aligned}
& \Delta V^{*,m}_t = V^{*,m}_t - V^{*,m-1}_t \\
&\quad = \sup_{t\leq\tau\leq T} \mathbb{E}_t\big[\max\{H^{\min\{k_\tau,m\}}_\tau + \mathbb{E}_\tau[V^{*,m-\min\{k_\tau,m\}}_{\tau+1}], \\
&\quad\ \ H^{\min\{k_\tau,m\}-1}_\tau + \mathbb{E}_\tau[V^{*,m-(\min\{k_\tau,m\}-1)}_{\tau+1}], \\
&\quad\ \ ..., H^1_\tau + \mathbb{E}_\tau[V^{*,m-1}_{\tau+1}]\}\big] - V^{*,m-1}_t \\
&\quad = \sup_{t\leq\tau\leq T} \mathbb{E}_t\big[\max\{H^{\min\{k_\tau,m\}}_\tau + \mathbb{E}_\tau[V^{*,m-\min\{k_\tau,m\}}_{\tau+1}], \\
&\quad\ \ H^{\min\{k_\tau,m\}-1}_\tau + \mathbb{E}_\tau[V^{*,m-(\min\{k_\tau,m\}-1)}_{\tau+1}], \\
&\quad\ \ ..., H^1_\tau + \mathbb{E}_\tau[V^{*,m-1}_{\tau+1}]\} - V^{*,m-1}_\tau + V^{*,m-1}_\tau\big] - V^{*,m-1}_t \\
&\quad = \sup_{t\leq\tau\leq T} \mathbb{E}_t\big[h^1_\tau + A^{*,m-1,min\{k_\tau-1,m-1\},2}_\tau + \mathbb{E}_\tau[V^{*,m-1}_{\tau+1}] \\
&\quad\ \ - V^{*,m-1}_\tau + V^{*,m-1}_\tau\big] - V^{*,m-1}_t .
\end{aligned}$$

In the last line we rewrote

$$\begin{aligned}
&\max\{H^{\min\{k_\tau,m\}}_\tau + \mathbb{E}_\tau[V^{*,m-\min\{k_\tau,m\}}_{\tau+1}], H^{\min\{k_\tau,m\}-1}_\tau + \mathbb{E}_\tau[V^{*,m-(\min\{k_\tau,m\}-1)}_{\tau+1}], \\
&\quad ..., H^1_\tau + \mathbb{E}_\tau[V^{*,m-1}_{\tau+1}]\}
\end{aligned}$$

as

$$h^1_\tau + A^{*,m-1,min\{k_\tau-1,m-1\},2}_\tau + \mathbb{E}_\tau[V^{*,m-1}_{\tau+1}]$$

using the same idea as in the proof of Proposition 1.

The term $\mathbb{E}_\tau[V^{*,m-1}_{\tau+1}] - V^{*,m-1}_\tau$ is just the previsible part of the option value process. Also from the Doob decomposition of $V^{*,m-1}_\tau$ we have

$$V^{*,m-1}_\tau = V^{*,m-1}_t + (M^{*,m-1}_\tau - M^{*,m-1}_t) - (D^{*,m-1}_\tau - D^{*,m-1}_t).$$

Thus

$$\begin{aligned}
\Delta V^{*,m}_t &= \sup_{t\le\tau\le T} \mathbb{E}_t\big[h^1_\tau + A^{*,m-1,min\{k_\tau-1,m-1\},2}_\tau - (D^{*,m-1}_{\tau+1} - D^{*,m-1}_\tau) \\
&\qquad + V^{*,m-1}_t + (M^{*,m-1}_\tau - M^{*,m-1}_t) \\
&\qquad\quad - (D^{*,m-1}_\tau - D^{*,m-1}_t)\big] - V^{*,m-1}_t \\
&= \sup_{t\le\tau\le T} \mathbb{E}_t\big[h^1_\tau + A^{*,m-1,min\{k_\tau-1,m-1\},2}_\tau - (D^{*,m-1}_{\tau+1} - D^{*,m-1}_t) \\
&\qquad + (M^{*,m-1}_\tau - M^{*,m-1}_t)\big] \\
&= \sup_{t\le\tau\le T} \mathbb{E}_t\big[h^1_\tau + A^{*,m-1,min\{k_\tau-1,m-1\},2}_\tau - D^{*,m-1}_{\tau+1}\big] + D^{*,m-1}_t,
\end{aligned}$$

as required. □

We are now ready to prove

Proposition 3 *The marginal value is a decreasing function of the number of exercise opportunities i.e. for $m \ge 1$*

$$\Delta V^{*,m+1}_t \le \Delta V^{*,m}_t, \qquad \forall t.$$

Proof: We will use induction by m. For $m = 1$ we have

$$\begin{aligned}
A^{*,1,min\{k_\tau-1,1\},2}_\tau - D^{*,1}_{\tau+1} + D^{*,1}_t &\le A^{*,1,min\{k_\tau-1,1\},1}_\tau - D^{*,1}_{\tau+1} + D^{*,1}_t \\
&= \begin{cases} -D^{*,1}_{\tau+1} + D^{*,1}_t, \; k_\tau = 1 \\ -D^{*,1}_\tau + D^{*,1}_t, \;\; k_\tau > 1 \end{cases}
\end{aligned}$$

where, in both cases, the right hand side is negative. Thus

$$\begin{aligned}
\Delta V^{*,2}_t &= \sup_{t\le\tau\le T} \mathbb{E}_t\big[h^1_\tau + A^{*,1,min\{k_\tau-1,1\}}_\tau - D^{*,1}_{\tau+1}\big] + D^{*,1}_t \\
&\le \sup_{t\le\tau\le T} \mathbb{E}_t\big[h^1_\tau\big] \\
&= \Delta V^{*,1}_t,
\end{aligned}$$

and the first step is proved.

By the inductive hypothesis we assume that $\Delta V^{*,l}_t \le \Delta V^{*,l-1}_t$, for any $l \le m$. Now we will prove that $\Delta V^{*,m+1}_t \le \Delta V^{*,m}_t$. First we will show that

$$D^{*,m}_{t+1} - D^{*,m}_t \ge D^{*,m-1}_{t+1} - D^{*,m-1}_t, \qquad \forall t, \tag{2}$$

which is equivalent to

$$A^{*,m,min\{k_t,m\},1}_t \ge A^{*,m-1,min\{k_t,m-1\},1}_t. \tag{3}$$

From the induction hypothesis we have

$$A_t^{*,m,min\{k_t,m-1\},1} \geq A_t^{*,m-1,min\{k_t,m-1\},1}.$$

By definition the summation in $A_t^{*,m,min\{k_t,m\},1}$ has at least as many terms as the summation in $A_t^{*,m,min\{k_t,m-1\},1}$ and hence (3) follows. By iterating (2) we have for any stopping time $\tau \in [t,T]$,

$$D_\tau^{*,m} - D_t^{*,m} \geq D_\tau^{*,m-1} - D_t^{*,m-1}, \tag{4}$$

Applying (4) we have

$$\begin{aligned} D_{\tau+1}^{*,m} &- D_t^{*,m} - (D_{\tau+1}^{*,m-1} - D_t^{*,m-1}) \\ &\geq D_{\tau+1}^{*,m} - D_\tau^{*,m} - (D_{\tau+1}^{*,m-1} - D_\tau^{*,m-1}) \\ &= A_\tau^{*,m,\min\{k_\tau,m\},1} - A_\tau^{*,m-1,\min\{k_\tau,m-1\},1}. \end{aligned}$$

Now we will show that

$$\begin{aligned} A_\tau^{*,m,\min\{k_\tau,m\},1} &- A_\tau^{*,m-1,\min\{k_\tau,m-1\},1} \\ &\geq A_\tau^{*,m,\min\{k_\tau-1,m\},1} - A_\tau^{*,m-1,\min\{k_\tau-1,m-1\},1}, \qquad \forall \tau. \end{aligned}$$

For $m < k_\tau$ we have equality, for $m = k_\tau$ the inequality is equivalent to

$$A_\tau^{*,m,m,1} \geq A_\tau^{*,m,m-1,1},$$

which is again trivially true. When $m > k_\tau$ the inequality is equivalent to

$$A_\tau^{*,m,k_\tau,1} - A_\tau^{*,m-1,k_\tau,1} \geq A_\tau^{*,m,k_\tau-1,1} - A_\tau^{*,m-1,k_\tau-1,1}, \qquad \forall \tau,$$

which follows from the fact that

$$\Delta V_\tau^{*,m-k_\tau+1} \leq \Delta V_\tau^{*,m-k_\tau}, \qquad \forall \tau.$$

