The Hausdorff spectrum of a class of multifractal processes

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Abstract

The Multifractal Embedded Branching Process (MEBP) process and Canonical Embedded Branching Process (CEBP) process were introduced by Decrouez and Jones (2012). The CEBP is a process in which the crossings of dyadic intervals constitute a branching process. An MEBP process is defined as a multifractal time-change of a CEBP process, where the time-change is such that both it and the CEBP can be simulated simultaneously in an on-line fashion. In this paper we investigate the scaling properties of the CEBP, the time-change, and the MEBP. Under various moment conditions, we show that CEBP processes have a constant modulus of continuity, obtain the Hausdorff spectrum of the time-change, and thus obtain the Hausdorff spectrum an an MEBP process.

Keywords: multifractal processes, Hausdorff spectrum, multifractal spectrum, singularity spectrum, branching processes

1 Introduction

Recently Decrouez and Jones [8] described a new class of multifractal processes, called Multifractal Embedded Branching Process (MEBP) processes. MEBPs are stochastic processes defined using a crossing tree. A crossing tree can be defined for any continuous one dimensional process. For such a process we define a sequence of stopping times—which we call crossing times—by looking at the process when it crosses points on the dyadic lattice $2^{-n}\mathbb{Z}$. Any crossing from $2^{-n}k$ to $2^{-n}(k \pm 1)$ can be decomposed into a sequence of crossings from $2^{-(n+1)}j$ to $2^{-(n+1)}(j \pm 1)$, for $j \in \{2k-1, 2k, 2k+1\}$. Accordingly any sample path can be decomposed into a labelled tree—the crossing tree—with labels giving the durations and directions of crossings on the dyadic lattice $2^{-n}\mathbb{Z}$, for all n.

An MEBP has a Galton-Watson branching process describing its crossing tree. For any suitable branching process there is a family of processes—identical up to a continuous time change—for which the spatial component of the crossing tree coincides with the branching process. We identify one of these as the Canonical Embedded Branching Process (CEBP), and then construct an MEBP from it using a multifractal time change. To construct the time change we use a multiplicative cascade on the (spatial component of) the crossing tree. The cascade defines a measure on the boundary of the tree, which we map onto $\mathbb{R}_+$ using the so called "branching measure" (in contrast to the way this is usually done, using a "splitting measure"). Taking partial integrals of the cascade measure gives us the time change process.

Multifractal processes find numerous applications in many diverse fields, including the study of natural phenomena in physics (hydrodynamic turbulence, the solar magnetic field), biology (human heart rate and gait), geology (earthquakes and fault repartitioning), but also man-made applications, such as signal and image processing (texture characterisation), financial markets (modelling of volatility) or computer network traffic, to cite but a few. See for example the reviews [16, 11], and the references therein. Multifractal processes have complex local dynamics. The dynamics of a process $X = \{X(t); t \in [0,T]\}$ at time t can be described using the local Hölder exponent $h_X(t)$, defined as

$$h_X(t) := \liminf_{\epsilon \to 0} \frac{1}{\log \epsilon} \log \sup_{|u-t|<\epsilon} |X(u) - X(t)| \quad 0 \le t \le T.$$

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When $h_X(t)$ is constant all along the sample path with probability 1, X is said to be monofractal. In contrast, there exist processes whose Hölder exponent behaves erratically, whereby in any interval of positive length we find a range of different exponents. For such processes the behaviour of $h_X(t)$ can be captured by the multifractal/singularity/Hausdorff spectrum D_X, a global description of the local fluctuations. $D_X(h)$ is defined as the Hausdorff dimension of the set of points with a given Hölder exponent h. For monofractal processes, $D_X(h)$ degenerates to a single point at some $h = H$ (so $D_X(H) = 1$, and the convention is to set $D_X(h) = -\infty$ for $h \neq H$). When the spectrum is non trivial for a range of values of h, the process is said to be multifractal.

As our time change is obtained by integrating a multifractal measure, we will also need a definition of the multifractal spectrum of a measure. Let $B(x, r)$ be a ball centred at $x \in \mathbb{R}^n$ with radius r. The local dimension of a finite measure μ at $x \in \mathbb{R}^n$ is defined, when the limit exists, as

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} \tag{1}$$

The multifractal or Hausdorff spectrum of μ at scale α, $D_\mu(\alpha)$, is defined as the Hausdorff dimension of the set of points with a given local dimension α. Measures for which the multifractal spectrum does not degenerate to a point are called multifractal measures. The term multifractal was originally used by Mandelbrot [17] as a description of measures arising in turbulence.

For many multifractal processes the local dynamics, as described by the multifractal spectrum, can be related to the global scaling. In such cases we say that a multifractal formalism holds. Let $\Delta_X(a, t)$ summarise the spatial displacement of X at time t and at a temporal scale a. For example we might just take $\Delta_X(a, t) = |X(t+a)-X(t)|$. If the expected time averages of $|\Delta_X(a, t)|^q$ scale like $c_q a^{\zeta_X(q)}$, then we call ζ_X a partition function. The multifractal formalism holds if we can write the multifractal spectrum in terms of the Legendre-Fenchel transform of ζ_X. Partition functions can be constructed, for example, using wavelets [3, 20, 2], or related multiresolution quantities such as wavelet leaders [12]. In what follows we introduce a novel set of multiresolution quantities, defined in terms of the crossing tree, which in our context can be viewed as a path-adapted multiresolution decomposition of the process. Using these we obtain a partition function for the time change process, and establish that a multifractal formalism holds.

Our principle aim in this paper is to describe the Hausdorff spectrum for a large class of MEBPs. This will involve two steps. We firstly show that the CEBP is a monofractal and calculate its Hölder exponent. This is done by obtaining a modulus of continuity result for the process in the style of Barlow and Perkins (1988), who obtained such a result for Brownian motion on the Sierpinski gasket. Some modifications are required however, as the process is not necessarily Markov. We then compute the Hausdorff spectrum of the time change, with methodology based on Biggins, Hambly and Jones (2011) for the lower bound, and Riedi (2003) for the upper bound. A simple transformation of this spectrum combined with the modulus of continuity result for the CEBP gives the Hausdorff spectrum for the MEBP.

A description of the construction of CEBP and MEBP is given in Sections 2 and 3. We work with a simpler subset of processes to those considered in [8], which allows us to streamline the construction. The modulus of continuity for CEBP processes is derived in Section 2.1, which tells us that they are monofractal. The multifractal spectrum of the time change is derived in Section 4, and a multifractal formalism established. Finally in Section 5 we put everything together to get the multifractal spectrum for MEBP processes.

Figure 1: A section of sample path and levels 3, 4 and 5 of its crossing tree. In the top frame we have joined the points T_k^n at each level, and in the bottom frame we have identified the k-th level n crossing with the point $(2^n, T_{k-1}^n)$ and linked each crossing to its subcrossings.

2 The Canonical Embedded Branching Process (CEBP) process

Let $X : \mathbb{R}^+ \to \mathbb{R}$ be a continuous process, with $X(0) = 0$. For $n \in \mathbb{Z}$ we define level n passage times T_k^n by putting $T_0^n = 0$ and

$$T_{k+1}^n = \inf\{t > T_k^n \mid X(t) \in 2^n\mathbb{Z},\ X(t) \neq X(T_k^n)\}.$$

The k-th level n (equivalently scale 2^n) crossing C_k^n is the sample path from T_{k-1}^n to T_k^n. That is, $C_k^n = \{(t, X(t)) \mid T_{k-1}^n \leq t < T_k^n\}$.

When passing from a coarse scale to a finer one, we decompose each level n crossing into a sequence of level $n-1$ crossings. To define the crossing tree, we associate nodes with crossings, and the children of a node are its subcrossings. The crossing tree is illustrated Figure 1, where the level 3, 4 and 5 crossings of a given sample path are shown.

In addition to indexing crossings by their level and position within each level, we will also use a tree indexing scheme. Let $\emptyset$ be the root of the tree, representing the first level 0 crossing. The first generation of children (which are level -1 crossings, of size 1/2) are labelled by i, $1 \leq i \leq Z_\emptyset$, where $Z_\emptyset$ is the number of children of $\emptyset$. The second generation (which are level -2 crossings, of size 1/4) are then labelled ij, $1 \leq j \leq Z_i$, where Z_i is the number of children of i. More generally, for $n \geq 0$, the index $\mathbf{i}$ of a level $-n$ crossing is a sequence of n positive integers, giving the offspring number of the level k crossing containing $\mathbf{i}$, for $k = -1, \ldots, -n$. Note that generation k of the tree corresponds to level $-k$ crossings. The length of an index $\mathbf{i} = i_1 \ldots i_n$ is denoted $|\mathbf{i}| = n$, and the k-th element is $\mathbf{i}[k] = i_k$. If $|\mathbf{i}| > m$, $\mathbf{i}|_m$ is the curtailment of $\mathbf{i}$ after m terms. Conventionally $|\emptyset| = 0$ and $\mathbf{i}|_0 = \emptyset$.