Thus

$$\begin{aligned} D_{\tau+1}^{*,m} - D_t^{*,m} &- (D_{\tau+1}^{*,m-1} - D_t^{*,m-1}) \\ &\geq A_\tau^{*,m,\min\{k_\tau-1,m\},1} - A_\tau^{*,m-1,\min\{k_\tau-1,m-1\},1} \\ &\geq A_\tau^{*,m,\min\{k_\tau-1,m\},2} - A_\tau^{*,m-1,\min\{k_\tau-1,m-1\},2}, \qquad \forall \tau \geq t. \end{aligned}$$

The last inequality follows from the ordering of $h_\tau^1, ..., h_\tau^{k_\tau}$ The last step is now

$$\begin{aligned} \Delta V_t^{*,m+1} &= \sup_{t \leq \tau \leq T} \mathbb{E}_t \big[h_\tau + A_\tau^{*,m,\min\{k_\tau-1,m\},2} - D_{\tau+1}^{*,m} \big] + D_t^{*,m} \\ &\leq \sup_{t \leq \tau \leq T} \mathbb{E}_t \big[h_\tau + A_\tau^{*,m-1,\min\{k_\tau-1,m-1\},2} - D_{\tau+1}^{*,m-1} \big] + D_t^{*,m-1} \\ &= \Delta V_t^{*,m}, \end{aligned}$$

and the proof is completed. □

Remark 1 In the proof of the Proposition 3 we proved that

$$D_{t+1}^{*,m} - D_t^{*,m} \geq D_{t+1}^{*,m-1} - D_t^{*,m-1},$$

for any $m > 1$. This inequality is equivalent to

$$\Delta D_{t+1}^{*,m} \geq \Delta D_t^{*,m}.$$

Remark 2 From

$$\mathbb{E}_t[\Delta V_{t+1}^{*,m}] \leq \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}] \leq ... \leq \mathbb{E}_t[\Delta V_{t+1}^{*,m-\min\{m,k_t\}+1}]$$

we can determine the optimal exercise strategy. At time t we exercise i times, where

$$i = \max(j : h_t^j \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-j+1}] | 1 \leq j \leq \min\{m, k_t\}). \tag{5}$$

Remark 3 From $h_t^i \geq h_t^{i+1} \geq ... \geq h_t^{i+l-1}$ and $\mathbb{E}_t[\Delta V_{t+1}^{*,m}] \leq \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}] \leq ... \leq \mathbb{E}_t[\Delta V_{t+1}^{*,m-l+1}]$ we have that

$$h_t^i - \mathbb{E}_t[\Delta V_{t+1}^{*,m}] \geq h_t^{i+1} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}] \geq ... \geq h_t^{i+l-1} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-l+1}].$$

We can now rewrite $A_t^{*,m,l,i}$ as

$$\begin{aligned} A_t^{*,m,l,i} &= \big[h_t^i - \mathbb{E}_t[\Delta V_{t+1}^{*,m}] + [h_t^{i+1} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}] \\ &\qquad + [... + [h_t^{i+l-1} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-l+1}]]_+ ...]_+]_+\big]_+ \\ &= \sum_{j=0}^{l-1} [h_t^{i+j} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-j}]]_+. \end{aligned}$$

Proposition 4 *For each $m \geq 1$, the marginal value process is a supermartingale*

$$\Delta V_t^{*,m} \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m}].$$

Proof: Using the Doob decompositions

$$V_t^{*,m} = V_0^{*,m} + M_t^{*,m} - D_t^{*,m}$$

and

$$V_t^{*,m-1} = V_0^{*,m-1} + M_t^{*,m-1} - D_t^{*,m-1},$$

we have

$$\Delta V_t^{*,m} = \Delta V_0^{*,m} + \Delta M_t^{*,m} - \Delta D_t^{*,m}. \tag{6}$$

By subtracting the conditional expectation of (6) at time $t + 1$ from (6) and using the fact that $\Delta M_t^{*,m}$ is a martingale we have

$$\Delta V_t^{*,m} - \mathbb{E}_t[\Delta V_{t+1}^{*,m}] = \Delta D_{t+1}^{*,m} - \Delta D_t^{*,m}.$$

From Remark 1 the last quantity is positive and the proposition follows. □

For simplicity of notation we introduce the expression:

$$B_t^m = \sum_{i=1}^{\min\{k_t,m\}} \min(h_t^i, \mathbb{E}_t[\Delta V_{t+1}^{*,m-i}])\mathbf{1}_{i=\max(j:h_t^j \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-j+1}])} \\ + h_t^1 \mathbf{1}_{h_t^1 \leq \mathbb{E}_t[\Delta V_{t+1}^{*,m}]}.$$

This is the minimum return of the m-th exercise right at time t.

A second expression for the marginal value as an optimal stopping problem is the following.

Proposition 5 *The marginal value $\Delta V_t^{*,m}$ can be written as*

$$\Delta V_t^{*,m} = \sup_{t \leq \tau \leq T} \mathbb{E}_t\left[B_\tau^m - D_\tau^{*,m-1}\right] + D_t^{*,m-1}.$$

Proof: From Proposition 1 and Remark 3

$$\begin{aligned} D_{t+1}^{*,m-1} - D_t^{*,m-1} &= A_t^{*,m-1,\min\{k_t,m-1\},1} \\ &= \sum_{j=0}^{\min(k_t,m-1)-1} [h_t^{j+1} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-1-j}]]_+. \end{aligned}$$

We also have

$$h_t^1 + A_t^{*,m-1,\min\{k_t-1,m-1\},2} = h_t^1 + \sum_{j=0}^{\min(k_t,m)-2} [h_t^{j+2} - \mathbb{E}_t[\Delta V_{t+1}^{*,m-1-j}]_+.$$

Thus

$$\begin{aligned} &h_t^1 + A_t^{*,m-1,\min\{k_t-1,m-1\},2} - D_{t+1}^{*,m-1} + D_t^{*,m-1} \\ &\quad = \sum_{i=1}^{k_t} \min(h_t^i, \mathbb{E}_t[\Delta V_{t+1}^{*,m-i}])\mathbf{1}_{i=\max(j:h_t^j \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-j+1}])} \\ &\quad + h_t^1 \mathbf{1}_{h_t^1 < \mathbb{E}_t[\Delta V_{t+1}^{*,m}]}, \qquad \forall t. \end{aligned}$$

If $m-1 < k_t$ the expression above is equal to just h_t^1. Now using Proposition 2 we finish the proof. □

4 The constant constraint case

In this section we will consider the case when the number of exercises of the option allowed is fixed and the same at each time point, that is $k_0 = k_1 = k_2 = ... = k_T = k$. Also the payoff from consecutive exercises at the same time point t will be the same h_t. To denote this option we will add an upper index $\mathbf{k}$ in the notation i.e. $V_t^{*,m,\mathbf{k}}$. In this case we will prove that the option is equivalent to k options with $\mathbf{k} = \mathbf{1}$, that is options which can only be exercised once at each time.

Proposition 6 *For any $i \geq 0$*

$$\Delta V_t^{*,ki+1,\mathbf{k}} = \Delta V_t^{*,ki+2,\mathbf{k}} = ... = \Delta V_t^{*,ki+k,\mathbf{k}}, \qquad \forall 0 \leq t \leq T.$$

Proof: We will prove this proposition by induction on i. For $i = 0$ we will use backward induction with respect to t. For $t = T$ we have

$$V_T^{*,j,\mathbf{k}} = jh_T, \qquad j = 1, 2, ..., k$$

and

$$\Delta V_T^{*,1,\mathbf{k}} = \Delta V_T^{*,2,\mathbf{k}} = ... = \Delta V_T^{*,k,\mathbf{k}}.$$

Now from the induction hypothesis for t we have that

$$\Delta V_{t+1}^{*,1,\mathbf{k}} = \Delta V_{t+1}^{*,2,\mathbf{k}} = ... = \Delta V_{t+1}^{*,k,\mathbf{k}}.$$

Using this for any $m \leq k$ we have that

$$V_t^{*,m,\mathbf{k}} = \begin{cases} mh_t, & h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,1,\mathbf{k}}] \\ \mathbb{E}_t[V_{t+1}^{*,m,\mathbf{k}}], & \mathbb{E}_t[\Delta V_{t+1}^{*,1,\mathbf{k}}] > h_t \end{cases}$$

and

$$V_t^{*,m-1,\mathbf{k}} = \begin{cases} (m-1)h_t, & h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,1,\mathbf{k}}] \\ \mathbb{E}_t[V_{t+1}^{*,m-1,\mathbf{k}}], & \mathbb{E}_t[\Delta V_{t+1}^{*,1,\mathbf{k}}] > h_t \end{cases}$$

Thus

$$\Delta V_t^{*,m,\mathbf{k}} = \begin{cases} h_t, & h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,1,\mathbf{k}}] \\ \mathbb{E}_t[\Delta V_{t+1}^{*,m,\mathbf{k}}], & \mathbb{E}_t[\Delta V_{t+1}^{*,1,\mathbf{k}}] > h_t \end{cases}$$

From here clearly

$$\Delta V_t^{*,1,\mathbf{k}} = \Delta V_t^{*,2,\mathbf{k}} = ... = \Delta V_t^{*,k,\mathbf{k}}.$$

Thus the induction on t is completed and we have the first step of the induction on m.

Now from the induction hypothesis we have

$$\Delta V_t^{*,ki+1,\mathbf{k}} = \Delta V_t^{*,ki+2,\mathbf{k}} = ... = \Delta V_t^{*,ki+k,\mathbf{k}}, \qquad \forall t.$$

We will prove that

$$\Delta V_t^{*,k(i+1)+1,\mathbf{k}} = \Delta V_t^{*,k(i+1)+2,\mathbf{k}} = ... = \Delta V_t^{*,k(i+1)+k,\mathbf{k}}, \qquad \forall t.$$

Again we will use backward induction on t. For $t = T$

$$\Delta V_T^{*,k(i+1)+1,\mathbf{k}} = \Delta V_T^{*,k(i+1)+2,\mathbf{k}} = ... = \Delta V_T^{*,k(i+1)+k,\mathbf{k}} = 0.$$

Now suppose we have

$$\Delta V_{t+1}^{*,k(i+1)+1,\mathbf{k}} = \Delta V_{t+1}^{*,k(i+1)+2,\mathbf{k}} = ... = \Delta V_{t+1}^{*,k(i+1)+k,\mathbf{k}},$$

we shall prove that

$$\Delta V_t^{*,k(i+1)+1,\mathbf{k}} = \Delta V_t^{*,k(i+1)+2,\mathbf{k}} = ... = \Delta V_t^{*,k(i+1)+k,\mathbf{k}}.$$

For $m > k$ we have

$$V_t^{*,m,\mathbf{k}} = \begin{cases} kh_t + \mathbb{E}_t[V_{t+1}^{*,m-k,\mathbf{k}}], & h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-k+1,\mathbf{k}}] \\ ... \\ h_t + \mathbb{E}_t[V_{t+1}^{*,m-1,\mathbf{k}}], & \mathbb{E}_t[\Delta V_{t+1}^{*,m-1,\mathbf{k}}] > h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m,\mathbf{k}}] \\ \mathbb{E}_t[V_{t+1}^{*,m,\mathbf{k}}], & \mathbb{E}_t[\Delta V_{t+1}^{*,m,\mathbf{k}}] > h_t \end{cases}$$

and

$$V_t^{*,m-1,\mathbf{k}} = \begin{cases} kh_t + \mathbb{E}_t[V_{t+1}^{*,m-k-1,\mathbf{k}}], & h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-k,\mathbf{k}}] \\ ... \\ h_t + \mathbb{E}_t[V_{t+1}^{*,m-2,\mathbf{k}}], & \mathbb{E}_t[\Delta V_{t+1}^{*,m-2,\mathbf{k}}] > h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-1,\mathbf{k}}] \\ \mathbb{E}_t[V_{t+1}^{*,m-1,\mathbf{k}}], & \mathbb{E}_t[\Delta V_{t+1}^{*,m-1,\mathbf{k}}] > h_t. \end{cases}$$

The marginal value then can be written as

$$\Delta V_t^{*,m,\mathbf{k}} = \begin{cases} \mathbb{E}_t[\Delta V_{t+1}^{*,m-k,\mathbf{k}}], & h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-k,\mathbf{k}}] \\ h_t, & \mathbb{E}_t[\Delta V_{t+1}^{*,m-k,\mathbf{k}}] > h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m,\mathbf{k}}] \\ \mathbb{E}_t[\Delta V_{t+1}^{*,m,\mathbf{k}}], & \mathbb{E}_t[\Delta V_{t+1}^{*,m,\mathbf{k}}] > h_t. \end{cases}$$

In this representation of $\Delta V_t^{*,m,\mathbf{k}}$, using both induction hypotheses, it is easy to see that

$$\Delta V_t^{*,k(i+1)+1,\mathbf{k}} = \Delta V_t^{*,k(i+1)+2,\mathbf{k}} = ... = \Delta V_t^{*,k(i+1)+k,\mathbf{k}}.$$

Thus the inner and the outer inductions, together with the proof, are completed. $\square$

Proposition 7 *For any $m \geq 1$*

$$V_t^{*,km,\mathbf{k}} = kV_t^{*,m,\mathbf{1}}.$$

Proof: We will prove this proposition by induction on m. We already have the first step of the induction $m = 1$ from Proposition 6. Now the induction hypothesis is

$$V_t^{*,k(m-1),\mathbf{k}} = kV_t^{*,m-1,\mathbf{1}}, \qquad \forall t.$$

We also use inner backward induction on t. For $t = T$

$$V_T^{*,km,\mathbf{k}} = kh_T = kV_T^{*,m,\mathbf{1}}.$$

By Proposition 6 all the middle cases in the right hand side of

$$V_t^{*,mk,\mathbf{k}} = \begin{cases} kh_t + \mathbb{E}_t[V_{t+1}^{*,mk-k,\mathbf{k}}], \; h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,mk-k+1,\mathbf{k}}] \\ \dots \\ h_t + \mathbb{E}_t[V_{t+1}^{*,mk-1,\mathbf{k}}], \quad \mathbb{E}_t[\Delta V_{t+1}^{*,mk-1,\mathbf{k}}] > h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,mk,\mathbf{k}}] \\ \mathbb{E}_t[V_{t+1}^{*,mk,\mathbf{k}}], \qquad\qquad \mathbb{E}_t[\Delta V_{t+1}^{*,mk,\mathbf{k}}] > h_t \end{cases}$$

will collapse and we are left with

$$V_t^{*,mk,\mathbf{k}} = \begin{cases} kh_t + \mathbb{E}_t[V_{t+1}^{*,(m-1)k,\mathbf{k}}], \; h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,mk,\mathbf{k}}] \\ \mathbb{E}_t[V_{t+1}^{*,mk,\mathbf{k}}], \qquad\qquad \mathbb{E}_t[\Delta V_{t+1}^{*,mk,\mathbf{k}}] > h_t. \end{cases}$$

We also have

$$kV_t^{*,m,\mathbf{1}} = \begin{cases} kh_t + k\mathbb{E}_t[V_{t+1}^{*,(m-1),\mathbf{1}}], \; h_t \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m,\mathbf{1}}] \\ k\mathbb{E}_t[V_{t+1}^{*,m,\mathbf{1}}], \qquad\qquad \mathbb{E}_t[\Delta V_{t+1}^{*,m,\mathbf{1}}] > h_t. \end{cases}$$

We clearly have an equality by the induction hypotheses and bearing in mind that $\Delta V_{t+1}^{*,mk,\mathbf{k}} = \Delta V_{t+1}^{*,m,\mathbf{1}}$. Thus we have completed the proof. □

5 The Dual Problem

In this section we will prove our main result, Theorem 1, on the dual representation for the price of the multiple exercise option which can be exercised k_t times at time t. We will essentially follow the ideas of Rogers [14] and Meinshausen and Hambly [13]. The structure of the proof is to obtain an initial useful dual representation for the marginal value in terms of B_t^m. Once we have this we can extend to the full version. The intuition behind the initial dual representation is that the optimal stopping time for the m-th exercise right τ_m^* is found by searching within the set of times that are either smaller than the optimal stopping time for the $m-1$-th exercise right τ_{m-1}^* or equal to it.

Theorem 2 *The marginal value* $\Delta V_s^{*,m}$ *is equal to*

$$\Delta V_s^{*,m} = \inf_{s\le\tau\le T}\inf_{M\in\boldsymbol{M}_s}\mathbb{E}_s\big[\max_{s\le t\le\tau}\big(B_t^m\mathbf{1}_{t<\tau}+\max(B_t^m,\mathbb{E}_t[\Delta V_{t+1}^{*,m-1}])\mathbf{1}_{t=\tau}-M_t\big)\big],$$

where the infima are taken over all stopping times τ *and over the set of integrable martingales vanishing at s,* $\boldsymbol{M}_s$*. The infimum is attained for the martingale* $\Delta M_t^{*,m}-\Delta M_s^{*,m}$ *and stopping time* τ *defined as* $\tau^*=\min\{t:D_{t+1}^{*,m-1}>0, t\ge s\}$.