A tree Ψ is a set of crossing indices, or nodes, such that: (a) $\emptyset \in \Psi$; (b) if a node $\mathbf{i}$ belongs to the tree then every ancestor node $\mathbf{i}|_k$, $k \le |\mathbf{i}|$, belongs to the tree; and (c) if $\mathbf{u} \in \Psi$, then $\mathbf{u}j \in \Psi$ for $j = 1, \ldots, Z_{\mathbf{u}}$ and $\mathbf{u}j \notin \Psi$ for $j > Z_{\mathbf{u}}$, where $Z_{\mathbf{u}}$ is the number of children of $\mathbf{u}$. Let $\Upsilon_\emptyset$ be the tree described by the crossing indices. Define $\Upsilon_{\mathbf{i}} = \{\mathbf{j} \in \Upsilon_\emptyset \,|\, |\mathbf{j}| \ge |\mathbf{i}| \text{ and } \mathbf{j}|_{|\mathbf{i}|} = \mathbf{i}\}$. The boundary of the tree is given by $\partial\Upsilon_\emptyset = \{\mathbf{i} \in \mathbb{N}^{\mathbb{N}} \,|\, \forall m \ge 0, \mathbf{i}|_m \in \Upsilon_\emptyset\}$. Let $\psi(\mathbf{i})$ be the position of node $\mathbf{i}$ within generation $|\mathbf{i}|$, so that crossing $\mathbf{i}$ is just $C^{-|\mathbf{i}|}_{\psi(\mathbf{i})}$. The nodes to the left and right of $\mathbf{i}$, corresponding to the crossings $C^{-|\mathbf{i}|}_{\psi(\mathbf{i})-1}$ and $C^{-|\mathbf{i}|}_{\psi(\mathbf{i})+1}$, will be denoted $\mathbf{i}-$ and $\mathbf{i}+$.

In general when we have quantities associated with crossings we will use tree indexing and level/position indexing interchangeably. So $Z_{\mathbf{i}} = Z^{-|\mathbf{i}|}_{\psi(\mathbf{i})}$, $T_{\mathbf{i}} = T^{-|\mathbf{i}|}_{\psi(\mathbf{i})}$, etc.

At present our tree indexing only applies to crossings contained within the first level 0 crossing. We extend the tree indexing to the whole process by indexing crossings relative to a *spine*, defined by the first crossing at each level. Clearly any crossing is contained in some super-crossing on the spine. For any n let $n{:}\emptyset$ be the index for C^n_1, which is the first crossing on level n of the spine. As before, the sub-crossings of C^n_1 define a tree, which we denote $\Upsilon_{n:\emptyset}$. Nodes in the tree $\Upsilon_{n:\emptyset}$ will be labelled $n{:}\mathbf{i}$, where $\mathbf{i}$ is the tree index relative to $n{:}\emptyset$. Thus $n{:}\mathbf{i}$ is in level $n - |\mathbf{i}|$ of the crossing tree, and a crossing previously labelled $\mathbf{i}$ is now labelled $0{:}\mathbf{i}$. Our definition of ψ also generalises, so node $n{:}\mathbf{i}$ corresponds to crossing $C^{n-|\mathbf{i}|}_{\psi(n:\mathbf{i})}$. If we omit the prefix, as will often be the case, then it is assumed that we are descending from the first level 0 crossing.

Note that this labelling is not unique, as $n{:}\mathbf{i} = (n+1){:}1\mathbf{i}$. To define the doubly infinite tree, consisting of the spine and all its descendants, we need to choose a unique index for each node. For nodes descended from $0{:}\emptyset$ we use an index starting from level 0 on the spine, otherwise we take the level at which the line of descent leaves the spine. That is, our doubly infinite tree is given by

$$\Upsilon = \{n{:}\mathbf{i} \,:\, C_{n:\mathbf{i}} \text{ is a crossing and either } n = 0 \text{ or } n \ge 1 \text{ and } \mathbf{i}[1] \ne 1\}.$$

We will write $\Upsilon(m)$ for all the level m nodes of the tree. That is $\Upsilon(m) = \{n{:}\mathbf{i} \in \Upsilon \,:\, n - |\mathbf{i}| = m\}$. Similarly, for $m \le n - |\mathbf{i}|$, $\Upsilon_{n:\mathbf{i}}(m) = \{n{:}\mathbf{j} \in \Upsilon_{n:\mathbf{i}} \,:\, n - |\mathbf{j}| = m\}$.

Let $\alpha^n_k \in \{+,-\}$ be the orientation of C^n_k, $+$ for up and $-$ for down. A level n up crossing is from $k2^n$ to $(k+1)2^n$, a down crossing is from $k2^n$ to $(k-1)2^n$, for some k. Subcrossing orientations have a particular structure. The level $n-1$ subcrossings that make up a level n parent crossing consist of *excursions* (up-down and down-up pairs) followed by a *direct crossing* (down-down or up-up pairs), whose direction depends on the parent crossing: if the parent crossing is up, then the subcrossings end up-up, otherwise, they end down-down.

Let $D^n_k = T^n_k - T^n_{k-1}$ be the duration of C^n_k. Clearly, to reconstruct the process we only need α^n_k and D^n_k for all n and k. The α^n_k encode the spatial behaviour of the process, and the D^n_k the temporal behaviour.

For any given crossing $C_{n:\mathbf{i}}$ there is a branching process embedded in its nested subcrossings. We define $W_{n:\mathbf{i}}(k) = |\{n{:}\mathbf{j} \in \Upsilon_{n:\mathbf{i}} \,:\, |\mathbf{j}| = |\mathbf{i}| + k\}|$ to be the number of level $n - |\mathbf{i}| - k$ level crossings which are sub-crossings of $C_{n:\mathbf{i}}$. If the subcrossing family sizes Z^m_k are i.i.d. then $\{W_{n:\mathbf{i}}(k)\}$ is clearly a Galton-Watson process, supercritical since $Z^m_k \ge 2$. In this case we call the process X an Embedded Branching Process (EBP) process.

For an EBP, let Z stand for a generic Z^m_k, let p be its probability mass function, and let $\mu = \mathbb{E}Z$. We make the following assumption

Assumption 2.1. *$Z \in 2\mathbb{N}$, Z is nontrivial (so $\mu = \mathbb{E}Z > 2$), and $\mathbb{E}(Z \log Z) < \infty$*

Assuming that the Z^m_k are i.i.d. and that Assumption 2.1 holds, standard results for supercritical Galton-Watson processes give us $\lim_{k\to\infty} \mu^{-k} W_{n:\mathbf{i}}(k) = W_{n:\mathbf{i}}$ exists a.s. and in mean, is continuous and strictly positive, and has mean 1. We have the following result from [8] Theorems 2.1 and 2.2.

Theorem 2.2. *For any offspring orientation distribution p satisfying Assumption 2.1, there exists a continuous process X defined on $\mathbb{R}_+$, such that, almost surely:*

1. *the Z_k^m are i.i.d. with distribution p;*
2. *given the number of subcrossings, the excursions are independent and equally likely to be up-down and down-up;*
3. *$D_k^m = \mu^m W_k^m$ for all m and k;*
4. *for each m the crossing durations D_k^m are all mutually independent, and D_k^m is independent of all Z_j^n for $n > m$;*
5. *let $H = \log 2/\log\mu$, then for all $c \in \{\mu^n : n \in \mathbb{Z}\}$,*

$$X(t) \overset{fdd}{=} c^{-H} X(ct), \tag{2}$$

where $\overset{fdd}{=}$ denotes equality for finite dimensional distributions (H is known as the Hurst index, and X is said to be discrete scale-invariant).

We call X the Canonical EBP (CEBP) process with offspring distribution p.

Remark 2.3. (1) The construction of a CEBP can be generalised to allow the distribution of Z_k^m and the pattern of excursions, to depend on the crossing orientation α_k^m.
(2) If we restrict ourselves to EBP processes whose crossing durations have the correct means and independence structure, then the CEBP is unique up to finite dimensional distributions.
(3) Brownian motion is a CEBP, with $p(2k) = \mathbb{P}(Z = 2k) = 2^{-k}$.

2.1 The modulus of continuity of a CEBP

The goal of this section is to establish that CEBP are monofractal processes with Hölder exponent $H = \log 2/\log\mu \in (0,1)$. We can motivate this result using a simple heuristic argument. For any $m{:}\mathbf{i} \in \partial\Upsilon_{m:\emptyset}$ we have

$$\begin{aligned}
&|X(T_{m:\mathbf{i}|_n-} + \mu^{m-n} W_{m:\mathbf{i}|_n}) - X(T_{m:\mathbf{i}|_n-})| \\
&= |X(T_{m:\mathbf{i}|_n}) - X(T_{m:\mathbf{i}|_n-})| \\
&= 2^{m-n} \\
&= (\mu^{m-n} W_{m:\mathbf{i}|_n})^{(m-n)\log 2/((m-n)\log\mu + \log W_{m:\mathbf{i}|_n})}
\end{aligned}$$

But, for suitable Z, we have from Liu [15] (see Lemma 4.8) that $\log W_{m:\mathbf{i}|_n}/(m-n) \to 0$, suggesting that the local Hölder exponent at $T_{m:\mathbf{i}} := \lim_{n\to\infty} T_{m:\mathbf{i}|_n}$ is $\log 2/\log\mu$.

Our approach is based on that used by Barlow & Perkins [4] to obtain the modulus of continuity of Brownian motion on the Sierpinski gasket, though complications arise because CEBP are not in general Markovian.

The basic idea is to use bounds on the crossing durations D_k^n to control how fast the process can move away from a given point. We will suppose throughout that X is a CEBP defined by the subcrossing number distribution p, which satisfies Assumption 2.1.

Let W stand for a generic W_k^n. We have the following from Biggins & Bingham [5].

Lemma 2.4. *Suppose that Assumption 2.1 holds. There exists strictly positive constants c_1, c_2, and c_3 such that for all $x > 0$*

$$\exp\left(-c_1 x^{-H/(1-H)}\right) \le \mathbb{P}(W < x) \le c_2 \exp\left(-c_3 x^{-H/(1-H)}\right).$$

Let $T_k^n(s)$ be the $k+1$-st level n crossing time greater than or equal to s, for $k \geq 0$. So if s is a level n crossing time then $T_0^n(s) = s$. The previous lemma gave us a bound on the duration of a crossing. The next lemma gives a lower bound on the time remaining in the current crossing. To establish this result we will need to make a further modest restriction to the class of CEBP we consider.

Assumption 2.5. *We assume that the subcrossing number distribution Z is such that there exists a ζ such that for all y*

$$Z + \zeta \geq_{st} Z - y \,|\, Z > y.$$

Here $\geq_{st}$ denotes stochastic domination. That is, for all y and z,

$$\mathbb{P}(Z - y > z \,|\, Z > y) \leq \mathbb{P}(Z + \zeta > z).$$

This condition clearly holds for Z bounded, and for Z that are NBU (New Better than Used, in which case $\zeta = 0$ suffices). Examples of NBU distributions include the negative binomial with shape ≥ 1 and the Poisson.

Lemma 2.6. *Let $\{\mathcal{F}_s\}_{s\geq 0}$ be the filtration generated by X.*

Suppose that Assumptions 2.1 and 2.5 hold, then there exists constants $c_4, c_5 > 0$ such that for all $x > 0$ and $n \in \mathbb{Z}$,

$$\mathbb{P}(T_0^n(s) - s \leq x \,|\, \mathcal{F}_s) \geq c_4 \exp\left(-c_5(\mu^{-n}x)^{-H/(1-H)}\right).$$

Proof. Note first that

$$T_0^{n+1}(s) = T_0^n(s) + \sum_{i=1}^{Z^{n+1}(s)} \mu^n W(i),$$

where $Z^{n+1}(s) \geq 0$ is the number of level n crossings from $T_0^n(s)$ to $T_0^{n+1}(s)$, and the $W(i)$ are independent and distributed as W. If $Z^{n+1}(s)$ is not zero then, conditioned on $\mathcal{F}_s$, it will be distributed as $Z - y \,|\, Z > y$, where Z has the subcrossing number distribution, and $y \geq 0$ is the number of level n crossings from the current level $n+1$ crossing which have already happened by time $T_0^n(s)$, including the current level n crossing. Our notation is illustrated in Figure 2.

Thus, from our assumption on Z, conditioning on $\mathcal{F}_s$ we have

$$\begin{aligned} T_0^{n+1}(s) - T_0^n(s) &\leq_{st} \sum_{i=1}^{Z+\zeta} \mu^n W(i) \\ &\stackrel{\mathrm{d}}{=} \mu^{n+1} W(0) + \sum_{i=1}^{\zeta} \mu^n W(i). \end{aligned}$$

As $n \downarrow -\infty$ we have $T_0^n(s) \downarrow s$ (this follows directly from [21] Theorem 1), whence

$$T_0^n(s) - s \leq_{st} \mu^n W(0,n) + \sum_{k=-\infty}^{n-1} \sum_{i=0}^{\zeta} \mu^k W(i,k),$$

where the $W(i,k)$ are i.i.d. with distribution W. Thus, for any $\theta > 0$ we have

$$\mathbb{E}(e^{-\theta(T_0^n(s)-s)} \,|\, \mathcal{F}_s) \geq \mathbb{E}e^{-\theta\mu^n W} \prod_{k=-\infty}^{n-1} \left(\mathbb{E}e^{-\theta\mu^k W}\right)^{\zeta}. \tag{3}$$

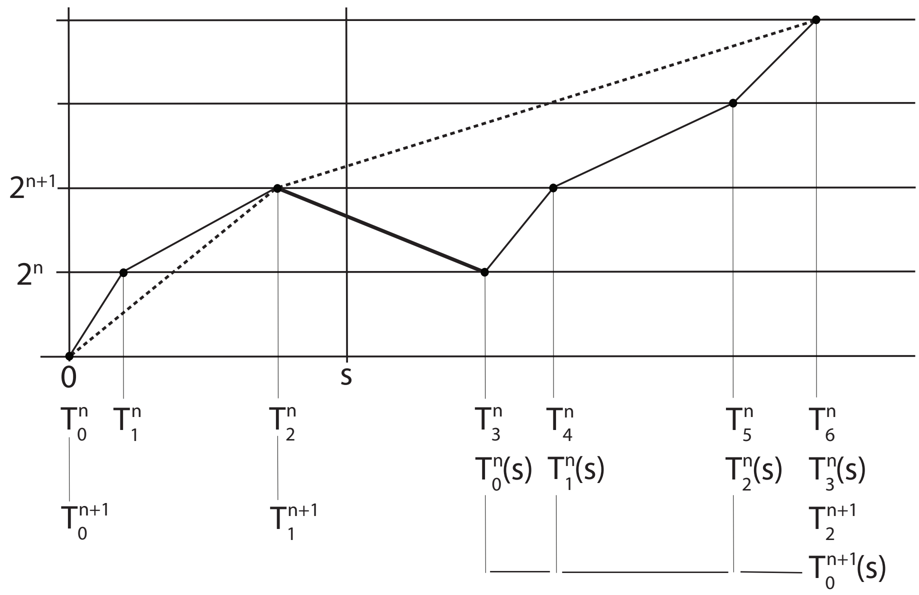


Figure 2: Notation from Lemma 2.6. The current level n crossing at time s is in bold. The first level n crossing time greater than s, denoted $T_0^n(s)$, corresponds to T_3^n, and the first level $(n+1)$ crossing time greater than s is $T_0^{n+1}(s) = T_6^n$. There are $Z^{n+1}(s) = 3$ level n crossings from $T_0^n(s)$ to $T_0^{n+1}(s)$, and $y = 1$ level n crossing from the current level $(n+1)$ crossing that has already happened at time s, including the current crossing in bold.