To prove this Theorem we require a preliminary result.

Lemma 3 *The marginal value* $\Delta V_t^{*,m}$ *satisfies the inequality*

$$\Delta V_t^{*,m}\le\max(B_t^m,\mathbb{E}_t[\Delta V_{t+1}^{*,m-1}]).$$

Proof: The inequality follows from the fact that the marginal value $\Delta V_t^{*,m}$ can be written in the following way:

$$\Delta V_t^{*,m}=\begin{cases}\mathbb{E}_t[\Delta V_{t+1}^{*,m-k_t}], & h_t^{k_t}\ge\mathbb{E}_t[\Delta V_{t+1}^{*,m-k_t}] \quad\text{and}\quad k_t<m\\ \min(h_t^i,\mathbb{E}_t[\Delta V_{t+1}^{*,m-i}]), & i=\max(j:h_t^j\ge\mathbb{E}_t[\Delta V_{t+1}^{*,m-j+1}]|1\le j\le\min\{m,k_t\})\\ \mathbb{E}_t[\Delta V_{t+1}^{*,m}], & \mathbb{E}_t[\Delta V_{t+1}^{*,m}]>h_t^1.\end{cases}$$

□

we now proceed to the proof of the Theorem.

Proposition 8 *For all stopping times* $\tau\le T$ *and all integrable martingales* M_t *that are null at time s*

$$\Delta V_s^{*,m}\le\mathbb{E}_s\big[\max_{s\le t\le\tau}\big(B_t^m\mathbf{1}_{t<\tau}+\max(B_t^m,\mathbb{E}_t[\Delta V_{t+1}^{*,m-1}])\mathbf{1}_{t=\tau}-M_t\big)\big].$$

Proof: Here we use Proposition 5 and split the interval for an intermediate stopping time, introducing stopping time θ

$$\begin{aligned}\Delta V_s^{*,m} &= \sup_{s\le\tau\le T}\mathbb{E}_s[B_\tau^m-D_\tau^{*,m-1}]\\ &= \sup_{s\le\theta\le\tau}\mathbb{E}_s[(B_\theta^m-D_\theta^{*,m-1})\mathbf{1}_{\theta<\tau}+\sup_{\tau\le\theta'\le T}\mathbb{E}_\tau[B_{\theta'}^m-D_{\theta'}^{*,m-1}]\mathbf{1}_{\theta=\tau}]\\ &\le \sup_{s\le\theta\le\tau}\mathbb{E}_s[B_\theta^m\mathbf{1}_{\theta<\tau}+(\sup_{\tau\le\theta'\le T}\mathbb{E}_\tau[B_{\theta'}^m-D_{\theta'}^{*,m-1}]+D_\tau^{*,m-1})\mathbf{1}_{\theta=\tau}].\end{aligned}$$

Here again by Proposition 5 we use $\Delta V_\tau^{*,m}$ instead of its representation and then by Lemma 3 we have

$$\begin{aligned}\Delta V_s^{*,m} &\le \sup_{s\le\theta\le\tau}\mathbb{E}_s[B_\theta^m\mathbf{1}_{\theta<\tau}+\Delta V_\theta^{*,m}\mathbf{1}_{\theta=\tau}]\\ &\le \sup_{s\le\theta\le\tau}\mathbb{E}_s[B_\theta^m\mathbf{1}_{\theta<\tau}+\max\{B_\theta^m,\mathbb{E}_\theta[\Delta V_{\theta+1}^{*,m-1}]\}\mathbf{1}_{\theta=\tau}].\end{aligned}$$

Now by introducing a martingale M_t vanishing at s and using the inequality for interchanging maximum and expectation

$$\begin{aligned}\Delta V_s^{*,m} &\leq \sup_{s\leq\theta\leq\tau} \mathbb{E}_s[B_\theta^m \mathbf{1}_{\theta<\tau} + \max\{B_\theta^m, \mathbb{E}_\theta[\Delta V_{\theta+1}^{*,m-1}]\}\mathbf{1}_{\theta=\tau} - M_\theta] \\ &\leq \mathbb{E}_s[\max_{s\leq t\leq\tau}(B_t^m \mathbf{1}_{t<\tau} + \max\{B_t^m, \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}]\}\mathbf{1}_{t=\tau} - M_t)],\end{aligned}$$

and the proof is completed. □

The second part of the proof of Theorem 2 is the following inequality.

Proposition 9 *The following inequality holds for the marginal value $\Delta V_s^{*,m}$*

$$\Delta V_s^{*,m} \geq \inf_{s\leq\tau\leq T}\inf_{M\in\boldsymbol{M}_s} \mathbb{E}_s[\max_{s\leq t\leq\tau}(B_t^m \mathbf{1}_{t<\tau} + \max\{B_t^m, \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}]\}\mathbf{1}_{t=\tau} - M_t)].$$

The infimum is attained for $\Delta M_t^{*,m} - \Delta M_s^{*,m}$ and $\tau^* = \min\{t : D_{t+1}^{*,m-1} > 0, t \geq s\}$.

Proof: Define a stopping time τ^* by $\tau^* = \min\{t : D_{t+1}^{*,m-1} > 0, t \geq s\}$. Substituting τ^* for τ and $\Delta M_t^{*,m} - \Delta M_s^{*,m}$ for M_t we have

$$\begin{aligned}&\inf_{s\leq\tau\leq T}\inf_{M\in\mathbf{M}_s} \mathbb{E}_s\big[\max_{s\leq t\leq\tau}\big(B_t^m \mathbf{1}_{t<\tau} + \max\{B_t^m, \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}]\}\mathbf{1}_{t=\tau} - M_t\big)\big] \\ &\leq \mathbb{E}_s\big[\max_{s\leq t\leq\tau^*}\big(B_t^m \mathbf{1}_{t<\tau^*} + \max\{B_t^m, \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}]\}\mathbf{1}_{t=\tau^*} - \Delta M_t^{*,m} + \Delta M_s^{*,m}\big)\big].\end{aligned}$$

By the definition of τ^* we have that $D_{\tau^*}^{*,m-1} = 0$. From

$$\begin{aligned}D_{\tau^*+1}^{*,m-1} - D_{\tau^*}^{*,m-1} &= A_{\tau^*}^{*,m-1,\min\{k_{\tau^*},m-1\},1} \\ &= \sum_{j=0}^{\min(k_{\tau^*},m-1)-1} [h_{\tau^*}^{j+1} - \mathbb{E}_{\tau^*}[\Delta V_{\tau^*+1}^{*,m-1-j}]]_+ > 0,\end{aligned}$$

it follows that

$$\max(B_{\tau^*}^m, \mathbb{E}_{\tau^*}[\Delta V_{\tau^*+1}^{*,m-1}]) = B_{\tau^*}^m.$$

Then we have

$$\begin{aligned}&\inf_{s\leq\tau\leq T}\inf_{M\in\mathbf{M}_s} \mathbb{E}_s\big[\max_{s\leq t\leq\tau}\big(B_t^m \mathbf{1}_{t<\tau} + \max\{B_t^m, \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}]\}\mathbf{1}_{t=\tau} - M_t\big)\big] \\ &\leq \mathbb{E}_s\big[\max_{s\leq t\leq\tau^*}\big(B_t^m \mathbf{1}_{t<\tau^*} + B_t^m \mathbf{1}_{t=\tau^*} - \Delta M_t^{*,m} + \Delta M_s^{*,m}\big)\big] \\ &= \mathbb{E}_s\big[\max_{s\leq t\leq\tau^*}\big(B_t^m - \Delta M_t^{*,m} + \Delta M_s^{*,m}\big)\big].\end{aligned}$$

From the representation

$$\Delta V_t^{*,m} = \sup_{t\leq\tau\leq T} \mathbb{E}_t\big[B_\tau^m - D_\tau^{*,m-1}\big] + D_t^{*,m-1},$$

and the positivity of $D_t^{*,m-1}$ we have

$$\begin{aligned}\Delta V_t^{*,m} &\geq \sup_{t\leq\tau\leq T} \mathbb{E}_t\big[B_\tau^m - D_\tau^{*,m-1}\big] \\ &\geq B_t^m - D_t^{*,m-1}.\end{aligned}$$