Using the lower bound in Lemma 2.4 for the left tail of W, it follows that for any $x > 0$

$$\begin{aligned}\mathbb{E}e^{-\theta\mu^k W} &\geq e^{-\theta x}\mathbb{P}(\mu^k W < x) \\ &\geq \exp\left(-\theta x - c_1(x\mu^{-k})^{-H/(1-H)}\right).\end{aligned}$$

For $x = (c_1/\theta)^{1-H}\mu^{kH}$ we get $\mathbb{E}e^{-\theta\mu^k W} \geq \exp\left(-c_2(\theta\mu^k)^H\right)$. Plugging this into (3) yields the bound

$$\mathbb{E}(e^{-\theta(T_0^n(s)-s)} \mid \mathcal{F}_s) \geq \exp\left(-c_3(\mu^n\theta)^H\right).$$

Applying Lemma 4.1 in [4] with $\theta = (c_4/x)^{1/(1-H)}2^{(2+k)/(1-H)}$, and readjusting the constants, gives the result. $\square$

Lemma 2.7. *Suppose that Assumptions 2.1 and 2.5 hold, then there exist constants $c_6, \ldots, c_9 > 0$ such that for all $\lambda > 0$ and any $s, t \geq 0$,*

$$\begin{aligned}c_6 \exp\left(-c_7(\lambda^{1/H}/t)^{H/(1-H)}\right) &\leq \mathbb{P}(|X(s+t) - X(s)| > \lambda \mid \mathcal{F}_s) \\ &\leq \mathbb{P}\left(\sup_{0\leq u\leq t}|X(s+u) - X(s)| > \lambda \mid \mathcal{F}_s\right) \leq c_8 \exp\left(-c_9(\lambda^{1/H}/t)^{H/(1-H)}\right).\end{aligned}$$

Proof. We start with the last inequality. Define $n \in \mathbb{Z}$ by $2^n \leq \lambda < 2^{n+1}$. If the maximum variation of X in the interval $[0, t]$ is at least λ, then necessarily there exists a k such that $s \leq T_{k-1}^{n-1} < T_k^{n-1} \leq s + t$. Thus, using Lemma 2.4,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq u\leq t}|X(s+u) - X(s)| > \lambda \mid \mathcal{F}_s\right) &\leq \mathbb{P}(D_k^{n-1} < t) \\ &= \mathbb{P}(W < \mu^{-(n-1)}t) \\ &\leq c_2 \exp\left(-c_3(t/\mu^{n-1})^{-H/(1-H)}\right) \qquad (4)\end{aligned}$$

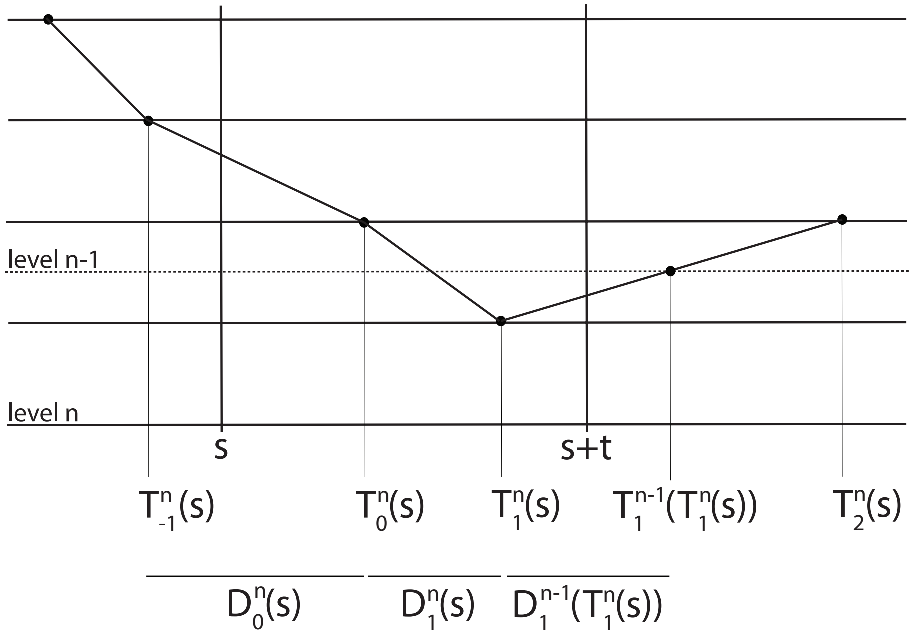


Figure 3: The points considered in case (a) of the proof of Lemma 2.7.

Re-expressing the last inequality in terms of λ and adjusting the constants yields the desired upper bound.

We now turn to the first inequality of the lemma. Our proof is based on Theorem 4.3 in [4], though more work is required because our process is not Markov. This time let $n \in \mathbb{Z}$ be such that $2^{n-2} \le \lambda < 2^{n-1}$. As before let $T^n_k(s)$ be the $k+1$-st level n crossing time of the process after time s, and also let $T^n_{-1}(s)$ be the first level n crossing time strictly before s.

Consider the possible level n movements of the process up to time $T^n_0(s)$. We take cases depending on the orientations of the two level n crossings leading up to $T^n_0(s)$.

$$\text{(a)} \;-- \quad \text{(b)} \;++ \quad \text{(c)} \;+- \quad \text{(d)} \;-+$$

Define $\pi = \mathbb{P}(Z > 2)$ (> 0 by Assumption 2.1).

In case (a) the next two level n crossings will have orientation $-+$ with probability $\pi/2$. Let $\alpha^n_{-1}(s)$, $\alpha^n_0(s)$, $\alpha^n_1(s)$, $\alpha^n_2(s)$, be respectively the orientations of the two crossings up to time $T^n_0(s)$ and the two crossings after time $T^n_0(s)$. Also let $T^{n-1}_1(T^n_1(s))$ be the next level $n-1$ crossing time after $T^n_1(s)$. Let $D^n_0(s) = T^n_0(s) - T^n_{-1}(s) \stackrel{\mathrm{d}}{=} \mu^n W$, $D^n_1(s) = T^n_1(s) - T^n_0(s) \stackrel{\mathrm{d}}{=} \mu^n W$ and $D^{n-1}_1(T^n_1(s)) = T^{n-1}_1(T^n_1(s)) - T^n_1(s) \stackrel{\mathrm{d}}{=} \mu^{n-1} W$. Note that they are independent and independent of the $\alpha_k^n(s)$. We have

$$\begin{aligned}
&\mathbb{P}(|X(s+t)-X(s)|>\lambda \mid \mathcal{F}_s, \alpha_{-1}^n(s)=-, \alpha_0^n(s)=-)\\
&\quad\geq \mathbb{P}(|X(s+t)-X(s)|>2^{n-1} \mid \mathcal{F}_s, \alpha_{-1}^n(s)=-, \alpha_0^n(s)=-)\\
&\quad\geq \pi/2\mathbb{P}(|X(s+t)-X(s)|>2^{n-1} \mid \mathcal{F}_s,\\
&\qquad\qquad \alpha_{-1}^n(s)=-, \alpha_0^n(s)=-, \alpha_1^n(s)=-, \alpha_2^n(s)=+)\\
&\quad\geq \pi/2\mathbb{P}(T_1^n(s)<s+t, T_1^{n-1}(T_1^n(s))>s+t \mid \mathcal{F}_s,\\
&\qquad\qquad \alpha_{-1}^n(s)=-, \alpha_0^n(s)=-, \alpha_1^n(s)=-, \alpha_2^n(s)=+)\\
&\quad\geq \pi/2\mathbb{P}(T_0^n(s)<s+t/2, D_1^n(s)<t/2, D_1^{n-1}(T_1^n(s))>t \mid \mathcal{F}_s,\\
&\qquad\qquad \alpha_{-1}^n(s)=-, \alpha_0^n(s)=-, \alpha_1^n(s)=-, \alpha_2^n(s)=+)\\
&\quad= \pi/2\mathbb{P}(T_0^n(s)<s+t/2 \mid \mathcal{F}_s, \alpha_{-1}^n(s)=-, \alpha_0^n(s)=-)\\
&\qquad\qquad \mathbb{P}(D_1^n(s)<t/2)\mathbb{P}(D_1^{n-1}(T_1^n(s))>t)
\end{aligned}$$

Thus from Lemmas 2.4 and 2.6 we have

$$\begin{aligned}
&\mathbb{P}(|X(s+t)-X(s)|>\lambda \mid \mathcal{F}_s, \alpha_{-1}^n(s)=-, \alpha_0^n(s)=-)\\
&\quad\geq c_1\exp\left(-c_2(\lambda^{1/H}/t)^{H/(1-H)}\right)\left(1-c_3\exp\left(-c_4(\lambda^{1/H}/t)^{H/(1-H)}\right)\right)
\end{aligned}$$

Choose K large enough such that the last term is at least $1/2$ when $\lambda^{1/H}/t\geq K$. Thus, since the LHS is decreasing in λ, we can find c_5 such that for all $t\in[0,1]$ and $\lambda\geq 0$, the LHS is bounded below by $c_5\exp\left(-c_2(\lambda^{1/H}/t)^{H/(1-H)}\right)$.

Cases (b) is analogous to case (a).