On the other hand

$$\Delta V_t^{*,m} = \Delta V_0^{*,m} + \Delta M_t^{*,m} - \Delta D_t^{*,m}$$

and

$$\Delta V_s^{*,m} = \Delta V_0^{*,m} + \Delta M_s^{*,m} - \Delta D_s^{*,m}.$$

Hence

$$B_t^m - D_t^{*,m-1} - \Delta M_t^{*,m} + \Delta M_s^{*,m} \leq \Delta V_s^{*,m} - \Delta D_t^{*,m} + \Delta D_s^{*,m}. \quad (7)$$

We have that $D_t^{*,m-1} = 0$ for $s \leq t \leq \tau^*$ so

$$\begin{aligned}&\inf_{s\leq\tau\leq T} \inf_{M\in\mathbf{M}_s} \mathbb{E}_s\big[\max_{s\leq t\leq\tau}\big(B_t^m\mathbf{1}_{t<\tau} + \max\{B_t^m, \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}]\}\mathbf{1}_{t=\tau} - M_t\big)\big] \\ &\leq \mathbb{E}_s\big[\max_{s\leq t\leq\tau^*}(\Delta V_s^{*,m} - \Delta D_t^{*,m} + \Delta D_s^{*,m})\big] \\ &\leq \Delta V_s^{*,m}.\end{aligned}$$

For the last step we used that $\Delta D_t^{*,m} > \Delta D_s^{*,m}$. Using Proposition 8 we can see that equality is attained for τ^* and $\Delta M_t^{*,m} - \Delta M_s^{*,m}$, and the proof is completed. $\square$

This result completes the proof of Theorem 2. This will allow us to prove the main result and we begin with some notation and a useful lemma.

Definition 4 We define for any set of stopping times $\{\tau_{m-1}, ..., \tau_1\}$ taking values in $t, ..., T$ and martingale $\mathcal{M}$ with $\mathcal{M}_t = 0$ the random variable

$$\Delta V_t^m = \max_{\substack{u=t,\dots,T \\ N_u^{m-1}<k_u}} (h_u^{N_u^{m-1}+1} - \mathcal{M}_u).$$

Lemma 4 *The following inequality holds for all* $1 \leq q < \min\{m, k_t\}$

$$\max(h_t^{q+1}, \mathbb{E}_t[\Delta V_{t+1}^{*,m-q}]) \geq B_t^m. \quad (8)$$

Proof: Suppose

$$\max(j : h_t^j \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-j+1}]) = l$$

this implies

$$\mathbb{E}_t[\Delta V_{t+1}^{*,m-l+1}] \leq h_t^l \quad , \quad \mathbb{E}_t[\Delta V_{t+1}^{*,m-l}] > h_t^{l+1}$$

and

$$B_t^m = \min(h_t^l, \mathbb{E}_t[\Delta V_{t+1}^{*,m-l}]).$$

Let consider two cases for $q+1$ and l. If $q+1 \le l$ we have

$$\max(h_t^{q+1}, \mathbb{E}_t[\Delta V_{t+1}^{*,m-q}]) \ge h_t^{q+1} \ge h_t^l \ge \min(h_t^l, \mathbb{E}_t[\Delta V_{t+1}^{*,m-l}]).$$

If $q+1 > l$ i.e. $q \ge l$

$$\max(h_t^{q+1}, \mathbb{E}_t[\Delta V_{t+1}^{*,m-q}]) \ge \mathbb{E}_t[\Delta V_{t+1}^{*,m-q}] \ge \mathbb{E}_t[\Delta V_{t+1}^{*,m-l}] \ge \min(h_t^l, \mathbb{E}_t[\Delta V_{t+1}^{*,m-l}]).$$

In either case the inequality we wanted to show holds.

□

We now establish Theorem 1 by proving two inequalities, which are formulated in Propositions 10 and 11.

Proposition 10 *For all sets of stopping times $\{\tau_{m-1}, ..., \tau_1\}$ and martingales $\mathcal{M} \in \boldsymbol{M}_s$*

$$\mathbb{E}_s[\Delta V_s^m] \ge \Delta V_s^{*,m} \qquad s = 0, ..., T.$$

Proof: We will use induction. For $m = 1$ the statement follows from [14]. For $m > 1$ we have

$$\begin{aligned}
&\mathbb{E}_s[\Delta V_s^m] \\
&\quad = \mathbb{E}_s\big[\max_{\substack{u=s,...,T \\ N_u^{m-1}(\tau_{m-1},...,\tau_1)<k_u}} (h_u^{N_u^{m-1}+1} - \mathcal{M}_u)\big] \\
&\quad = \mathbb{E}_s\big[\max(\max_{s\le t<\tau_{m-1}} (h_t^{N_t^{m-1}+1} - \mathcal{M}_t), (h_{\tau_{m-1}}^{N_{\tau_{m-1}}^{m-1}+1} - \mathcal{M}_{\tau_{m-1}}) \mathbf{1}_{N_{\tau_{m-1}}^{m-1}<k_{\tau_{m-1}}}, \\
&\qquad \sum_{l=1}^{\min\{k_{\tau_{m-1}},m-1\}} \max_{\substack{u=\tau_{m-1}+1,...,T \\ N_u^{m-l-1}(\tau_{m-l-1},...,\tau_1)<k_u}} (h_u^{N_u^{m-l-1}+1} - \mathcal{M}_u) \mathbf{1}_{N_{\tau_{m-1}}^{m-1}=l})\big].
\end{aligned}$$

Here by using the tower property and by interchanging the expectation and the maximum we have

$$\begin{aligned}
&\mathbb{E}_s[\Delta V_s^m] \\
&\quad \ge \mathbb{E}_s\big[\mathbb{E}_{\tau_{m-1}+1}\big[\max(\max_{s\le t<\tau_{m-1}} (h_t^{N_t^{m-1}+1} - \mathcal{M}_t), (h_{\tau_{m-1}}^{N_{\tau_{m-1}}^{m-1}+1} - \mathcal{M}_{\tau_{m-1}}) \mathbf{1}_{N_{\tau_{m-1}}^{m-1}<k_{\tau_{m-1}}}, \\
&\qquad \sum_{l=1}^{\min\{k_{\tau_{m-1}},m-1\}} \max_{\substack{u=\tau_{m-1}+1,...,T \\ N_u^{m-l-1}(\tau_{m-l-1},...,\tau_1)<k_u}} (h_u^{N_u^{m-l-1}+1} - \mathcal{M}_u) \mathbf{1}_{N_{\tau_{m-1}}^{m-1}=l})\big]\big] \\
&\quad \ge \mathbb{E}_s\big[\max(\max_{s\le t<\tau_{m-1}} (h_t^{N_t^{m-1}+1} - \mathcal{M}_t), (h_{\tau_{m-1}}^{N_{\tau_{m-1}}^{m-1}+1} - \mathcal{M}_{\tau_{m-1}}) \mathbf{1}_{N_{\tau_{m-1}}^{m-1}<k_{\tau_{m-1}}}, \\
&\qquad \sum_{l=1}^{\min\{k_{\tau_{m-1}},m-1\}} \mathbb{E}_{\tau_{m-1}+1}\big[\max_{\substack{u=\tau_{m-1}+1,...,T \\ N_u^{m-l-1}(\tau_{m-l-1},...,\tau_1)<k_u}} (h_u^{N_u^{m-l-1}+1} - \mathcal{M}_u + \mathcal{M}_{\tau_{m-1}+1})\big] \mathbf{1}_{N_{\tau_{m-1}}^{m-1}=l} \\
&\qquad - \mathcal{M}_{\tau_{m-1}+1})\big].
\end{aligned}$$

By the induction hypothesis

$$\mathbb{E}_s[\Delta V_s^m] \geq \mathbb{E}_s\big[\max(\max_{s\leq t<\tau_{m-1}}(h_t^{N_t^{m-1}+1}-\mathcal{M}_t),(h_{\tau_{m-1}}^{N_{\tau_{m-1}}^{m-1}+1}-\mathcal{M}_{\tau_{m-1}})\mathbf{1}_{N_{\tau_{m-1}}^{m-1}<k_{\tau_{m-1}}},\\ \sum_{l=1}^{\min\{k_{\tau_{m-1}},m-1\}}\Delta V_{\tau_{m-1}+1}^{*,m-l}\mathbf{1}_{N_{\tau_{m-1}}^{m-1}=l}-\mathcal{M}_{\tau_{m-1}+1})\big].$$