In case (c) we distinguish two further possibilities: (c1) the next two level n crossings form an excursion (either $-+$ or $+-$); and (c2) the next two level n crossings form a direct crossing (either $--$ or $++$). In case (c1) with probability 1/2 the excursion will be $-+$, in which case we can proceed as in case (a) to get a bound of the same form. In case (c2) if the direct crossing is $--$ then the approach of case (a) again suffices, however if the direct crossing is $++$ then we need to modify the argument a little. In this case we wish to bound $\mathbb{P}(|X(s+t)-X(s)|>\lambda \mid \mathcal{F}_s, \alpha_{-1}^n(s)=+, \alpha_0^n(s)=-, \alpha_1^n(s)=+, \alpha_2^n(s)=+)$. With probability $\pi/2$ the next pair of level n crossings are the excursion $+-$. Let $T_1^{n-1}(T_3^n(s))$ be the next level $n-1$ crossing time after $T_3^n(s)$, then we get

$$\begin{aligned}
&\mathbb{P}(|X(s+t)-X(s)|>\lambda \mid \mathcal{F}_s, \alpha_{-1}^n(s)=+, \alpha_0^n(s)=-, \alpha_1^n(s)=+, \alpha_2^n(s)=+)\\
&\quad\geq \pi/2\mathbb{P}(|X(s+t)-X(s)|>2^{n-1} \mid \mathcal{F}_s,\\
&\qquad\qquad \alpha_{-1}^n(s)=+, \alpha_{0,s}^n=-, \alpha_1^n(s)=+, \alpha_2^n(s)=+, \alpha_3^n(s)=+, \alpha_4^n(s)=-)\\
&\quad\geq \pi/2\mathbb{P}(T_3^n(s)<s+t, T_1^{n-1}(T_3^n(s))>s+t \mid \mathcal{F}_s,\\
&\qquad\qquad \alpha_{-1}^n(s)=+, \alpha_{0,s}^n=-, \alpha_1^n(s)=+, \alpha_2^n(s)=+, \alpha_3^n(s)=+, \alpha_4^n(s)=-)\\
&\quad\geq \pi/2\mathbb{P}(T_0^n(s)<s+t/4, D_1^n(s)<t/4, D_2^n(s)<t/4, D_3^n(s)<t/4, D_1^{n-1}(T_3^n(s))>t \mid \mathcal{F}_s,\\
&\qquad\qquad \alpha_{-1}^n(s)=+, \alpha_{0,s}^n=-, \alpha_1^n(s)=+, \alpha_2^n(s)=+, \alpha_3^n(s)=+, \alpha_4^n(s)=-)\\
&\quad= \pi/2\mathbb{P}(T_0^n(s)<s+t/2 \mid \mathcal{F}_s, \alpha_{-1}^n(s)=+, \alpha_0^n(s)=-)\\
&\qquad\qquad \mathbb{P}(D_1^n(s)<t/4)\mathbb{P}(D_2^n(s)<t/4)\mathbb{P}(D_3^n(s)<t/4)\mathbb{P}(D_1^{n-1}(T_3^n(s))>t).
\end{aligned}$$

This can be bounded below in the same way as in case (a).

Case (d) is analogous to case (c).

Finally, for general $t\geq 0$, let m be such that $\mu^{-m}t\leq 1$. Then, noting that $(2^{-m}\lambda)^{1/H}/(t\mu^{-m})=$

$\lambda^{1/H}/t$, by the discrete scaling of X,

$$\begin{aligned}\mathbb{P}(|X(s+t)-X(s)|>\lambda \mid \mathcal{F}_s) &= \mathbb{P}(|X(\mu^{-m}(s+t))-X(\mu^{-m}s)|>2^{-m}\lambda \mid \mathcal{F}_{\mu^{-m}s}) \\ &\geq c_5 \exp\left(-c_6((2^{-m}\lambda)^{1/H}/(\mu^{-m}t))^{H/(1-H)}\right) \\ &= c_5 \exp\left(-c_6(\lambda^{1/H}/t)^{H/(1-H)}\right),\end{aligned}$$

which concludes the proof of the lemma. □

We are now able to establish the modulus of continuity of the CEBP.

Theorem 2.8. *Suppose that Assumptions 2.1 and 2.5 hold. Let $h_H(\delta)=\delta^H|\log\delta|^{1-H}$, then there exist constants c_{10}, $c_{11}>0$ such that*

$$\begin{aligned}c_{10} &\leq \liminf_{\delta\to 0} \sup_{s,t\in[0,1],|t-s|<\delta} \frac{|X(t)-X(s)|}{h_H(t-s)} \\ &\leq \limsup_{\delta\to 0} \sup_{s,t\in[0,1],|t-s|<\delta} \frac{|X(t)-X(s)|}{h_H(t-s)} \leq c_{11}.\end{aligned}$$

Proof. Consider first the lower bound. Fix $c_1>0$, then for any $l>0$ and $m=0,1,\dots,2^l-1$, put

$$A_{m,l}=\left\{|X((m+1)2^{-l})-X(m2^{-l})|>c_1 l^{1-H}2^{-lH}\right\}.$$

By Lemma 2.7 we have $\mathbb{P}(A_{m,l}\mid\mathcal{F}_{m2^{-l}})\geq c_2e^{-c_3l}$, where $c_3\propto c_1^{1/(1-H)}$. By repeatedly conditioning we have

$$\begin{aligned}\mathbb{P}\left(\bigcap_{m=0}^{2^l-1}A^c_{m,l}\right) &= \prod_{m=0}^{2^l-1}\mathbb{P}(A^c_{m,l}\mid\mathcal{F}_{m2^{-l}}) = \prod_{m=0}^{2^l-1}(1-\mathbb{P}(A_{m,l}\mid\mathcal{F}_{m2^{-l}})) \\ &\leq \left(1-c_2e^{-c_3l}\right)^{2^l} = \left(1-\frac{c_2e^{-c_3l}2^l}{2^l}\right)^{2^l} \\ &\leq c_4\exp\left(-c_2e^{(\log 2-c_3)l}\right)\end{aligned}$$

We can choose c_1 so that $\log 2-c_3>0$, in which case the RHS above tends to 0 as $l\to\infty$, and we have

$$\mathbb{P}\left(|X(t+2^{-l})-X(t)|\leq c_5h_H(2^{-l}),\ \forall l>0,\ t\in[0,1-2^{-l}]\right)=0,$$

which establishes the lower bound.

For the upper bound we proceed in a similar manner, though we can no longer just consider points on the lattice $2^{-l}\mathbb{Z}$. For $l>0$ and $m=0,\dots,2^l-1$, let $I_{m,l}=[m2^{-l},(m+1)2^{-l})$, and define

$$\begin{aligned}\Phi_{m,l} &= \sup_{t\in I_{m,l}}|X(t)-X(m2^{-l})| \\ B_{m,l} &= \{\Phi_{m,l}>c_1l^{1-H}2^{-lH}\}\end{aligned}$$

From our estimate in Lemma 2.7 we have $\mathbb{P}(B_{m,l}\mid\mathcal{F}_{m2^{-l}})\leq c_2e^{-c_3l}$, where $c_3\propto c_1^{1/(1-H)}$. Thus, repeatedly conditioning on $\mathcal{F}_{m2^{-l}}$ for $m = 2^l - 1, \ldots, 0$, we have

$$\begin{aligned}
\mathbb{P}(B_{m,l} \text{ for some } 0 \leq m < 2^l) &= 1 - \mathbb{P}(B_{m,l}^c \text{ for all } 0 \leq m < 2^l) \\
&\leq 1 - (1 - c_2 e^{-c_3 l})^{2^l} \\
&= 1 - \left(1 - \frac{c_2 e^{-c_3 l} 2^l}{2^l}\right)^{2^l} \\
&\leq 1 - \exp\left(-c_4 e^{-(c_3 - \log 2) l}\right) \\
&\leq c_4 e^{-(c_3 - \log 2) l}.
\end{aligned}$$

Here we have chosen c_1 so that $c_3 - \log 2 > 0$.

Applying the Borel-Cantelli lemma, we see that there exists an L such that with probability 1

$$\Phi_{m,l} \leq c_1 l^{1-H} 2^{-lH} \text{ for all } l > L \text{ and } 0 \leq m < 2^l.$$

Now let $s \in I_{m,l}$ and suppose that t is such that $s < t$ and $|s-t| < 2^{-l}$. Then $t \in I_{m,l} \cup I_{m+1,l}$ and we have, with probability 1,

$$\begin{aligned}
|X(t) - X(s)| &\leq |X(t) - X((m+1)2^{-l})| + |X((m+1)2^{-l}) - X(m2^{-l})| \\
&\quad + |X(m2^{-l}) - X(s)| \\
&\leq 3c_1 l^{1-H} 2^{-lH}.
\end{aligned}$$

If we take $2^{-(l+1)} \leq \delta \leq 2^{-l}$, then we have, with probability 1,

$$\sup_{s,t \in [0,1], |s-t| < \delta} |X(s) - X(t)| \leq c_5 l^{1-H} 2^{-lH} \leq c_6 h_H(\delta),$$

as required. □

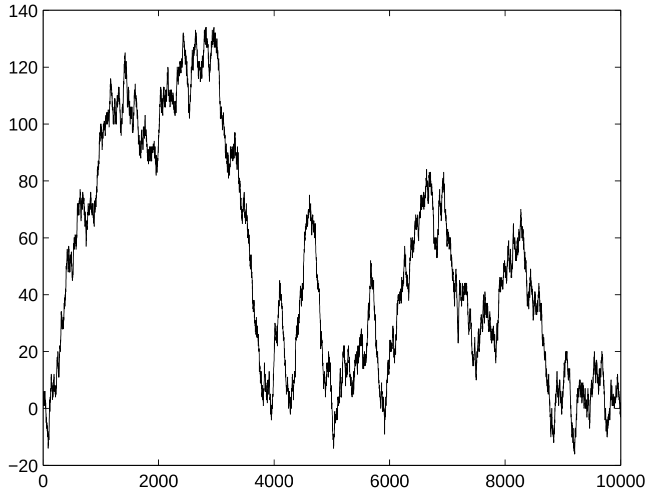
Remark 2.9. *In the special case where the CEBP reduces to a Brownian motion, the existence of the limit in Theorem 2.8 follows from Levy's modulus of continuity theorem.*

The argument above works for s and t in any bounded interval, not just $[0,1]$. This allows us to state the following corollary for all $t \in [0,\infty)$.