Here again by using the tower property and by interchanging the expectation and the maximum we have

$$\begin{aligned}
&\mathbb{E}_s[\Delta V_s^m]\\
&\geq \mathbb{E}_s\big[\mathbb{E}_{\tau_{m-1}}\big[\max(\max_{s\leq t<\tau_{m-1}}(h_t^{N_t^{m-1}+1}-\mathcal{M}_t),(h_{\tau_{m-1}}^{N_{\tau_{m-1}}^{m-1}+1}-\mathcal{M}_{\tau_{m-1}})\mathbf{1}_{N_{\tau_{m-1}}^{m-1}<k_{\tau_{m-1}}},\\
&\qquad \sum_{l=1}^{\min\{k_{\tau_{m-1}},m-1\}}\Delta V_{\tau_{m-1}+1}^{*,m-l}\mathbf{1}_{N_{\tau_{m-1}}^{m-1}=l}-\mathcal{M}_{\tau_{m-1}+1})\big]\big]\\
&\geq \mathbb{E}_s\big[\max(\max_{s\leq t<\tau_{m-1}}(h_t^{N_t^{m-1}+1}-\mathcal{M}_t),(h_{\tau_{m-1}}^{N_{\tau_{m-1}}^{m-1}+1}-\mathcal{M}_{\tau_{m-1}})\mathbf{1}_{N_{\tau_{m-1}}^{m-1}<k_{\tau_{m-1}}},\\
&\qquad \sum_{l=1}^{\min\{k_{\tau_{m-1}},m-1\}}(\mathbb{E}_{\tau_{m-1}}[\Delta V_{\tau_{m-1}+1}^{*,m-l}]-\mathcal{M}_{\tau_{m-1}})\mathbf{1}_{N_{\tau_{m-1}}^{m-1}=l})\big]\\
&= \mathbb{E}_s\big[\max(\max_{s\leq t<\tau_{m-1}}(h_t^{N_t^{m-1}+1}-\mathcal{M}_t),\\
&\qquad \sum_{l=1}^{\min\{k_{\tau_{m-1}}-1,m-1\}}(\max(h_{\tau_{m-1}}^{l+1},\mathbb{E}_{\tau_{m-1}}[\Delta V_{\tau_{m-1}+1}^{*,m-l}])-\mathcal{M}_{\tau_{m-1}})\mathbf{1}_{N_{\tau_{m-1}}^{m-1}=l}\\
&\qquad +(\mathbb{E}_{\tau_{m-1}}[\Delta V_{\tau_{m-1}+1}^{*,m-\min\{k_{\tau_{m-1}},m\}}]-\mathcal{M}_{\tau_{m-1}})\mathbf{1}_{N_{\tau_{m-1}}^{m-1}=\min\{k_{\tau_{m-1}},m\}})\big].
\end{aligned}$$

We use $h_t^{N_t^{m-1}+1}=h_t^1\geq B_t^m$, for $t<\tau_{m-1}$; $\mathbb{E}_t[\Delta V_{t+1}^{*,m-k_{\tau_{m-1}}}]\geq B_t^m$ for $k_{\tau_{m-1}}\leq m$; $\mathbb{E}_t[\Delta V_{t+1}^{*,m-l}]\geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}]$ for $l\geq 1$ and Lemma 4, and then Theorem 2, to see that

$$\begin{aligned}
\mathbb{E}_s[\Delta V_s^m] &\geq \mathbb{E}_s\big[\max_{s\leq t\leq\tau_{m-1}}\big(B_t^m\mathbf{1}_{t<\tau_{m-1}}+\max(B_t^m,\mathbb{E}_t[\Delta V_{t+1}^{*,m-1}])\mathbf{1}_{t=\tau_{m-1}}-\mathcal{M}_t\big)\big]\\
&\geq \inf_{s\leq\tau\leq T}\inf_{\mathcal{M}\in\mathbf{M}_s}\mathbb{E}_s\big[\max_{s\leq t\leq\tau}\big(B_t^m\mathbf{1}_{t<\tau}+\max(B_t^m,\mathbb{E}_t[\Delta V_{t+1}^{*,m-1}])\mathbf{1}_{t=\tau}-\mathcal{M}_t\big)\big]\\
&= \Delta V_s^{*,m}
\end{aligned}$$

as required. □

Proposition 11 *Under the optimal set of stopping times $\{\tau_{m-1}^*,...,\tau_1^*\}$ and martingales $\mathcal{M}_t^*$ defined by*

$$\mathcal{M}_{t+1}^*-\mathcal{M}_t^*=\sum_{l=0}^{m-1}(\Delta M_{t+1}^{*,m-l}-\Delta M_t^{*,m-l})\mathbf{1}_{\bar{N}_t^*=l},$$

and

$$\mathcal{M}^*_s = 0$$

where $\bar{N}^*_t = \bar{N}^{m-1}_t(\tau^*_{m-1}, \dots, \tau^*_1)$, *we have*

$$\Delta V^m_s \leq \Delta V^{*,m}_s \qquad s = 0, ..., T.$$

Proof: The proof is again by induction. For $m = 1$ the statement follows from [14]. For $m > 1$ we have

$$\begin{aligned}
\Delta V^m_s = & \max_{\substack{u=s,\dots,T \\ N^{m-1}_u(\tau^*_{m-1},\dots,\tau^*_1)<k_u}} (h^{N^{m-1}_u+1}_u - \mathcal{M}^*_u) \\
= & \max\Big(\max_{s\leq t<\tau^*_{m-1}} (h^{N^{m-1}_t+1}_t - \mathcal{M}^*_t), (h^{N^{m-1}_{\tau^*_{m-1}}+1}_{\tau^*_{m-1}} - \mathcal{M}^*_{\tau^*_{m-1}})\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}<k_{\tau^*_{m-1}}}, \\
& \sum_{l=1}^{\min\{k_{\tau^*_{m-1}},m-1\}} \max_{\substack{u=\tau^*_{m-1}+1,\dots,T \\ N^{m-l-1}_u(\tau^*_{m-l-1},\dots,\tau^*_1)<k_u}} (h^{N^{m-1}_u+1}_u - \mathcal{M}^*_u)\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}=l}\Big) \\
= & \max\Big(\max_{s\leq t<\tau^*_{m-1}} (h^{N^{m-1}_t+1}_t - \mathcal{M}^*_t), (h^{N^{m-1}_{\tau^*_{m-1}}+1}_{\tau^*_{m-1}} - \mathcal{M}^*_{\tau_{m-1}})\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}<k_{\tau^*_{m-1}}} \\
& \sum_{l=1}^{\min\{k_{\tau^*_{m-1}},m-1\}} \Big(\max_{\substack{u=\tau^*_{m-1}+1,\dots,T \\ N^{m-l-1}_u(\tau^*_{m-l-1},\dots,\tau^*_1)<k_u}} (h^{N^{m-l-1}_u+1}_u - \mathcal{M}^*_u + \mathcal{M}^*_{\tau^*_{m-1}+1}) - \mathcal{M}^*_{\tau^*_{m-1}+1}\Big)\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}=l}\Big).
\end{aligned}$$

We now use the induction hypothesis

$$\begin{aligned}
\Delta V^m_s \leq & \max\Big(\max_{s\leq t<\tau^*_{m-1}} (h^{N^{m-1}_t+1}_t - \mathcal{M}^*_t), (h^{N^{m-1}_{\tau^*_{m-1}}+1}_{\tau^*_{m-1}} - \mathcal{M}^*_{\tau^*_{m-1}})\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}<k_{\tau^*_{m-1}}}, \\
& \sum_{l=1}^{\min\{k_{\tau^*_{m-1}},m-1\}} (\Delta V^{*,m-l}_{\tau^*_{m-1}+1} - \mathcal{M}^*_{\tau^*_{m-1}+1})\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}=l}\Big).
\end{aligned}$$

Using

$$\Delta M^{*,m-l}_{t+1} - \Delta M^{*,m-l}_t = \Delta V^{*,m-l}_{t+1} - \mathbb{E}_t[\Delta V^{*,m-l}_{t+1}],$$

the definition of the optimal martingale

$$\mathcal{M}^*_{t+1} - \mathcal{M}^*_t = \sum_{l=0}^{m-1} (\Delta M^{*,m-l}_{t+1} - \Delta M^{*,m-l}_t)\mathbf{1}_{\bar{N}^*_t=l}$$

and $\bar{N}^*_{\tau^*_{m-1}} = N^*_{\tau^*_{m-1}}$ we obtain

$$\Delta V^m_s \le \max\Big(\max_{s\le t<\tau^*_{m-1}} (h_t^{N_t^{m-1}+1} - \mathcal{M}^*_t), \sum_{l=1}^{\min\{k_{\tau^*_{m-1}}-1, m-1\}} (h^{l+1}_{\tau^*_{m-1}} - \mathcal{M}^*_{\tau^*_{m-1}})\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}=l},$$
$$\sum_{l=1}^{\min\{k_{\tau^*_{m-1}}, m-1\}} (\mathbb{E}_{\tau^*_{m-1}}[\Delta V^{*,m-l}_{\tau^*_{m-1}+1}] - \mathcal{M}^*_{\tau^*_{m-1}})\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}=l}\Big)$$
$$\le \max_{s\le t\le\tau^*_{m-1}} \Big(h_t^{N_t^{m-1}+1}\mathbf{1}_{t<\tau^*_{m-1}}$$
$$+ \sum_{l=1}^{\min\{k_{\tau^*_{m-1}}-1, m-1\}} \max(h^{l+1}_{\tau^*_{m-1}}, \mathbb{E}_{\tau^*_{m-1}}[\Delta V^{*,m-l}_{\tau^*_{m-1}+1}])\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}=l}$$
$$+ \mathbb{E}_{\tau^*_{m-1}}[\Delta V^{*,m-\min\{k_{\tau^*_{m-1}},m\}}_{\tau^*_{m-1}+1}]\mathbf{1}_{N^{m-1}_{\tau^*_{m-1}}=\min\{k_{\tau^*_{m-1}},m\}} - \mathcal{M}^*_t\Big).$$

Using the definition of the optimal martingale and the fact that for $t < \tau^*_{m-1}$

$$\mathbf{1}_{N^*_t=l} = \begin{cases} 1, \, l = 0 \\ 0, \, l \neq 0 \end{cases}$$

we get that $\mathcal{M}^*_t = \Delta M^{*,m}_t$, for $t \le \tau^*_{m-1}$. Under the optimal exercise policy the conditions for exercising l out of $m-1$ rights at time t can be written as

$$N^*_t = l \iff \mathbb{E}_t[\Delta V^{*,m-l}_{t+1}] \le h^l_t \quad \text{and} \quad \mathbb{E}_t[\Delta V^{*,m-l-1}_{t+1}] > h^{l+1}_t.$$

This implies that either l or $l+1$ out of m rights are exercised at time t. The corresponding conditions are

$$\max(j : h^j_t \ge \mathbb{E}_t[\Delta V^{*,m-j+1}_{t+1}]) = l \iff \mathbb{E}_t[\Delta V^{*,m-l+1}_{t+1}] \le h^l_t \quad \text{and} \quad \mathbb{E}_t[\Delta V^{*,m-l}_{t+1}] > h^{l+1}_t$$

or

$$\max(j : h^j_t \ge \mathbb{E}_t[\Delta V^{*,m-j+1}_{t+1}]) = l+1 \iff \mathbb{E}_t[\Delta V^{*,m-l}_{t+1}] \le h^{l+1}_t \quad \text{and} \quad \mathbb{E}_t[\Delta V^{*,m-l-1}_{t+1}] > h^{l+2}_t$$

Thus we get

$$\max(h^{l+1}_t, \mathbb{E}_t[\Delta V^{*,m-l}_{t+1}])\mathbf{1}_{N^*_t=l} \le (\min(h^l_t, \mathbb{E}_t[\Delta V^{*,m-l}_{t+1}])\mathbf{1}_{\max(j:h^j_t\ge\mathbb{E}_t[\Delta V^{*,m-j+1}_{t+1}])=l}$$
$$+ \min(h^{l+1}_t, \mathbb{E}_t[\Delta V^{*,m-l-1}_{t+1}])\mathbf{1}_{\max(j:h^j_t\ge\mathbb{E}_t[\Delta V^{*,m-j+1}_{t+1}])=l+1})\mathbf{1}_{N^*_t=l}.$$

If $l = k_t$ and $k_t \le m$

$$N^*_t = k_t \iff \mathbb{E}_t[\Delta V^{*,m-k_t}_{t+1}] \le h^{k_t}_t$$

and

$$\max(j : h^j_t \ge \mathbb{E}_t[\Delta V^{*,m-j+1}_{t+1}]) = k_t$$

We also have

$$h_t^1 = B_t^m, \qquad t < \tau_{m-1}^*.$$

From here we can write

$$\Delta V_s^m \leq \max_{s \leq t \leq \tau_{m-1}^*} \left(B_t^m - \Delta M_t^{*,m} + \Delta M_s^{*,m} \right) \leq \Delta V_s^{*,m}.$$

The last step here came from (7). $\square$

Thus, combining Propositions 10 and 11, we have proved Theorem 1.

6 Practical Application

In this section we will introduce methods to compute lower and upper bounds for the price of the multiple exercise claims, using some of the theoretical properties we have derived. The lower bound is calculated by a generalization of the regression method introduced by Longstaff and Schwartz[12] and the upper bound calculation relies on the dual representation in Theorem 1.

With the ordering of the continuation values

$$\mathbb{E}_t[\Delta V_{t+1}^{*,m}] \leq \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}], \qquad \forall m, \forall t$$

we can determine the optimal exercise strategy at time level t as

if $i = \max(j : h_t^j \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-j+1}] | 1 \leq j \leq \min\{m, k_t\})$ - we exercise i times.
if $\mathbb{E}_t[\Delta V_{t+1}^{*,m}] > h_t^1$ - do not exercise.

In terms of the marginal continuation values we can write the strategy as

if $i = \max(j : h_t^j \geq \Delta C_t^{*,m-j+1} | 1 \leq j \leq \min\{m, k_t\})$ - we exercise i times.
if $\Delta C_t^{*,m} > h_t^1$ - do not exercise.

We will start with the calculation of the lower bound. The idea here is to work backwards in time, approximating the marginal continuation value with a linear combination of basis functions. In this way an approximation of the optimal exercise strategy is found and consequently a lower bound for the option price.

Lower bound algorithm

Let $\psi_i : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, \ldots, l$ be the basis functions used for the regression.

1. For all times $t \in \{0, 1, 2, ..., T\}$ and at each point $x \in \mathbb{R}^d$ we define $\Delta \hat{C}_t^m(x)$, an approximation to the $m-$th marginal continuation value $\Delta C_t^{*,m}$, by

$$\Delta \hat{C}_t^m(x) = \sum_{i=1}^{l} c_{t,i}^m \psi_i(x).$$

Of course the optimal continuation values are not known. Thus we use non-optimal continuation values C_t^m defined as follows.

2. Suppose that, working backwards in time and forward from one exercise opportunity, approximations $\Delta\hat{C}^{m}_{t+1}, \Delta\hat{C}^{m-1}_{t+1}, ..., \Delta\hat{C}^{m-\min\{m,k_{t+1}\}+1}_{t+1}$ to the $m-$th, $m-1$,...,$m-\min\{m,k_{t+1}\}+1$ marginal continuation value functions have been obtained. Then for path j define the approximate continuation value $C_t^{m,(j)}$ to be

$$C_t^{m,(j)} = \begin{cases} H_{t+1}^{\min\{k_{t+1}^{(j)},m\}}(X_{t+1}^{(j)}) + C_{t+1}^{m-\min\{k_{t+1}^{(j)},m\},(j)}, \\ \qquad \text{if } h_{t+1}^{\min\{k_{t+1}^{(j)},m\}}(X_{t+1}^{(j)}) \geq \Delta\hat{C}_{t+1}^{m-\min\{k_{t+1}^{(j)},m\}+1}(X_{t+1}^{(j)}) \\ H_{t+1}^{\min\{k_{t+1}^{(j)},m\}-1}(X_{t+1}^{(j)}) + C_{t+1}^{m-\min\{k_{t+1}^{(j)},m\}+1,(j)}, \\ \qquad \text{if } \min\{k_{t+1}^{(j)},m\}-1 = \max(i : h_{t+1}^{i}(X_{t+1}^{(j)}) \geq \Delta\hat{C}_{t+1}^{m-i+1}(X_{t+1}^{(j)}) | 1 \leq i \leq \min\{m,k_{t+1}\}) \\ \qquad \vdots \\ C_{t+1}^{m,(j)}, \\ \qquad \text{if } \Delta\hat{C}_{t+1}^{m}(X_{t+1}^{(j)}) > h_{t+1}(X_{t+1}^{(j)}) \end{cases}$$

The non-optimal $m-$th marginal continuation values are also defined by

$$\Delta C_t^{m,(j)} = C_t^{m,(j)} - C_t^{m-1,(j)}.$$

3. Let $\psi = (\psi_1, \psi_2, ..., \psi_l)$ and $\bar{c}_t^m = (c_{t,1}^m, c_{t,2}^m, ..., c_{t,l}^m)$. If n paths of the Markov chain are simulated, an estimate for the regression coefficients would be

$$\bar{c}_t^m = \arg\min_{c \in \mathbb{R}^l} \sum_{j=1}^{n} \left(\Delta C_t^{m,(j)} - \sum_{i=1}^{l} c_{t,i}^m \psi_i(X_t^{(j)})\right)^2.$$

The explicit formulas for the coefficients are

$$\begin{aligned} \bar{c}_t^m &= \Psi^{-1} v^m, \\ \Psi_{l,p} &= \frac{1}{n} \sum_{j=1}^{n} \psi_l(X_t^{(j)}) \psi_p(X_t^{(j)}), \\ v_l^m &= \frac{1}{n} \sum_{j=1}^{n} \psi_l(X_t^{(j)}) \Delta C_t^{m,(j)}. \end{aligned}$$

4. Once the coefficients $c_{t,1}^m, c_{t,2}^m, ..., c_{t,l}^m$ are obtained we can approximate the m-th marginal continuation value, and from there the stopping rule, at any point in the state space. We work backwards in time until we reach $t = 0$.