Corollary 2.10. *Suppose that Assumptions 2.1 and 2.5 hold, then the CEBP is a monofractal, in that $\mathbb{P}$-a.s. the Holder exponent $h_X(t) = H$ for all $t \in [0,\infty)$.*

3 The Multifractal Embedded Branching Process (MEBP) process

The Multifractal Embedded Branching process (MEBP processes) is constructed as the time change of a CEBP process. The time change is constructed from a cascade process defined on the embedded branching process. (This allows us to construct a Markov approximation to the process, which can be simulated online. See [8] for details.) The cascade process defines a measure, ν, on the boundary of the (doubly infinite) tree Υ. The measure ν is then mapped to a measure ζ on $\mathbb{R}_+$, with which we define a chronometer $\mathcal{M}$ (a non-decreasing process) by $\mathcal{M}(t) = \zeta([0,t])$. The MEBP process is then given by $Y = X \circ \mathcal{M}^{-1}$, where X is the CEBP. The crossing trees of X and Y have the same spatial structure, but have different crossing durations. In Figure 4 we plot a realisation of an MEBP process and its associated CEBP.



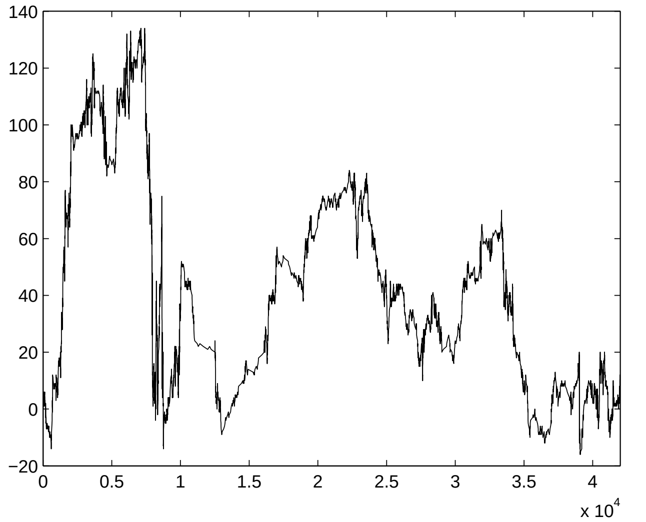


Figure 4: Top figure: CEBP process where the offspring consist of a geometric(0.6) number of excursions, each up-down or down-up with equal probability, followed by either an up-up or down-down direct crossing (compare this with Brownian motion, for which there are a geometric(0.5) number of excursions). Bottom figure: MEBP process obtained from a multifractal time change of the top CEBP process, with i.i.d. gamma distributed weights.

With each crossing C_k^m we associate a random weight R_k^m. We then define cumulative weights ρ_k^m as follows (using our tree indexing):

$$\rho_{n:\mathbf{i}} = \begin{cases} \prod_{k=1}^{|\mathbf{i}|} R_{n:\mathbf{i}|_k} / \prod_{k=0}^{n-1} R_{k:\emptyset} & n > 0 \\ \prod_{k=1}^{|\mathbf{i}|} R_{n:\mathbf{i}|_k} & n = 0 \\ \prod_{k=1}^{|\mathbf{i}|} R_{n:\mathbf{i}|_k} \prod_{k=1}^{|n|} R_{-k:\emptyset} & n < 0 \end{cases}$$

Here we have used the convention that $\prod_{k=1}^{0} x_k = 1$, to deal with the case $|\mathbf{i}| = 0$.

The cumulative weight at a crossing $C_{n:\mathbf{i}}$ is the product of the weights down the line of descent from the spine to the crossing, scaled by the product of the weights on the spine up to level n.

Assumption 3.1. *Assume that the $R_k^m \in (0, \infty)$ are i.i.d. Let R stand for a generic R_k^m. Define*

$$m(\theta) = \mathbb{E} \sum_{k=1}^{Z_\emptyset} (R_k)^\theta = \mu \mathbb{E} R^\theta.$$

We suppose that $m(\theta) < \infty$ in an open neighbourhood of 1, $m(1) = 1$, $m'(1) < 0$, and

$$\mathbb{E} \left(\sum_{k=1}^{Z_\emptyset} R_k \log \sum_{k=1}^{Z_\emptyset} R_k \right) < \infty.$$

As we are assuming independent weights, the condition $m(1) = 1$ implies $\mathbb{E}R = 1/\mu$. For each crossing $C_{n:\mathbf{i}}$ there is an associated branching random walk (BRW) with branching structure given by $\Upsilon_{n:\mathbf{i}}$, and the position of $n{:}\mathbf{j} \in \Upsilon_{n:\mathbf{i}}$ given by $-\log(\rho_{n:\mathbf{j}}/\rho_{n:\mathbf{i}})$. Let

$$\mathcal{W}_{n:\mathbf{i}}(k) = \sum_{\mathbf{j} \in \Upsilon_{n:\mathbf{i}},\, |\mathbf{j}| = |\mathbf{i}| + k} \frac{\rho_{n:\mathbf{j}}}{\rho_{n:\mathbf{i}}}.$$

Standard results on martingales in BRWs [7] give us the following.

Lemma 3.2. *Under Assumptions 2.1 and 3.1, $\mathcal{W}_{n:\mathbf{i}}(k)$ converges almost surely and in mean to some $\mathcal{W}_{n:\mathbf{i}}$. Moreover, the $\mathcal{W}_{n:\mathbf{i}}$ are identically distributed with mean 1. For $|\mathbf{i}| = k$ the $\mathcal{W}_{n:\mathbf{i}}$ are mutually independent, and $\mathcal{W}_{n:\mathbf{i}}$ is independent of $Z_{n:\mathbf{j}}$ and $R_{n:\mathbf{j}}$ for any $\mathbf{j}$ not a descendant of $\mathbf{i}$.*

For all nodes $n{:}\mathbf{i}$,

$$\mathcal{W}_{n:\mathbf{i}} = \sum_{j=1}^{Z_{n:\mathbf{i}}} R_{n:\mathbf{i}j} \mathcal{W}_{n:\mathbf{i}j} \tag{5}$$

We can now define the measure ν on $\partial\Upsilon$ by

$$\nu(\partial\Upsilon_{n:\mathbf{i}}) = \rho_{n:\mathbf{i}} \mathcal{W}_{n:\mathbf{i}}.$$

By Carathéodory's Extension Theorem, we can uniquely extend ν to the sigma algebra generated by these cylinder sets. The measure ζ is a mapping of ν from the doubly infinite tree to $\mathbb{R}_+$. Let T_k^n denote the k-th level n passage time of the CEBP process X, then we put

$$\zeta((T_{k-1}^n, T_k^n]) := \nu(\partial\Upsilon_k^n) = \rho_k^n \mathcal{W}_k^n.$$

Putting $\zeta(\{0\}) = 0$, this gives us $\zeta([0, T_k^n])$ for all $n \in \mathbb{Z}$ and $k \geq 0$.

For arbitrary $t \in \mathbb{R}_+$, let $m{:}\mathbf{i} \in \partial\Upsilon$ be such that $t \in (T_{m:\mathbf{i}|_n-}, T_{m:\mathbf{i}|_n}]$ for all $n \geq 0$. Noting that $\{T_{m:\mathbf{i}|_n}\}_n$ is a non-increasing sequence, we define $\zeta([0,t]) = \lim_{n\to\infty} \zeta([0, T_{m:\mathbf{i}|_n}])$. We define $\mathcal{M}(t) = \zeta([0,t])$, and define the MEBP process Y as

$$Y = X \circ \mathcal{M}^{-1}.$$

It is shown in [8] that $\mathcal{M}$ is continuous with no flat spots.

Put $\mathcal{T}_k^n = \mathcal{M}(T_k^n) = \sum_{j=1}^k \rho_j^n \mathcal{W}_j^n$, then $Y(\mathcal{T}_k^n) = X(T_k^n)$, so $\mathcal{T}_k^n$ is the k-th level n crossing time for Y, and $\mathcal{D}_k^n = \rho_k^n \mathcal{W}_k^n$ the k-th level n crossing duration. Note that if we take constant weights equal to $1/\mu$, then $\mathcal{T}_k^n = T_k^n$ and $Y = X$.

Mandelbrot, Fisher and Calvet [19] described a class of multifractal processes such that

$$Y(at) \stackrel{fdd}{=} M(a)Y(t) \text{ and } M(ab) \stackrel{d}{=} M_1(a)M_2(b),$$

where M_1 and M_2 are independent copies of M. Write A for M^{-1} then we can re-express the scaling rule for Y as

$$Y(A(a)t) \stackrel{fdd}{=} aY(t) \text{ and } A(ab) \stackrel{d}{=} A_1(a)A_2(b), \tag{6}$$

where A_1 and A_2 are independent copies of A. For our MEBP Y we have, for any $n \in \mathbb{Z}$,

$$Y(t) \stackrel{fdd}{=} 2^{-n} Y(t\rho_1^n).$$

This is of the same form as (6), with $A(2^n) = \rho_1^n = \prod_{k=1}^{-n\vee 0} R_{-k:\emptyset} / \prod_{k=0}^{n-1\vee 0} R_{k:\emptyset}$, however it only holds for $a, b \in 2^{\mathbb{Z}}$.

4 Hausdorff spectrum of ζ

Using the natural metric on $\partial\Upsilon$, the multifractal spectrum of ν is already known [14]. However when we map ν to ζ we lose the strong separation between the components of $\partial\Upsilon$, making life more difficult. In particular, the points T_k^n all correspond to two distinct points on $\partial\Upsilon$. One of these will have an index ending $111\ldots$ and the other will end $Z_{m:\mathbf{j}|_n} Z_{m:\mathbf{j}|_{n+1}} Z_{m:\mathbf{j}|_{n+2}} \ldots$, for some $m{:}\mathbf{j}$. These points are countably dense in $\mathbb{R}_+$, but we will show that none-the-less they do not effect the spectrum of ζ.

Hausdorff dimension is not suited to unbounded sets, so we restrict ζ to the finite interval $[0, T_1^0]$, corresponding to $\partial\Upsilon_{0:\emptyset}$. As ζ is self-similar, its Hausdorff spectrum will be the same when restricted to any finite interval. We will not be using the doubly infinite tree in this section and simply write $\Upsilon_\emptyset$ for $\Upsilon_{0:\emptyset}$ and $\mathbf{i}$ for $0{:}\mathbf{i}$.

Our approach for the lower bound is based on that of [6], where the multifractal spectrum of a random self-similar measure is obtained under fairly weak assumptions. For the upper bound we take the approach of Riedi [22].

4.1 Lower bound

Fix α and let F_α be the set of points in $[0, T_1^0]$ with ζ local dimension α: $F_\alpha = \{x \in [0, T_1^0] : d_\zeta(x) = \alpha\}$, where $d_\zeta(x)$ was defined in (1). We obtain a lower bound on the Hausdorff dimension of F_α using the approach of [1]. That is, if we can find a Borel measure λ such that $\lambda(F_\alpha) > 0$ and

$$\limsup_{r\to 0} \frac{\lambda(B(x,r))}{r^d} < \infty, \quad \forall x \in F_\alpha,$$

then $\dim_H(F_\alpha)$, the Hausdorff dimension of F_α, is $\geq d$.

For any given node $\mathbf{i} \in \Upsilon_\emptyset$ and $\theta \in \mathbb{R}$ we have a non-negative martingale

$$\mathcal{W}_{\mathbf{i}}^{(\theta)}(k) = m(\theta)^{-k} \sum_{\mathbf{j}\in\Upsilon_{\mathbf{i}},\, |\mathbf{j}|=|\mathbf{i}|+k} \left(\frac{\rho_{\mathbf{j}}}{\rho_{\mathbf{i}}}\right)^\theta$$

Let $\mathcal{W}_{\mathbf{i}}^{(\theta)}$ be the a.s. limit of this martingale (so $\mathcal{W}_{\mathbf{i}} = \mathcal{W}_{\mathbf{i}}^{(1)}$). We make the following assumption (a strengthening of Assumption 3.1).

Assumption 4.1. *Recall $m(\theta) = \mu \mathbb{E} R^\theta$. Let $I_m \subset (0,\infty)$ be the interior of the set of $\theta > 0$ for which $m(\theta) \in (0,\infty)$. We suppose that $1 \in I_m$, $m(1) = 1$, and that for all $\theta \in I_m$, $m'(\theta) = \mu \mathbb{E} R^\theta \log R$ exists and $\log m(\theta) > \theta m'(\theta)/m(\theta)$ (equivalently $\mathbb{E}R^\theta \log(\mathbb{E}R^\theta) + (\log\mu)/\mu > \mathbb{E}R^\theta \log R^\theta$). We also suppose that for all $\theta \in I_m$ we can find an $\epsilon > 0$ for which*

$$\mathbb{E}\left(\sum_{k=1}^{Z_\emptyset} R_k^\theta\right)^{1+\epsilon} < \infty. \tag{7}$$

Lemma 4.2. *Under Assumptions 2.1 and 4.1, for all $\theta \in I_m$, $\mathcal{W}_{\mathbf{i}}^{(\theta)}(k)$ converges almost surely and in mean to some $\mathcal{W}_{\mathbf{i}}^{(\theta)}$. Moreover, the $\mathcal{W}_{\mathbf{i}}^{(\theta)}$ are identically distributed with mean 1, for any $\theta \in I_m$ there is an $\epsilon > 0$ such that $\mathbb{E}{\mathcal{W}_{\mathbf{i}}^{(\theta)}}^{1+\epsilon} < \infty$, and $\mathbb{E}\mathcal{W}_{\mathbf{i}}^{(\theta)} |\log W_{\mathbf{i}}| < \infty$.*

For $|\mathbf{i}| = k$ the $\mathcal{W}_{\mathbf{i}}^{(\theta)}$ are mutually independent, and $\mathcal{W}_{\mathbf{i}}^{(\theta)}$ is independent of $Z_{\mathbf{j}}$ and $R_{\mathbf{j}}$ for any $\mathbf{j}$ not a descendant of $\mathbf{i}$. For all nodes $\mathbf{i}$,

$$\mathcal{W}_{\mathbf{i}}^{(\theta)} = m(\theta)^{-1} \sum_{j=1}^{Z_{\mathbf{i}}} R_{\mathbf{i}j}^\theta \mathcal{W}_{\mathbf{i}j}^{(\theta)} \tag{8}$$

Proof. The mean convergence follows from Theorem 7.1 of [7], and the limit necessarily has mean 1.

That $\mathbb{E}{\mathcal{W}_{\mathbf{i}}^{(\theta)}}^{1+\epsilon} < \infty$ follows from [10] Proposition 4 (see [6] Lemma 9.1), upon noting that the condition $\theta m'(\theta) < m(\theta)\log m(\theta)$ implies that for some $\epsilon > 0$ we have $m(\theta(1+\epsilon)) < m(\theta)^{1+\epsilon}$.

That $\mathbb{E}\mathcal{W}_{\mathbf{i}}^{(\theta)} |\log W_{\mathbf{i}}| < \infty$ follows from [6] Lemma 9.6 (which is just an application of Hölder's inequality), provided $\mathbb{E}W^{-\epsilon} < \infty$ for all $\epsilon > 0$ sufficiently small. However, from [5] Theorem 3, the left tail of W decays exponentially, so all negative moments are finite.

The remainder of the lemma follows from the branching structure of the $\mathcal{W}_{\mathbf{i}}^{(\theta)}(k)$. □

We will suppose that Assumptions 2.1 and 4.1 hold for the remainder of this subsection. Generalising our definition of ν, for each $\theta \in I_m$ we define a measure $\nu^{(\theta)}$ on $\partial\Upsilon_\emptyset$ by

$$\nu^{(\theta)}(\partial\Upsilon_{\mathbf{i}}) = \rho_{\mathbf{i}}^\theta m(\theta)^{-|\mathbf{i}|} \mathcal{W}_{\mathbf{i}}^{(\theta)}.$$

We then project this from $\partial\Upsilon_\emptyset$ to $[0, T_1^0]$ to get the measure $\zeta^{(\theta)}$

$$\zeta^{(\theta)}((T_{k-1}^n, T_k^n]) := \nu^{(\theta)}(\partial\Upsilon_k^n) = \rho_k^n m(\theta)^n \mathcal{W}_k^{(\theta),n}.$$

For a suitable choice of θ, $\zeta^{(\theta)}$ is the measure we use to bound $\dim(F_\alpha)$. To show it has the correct properties we need a number of lemmas.