We can now move to the problem of estimating the upper bound. In order to use the dual representation from Theorem 1 we need approximations to the optimal martingale $\mathcal{M}_t^{*,m}$ and the stopping times $\tau_{m-1}, ..., \tau_1$.

Upper bound algorithm

1. We will assume that we already have found an approximation to the marginal continuation value (for instance the technique described earlier for the lower bound approximation). From there an approximation to the stopping times can be found.
2. We will give an approximation of the optimal martingale $\mathcal{M}_t^{*,m}$. We start by approximating the marginal martingales. We can write the martingale differences $\Delta M_{t+1}^{*,m} - \Delta M_t^{*,m}$ as

$$\begin{aligned}\Delta DM_{t+1}^{*,m} :&= \Delta M_{t+1}^{*,m} - \Delta M_t^{*,m} \\ &= \Delta V_{t+1}^{*,m} - \Delta V_t^{*,m} + \Delta D_{t+1}^{*,m} - \Delta D_t^{*,m} \\ &= \Delta V_{t+1}^{*,m} - \mathbb{E}_t[\Delta V_{t+1}^{*,m}].\end{aligned}$$

3. Using $\Delta \hat{C}_t^m$, our approximation of the marginal continuation value $\Delta C_t^{*,m}$ using regression, we have

$$\Delta \hat{V}_{t+1}^m(x) = \begin{cases} \Delta \hat{C}_{t+1}^{m-k_{t+1}}(x), & h_{t+1}^{k_{t+1}}(x) \geq \Delta \hat{C}_{t+1}^{m-k_{t+1}}(x) \quad \text{and} \quad m > k_{t+1} \\ \min(h_{t+1}^i(x), \Delta \hat{C}_{t+1}^{m-i}(x)), & i = \max(j : h_{t+1}^j(x) \geq \Delta \hat{C}_t^{m-j+1}(x) | 1 \leq j \leq \min\{m, k_{t+1}\}) \\ \Delta \hat{C}_{t+1}^m(x), & \Delta \hat{C}_{t+1}^m(x) > h_{t+1}^1(x) \end{cases} \tag{9}$$

4. We define the approximation of the $m-$th marginal martingale difference, $\Delta DM_{t+1}^{*,m,\mathbf{k}}$, by

$$\Delta DM_{t+1}^m = \Delta \hat{V}_{t+1}^m - \hat{\mathbb{E}}_t[\Delta \hat{V}_{t+1}^m] \tag{10}$$

where $\Delta \hat{V}_t^m$ is given by (9) and $\hat{\mathbb{E}}_t[\Delta \hat{V}_{t+1}^m]$ is calculated as an average over K i.i.d. subpaths advancing one time-step.
5. We define the approximation of $D\mathcal{M}_{t+1}^{*,m} = \mathcal{M}_{t+1}^{*,m} - \mathcal{M}_t^{*,m}$ by

$$D\mathcal{M}_{t+1}^m = \sum_{l=0}^{m-1} \Delta DM_{t+1}^{m-l} \mathbf{1}_{\bar{N}_t = l}. \tag{11}$$

6. We define the approximation of the martingale $\mathcal{M}_{t+1}^{*,m}$ by

$$\mathcal{M}_{t+1}^m = D\mathcal{M}_{t+1}^m + D\mathcal{M}_t^m + ... + D\mathcal{M}_1^m. \tag{12}$$

7. Now using the approximations of the stopping time and the martingale instead of the optimal ones we obtain an upper bound for the option price.

7 Numerical example

The numerical example we consider is pricing a swing contract with the following features:

1 The swing option has maturity T days and can be exercised on days $1, 2, ..., T$.

Table 1 Lower and upper bounds compared with option with one exercise right per time.

Exercise possibilities	1 ex. right per time	lower	upper	standard dev upper bound	relative difference	upper bound marginal value	CPU time in sec
1	4.77	4.77	4.79	0.0065	0.004	4.79	44
2	9.06	9.37	9.39	0.0090	0.002	4.60	85
3	13.03	13.65	13.68	0.0106	0.002	4.29	135
4	16.83	17.73	17.83	0.0119	0.006	4.15	186
5	20.52	21.70	21.84	0.0128	0.006	4.01	242
6	24.08	25.52	25.74	0.0138	0.008	3.90	308
7	27.52	29.32	29.54	0.0144	0.008	3.80	346
8	30.89	32.98	33.26	0.015	0.008	3.72	396
9	34.18	36.59	36.91	0.0156	0.009	3.65	448
10	37.37	40.08	40.50	0.0162	0.010	3.59	509
15	52.80	57.05	57.66	0.018	0.011	3.34	767
20	67.13	72.95	73.83	0.02	0.012	3.17	1018
25	80.74	88.12	89.27	0.0212	0.013	3.04	1312

2 It can be exercised up to k_t times on day t and the total number of exercise rights is m .
3 When exercising the option, its holder buys a certain number of units (say in 1MWh) of electricity for a prespecified fixed price K.

The underlying process X is the electricity spot price. We model X as the exponential of a discrete mean reverting process

$$\log X_{t+1} = (1-\alpha)\log X_t + \sigma W_t, \tag{13}$$

where $X_0 = 1, \sigma = 0.5$ and $\alpha = 0.9$ and W is a Brownian motion, so in this context a random walk with normally distributed increments. The payoff is taken to be the spot price itself. The basis functions used for approximating the marginal continuation values are

$$\Psi = \{1, \log X\}. \tag{14}$$

We look at options with a lifetime $T = 1000$. The second column of Table 1 is for options with one exercise right per time. The third and fourth columns are with lower and upper bounds for the marginal values of options that can be exercised up to one time on each weekend day and up to two times on each weekday. The fifth column is the standard deviation of the upper bound. The sixth column shows computation times. We use 10000 pre-simulation path to determine the stopping strategy and 20000 paths for the lower bound, given the stopping strategy. For the upper bound we use 1000 paths and 50 inner paths for the martingale approximation. The implementation was done in matlab and the calculation was run on a 1.5 GHz processor. We note that the computation time grows roughly linearly with the number of exercise opportunities available.

We can see that for 1-25 exercise opportunities the upper and lower bounds for the option values are quite close. In fact the relative difference between the

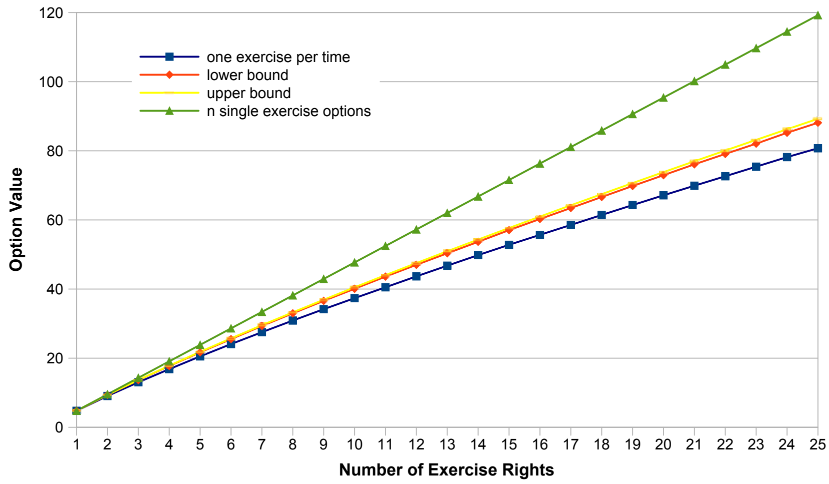

Fig. 1 Lower and upper bounds for the option prices.

upper and the lower bound remains below 1.5%. In the Figure we compare the bounds on the price with either having only one exercise possibility at each time point, or with buying the same number of single exercise options.

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