Lemma 4.3. *Let*

$$\begin{aligned} G_1 &= \left\{ \mathbf{i} \in \partial\Upsilon_\emptyset : \lim_{k\to\infty} \frac{\log W_{\mathbf{i}|_k}}{k} = 0 \right\} \\ G_2 &= \left\{ \mathbf{i} \in \partial\Upsilon_\emptyset : \lim_{k\to\infty} \frac{\log \mathcal{W}_{\mathbf{i}|_k}^{(\theta)}}{k} = 0 \right\} \end{aligned}$$

Then for all $\theta \in I_m$ we have

$$\nu^\theta(G_1 \cap G_2) = \nu^\theta(\partial\Upsilon_\emptyset),\ \mathbb{P}\text{-}a.s.$$

Proof. We consider G_1 first. Let $E_{\epsilon,k} = \{\mathbf{i} \in \partial\Upsilon_\emptyset : |\log W_{\mathbf{i}|_k}| > k\epsilon\}$. For any l we have

$$A_\epsilon = \left\{\mathbf{i} \in \partial\Upsilon_\emptyset : \limsup_{k\to\infty} \frac{|\log W_{\mathbf{i}|_k}|}{k} > \epsilon\right\} \subset \bigcup_{k\geq l} E_{\epsilon,k}.$$

Hence

$$\nu^{(\theta)}(A_\epsilon) \leq \lim_{l\to\infty} \sum_{k\geq l} \nu^{(\theta)}(E_{\epsilon,k}).$$

Now

$$\begin{aligned}
\mathbb{E}\nu^{(\theta)}(E_{\epsilon,k}) &= \mathbb{E} \sum_{\mathbf{i}\in\Upsilon_\emptyset,\, |\mathbf{i}|=k} \rho_{\mathbf{i}}^\theta m(\theta)^{-k} \mathcal{W}_{\mathbf{i}}^{(\theta)} I_{\{|\log W_{\mathbf{i}}|>k\epsilon\}} \\
&= \mathbb{E} \sum_{\mathbf{i}\in\Upsilon_\emptyset,\, |\mathbf{i}|=k} \rho_{\mathbf{i}}^\theta m(\theta)^{-k} \mathbb{E}\mathcal{W}_{\mathbf{i}}^{(\theta)} I_{\{|\log W_{\mathbf{i}}|>k\epsilon\}} \\
&= \mathbb{E}\mathcal{W}_{\mathbf{i}}^{(\theta)} I_{\{|\log W_{\mathbf{i}}|>k\epsilon\}}
\end{aligned}$$

so

$$\begin{aligned}
\mathbb{E}\sum_k \nu^{(\theta)}(E_{\epsilon,k}) &= \mathbb{E}\mathcal{W}_{\mathbf{i}}^{(\theta)} \sum_k I_{\{\epsilon^{-1}|\log W_{\mathbf{i}}|>k\}} \\
&\leq \mathbb{E}\mathcal{W}_{\mathbf{i}}^{(\theta)}(1+\epsilon^{-1}|\log W_{\mathbf{i}}|) \\
&< \infty \text{ (from Lemma 4.2).}
\end{aligned}$$

Thus $\mathbb{E}\nu^{(\theta)}(A_\epsilon) = 0$ and hence $\nu^{(\theta)}(\partial\Upsilon_\emptyset \backslash G_1) = 0$, $\mathbb{P}$-a.s.

For G_2, redefine $E_{\epsilon,k} = \left\{\mathbf{i} \in \partial\Upsilon_\emptyset : |\log \mathcal{W}_{\mathbf{i}|_k}^{(\theta)}| > k\epsilon\right\}$ and $A_\epsilon = \left\{\mathbf{i} \in \partial\Upsilon_\emptyset : \limsup_{k\to\infty} |\log \mathcal{W}_{\mathbf{i}|_k}^{(\theta)}|/k > \epsilon\right\}$, then proceed in the same manner as above to show that

$$\mathbb{E}\sum_k \nu^{(\theta)}(E_{\epsilon,k}) \leq \mathbb{E}\mathcal{W}_{\mathbf{i}}^{(\theta)}(1+\epsilon^{-1}|\log \mathcal{W}_{\mathbf{i}}^{(\theta)}|).$$

In this case the RHS is finite when $\mathbb{E}\mathcal{W}_{\mathbf{i}}^{(\theta)} \log_+ \mathcal{W}_{\mathbf{i}}^{(\theta)} < \infty$, which certainly holds when $\mathcal{W}_{\mathbf{i}}^{(\theta)}$ has a finite $1+\epsilon$ moment. □

Lemma 4.4. *Let*

$$G_3 = \left\{\mathbf{i} \in \partial\Upsilon : \lim_{k\to\infty} \frac{\log \rho_{\mathbf{i}|_k}}{k} = -\alpha \log \mu\right\},$$

then, for θ such that

$$\alpha = -\frac{\mathbb{E}R^\theta \log R}{(\log\mu)\mathbb{E}R^\theta}$$

we have $\nu^{(\theta)}(G_3) = \nu^{(\theta)}(\partial\Upsilon_\emptyset)$, $\mathbb{P}$-a.s.

Proof. The approach we use can be found in a number of places, see for example [6] Lemma 8.6, so we will only sketch the argument here.

For θ as above, let $\bar{\nu}^{(\theta)}$ be the measure on $\mathbb{N}^{\mathbb{N}}$ given by

$$\bar{\nu}^{(\theta)}(A) = \mathbb{E}\nu^{(\theta)}(A \cap \partial\Upsilon_\emptyset)$$

and let $\bar{\mathbb{E}}^{(\theta)}$ be the corresponding expectation operator. Define the random variable X_k on $\mathbb{N}^{\mathbb{N}}$ by

$$X_k(\mathbf{i}) = \log R_{\mathbf{i}|_k}$$

then it follows from the branching structure of $\nu^{(\theta)}$ and the independence of the $R_{\mathbf{i}|_k}$, that the $\{X_k\}$ are i.i.d. under $\bar{\nu}^{(\theta)}$. Thus by the SLLN, $\bar{\nu}^{(\theta)}$-a.s. we have

$$\begin{aligned} \frac{\sum_{j=1}^{k} X_j}{k} &= \frac{\log \rho_{\mathbf{i}|k}}{k} \\ &\to \bar{\mathbb{E}}^{(\theta)} X_1 \\ &= \frac{\mathbb{E}R^\theta \log R}{\mathbb{E}R^\theta} \end{aligned}$$

It follows immediately that $\mathbb{E}\nu^{(\theta)}(\partial\Upsilon_\emptyset \backslash G_3) = 0$, as required. □

The map from $\partial\Upsilon$ to $[0, T_1^0]$ that sends ν to ζ is just $\mathbf{i} \mapsto T_\mathbf{i} = \lim_{n\to\infty} T_{\mathbf{i}|_n}$, which we will denote T. As noted earlier, T is not one-to-one. For $\mathbf{i} \in \Upsilon_\emptyset$ we will write $\mathbf{i}(n,1)$ for the index $\mathbf{i}$ followed by n ones, and $\mathbf{i}(n,Z)$ for the index $\mathbf{i}$ followed by $Z_\mathbf{i}$, $Z_{\mathbf{i}(1,Z)}$, $Z_{\mathbf{i}(2,Z)}, \ldots, Z_{\mathbf{i}(n-1,Z)}$.

For $\mathbf{i} \in \Upsilon_\emptyset$ and any n we have $T_\mathbf{i} = T_{\mathbf{i}(n,Z)} = T_{\mathbf{i}(\infty,Z)}$. Recall that $\mathbf{i}-$ is the crossing immediately before $\mathbf{i}$, thus $T_{\mathbf{i}-(\infty,Z)} = T_{\mathbf{i}(\infty,1)}$. Write $J_\mathbf{i}$ for the interval $(T_{\mathbf{i}-}, T_\mathbf{i}]$. Suppose that $|\mathbf{i}| = m$, so that $J_\mathbf{i}$ is at level $-m$, then for any $n > m$, the first level $-n$ interval contained in $J_\mathbf{i}$ is $J_{\mathbf{i}(n-m,1)}$ (with right hand limit $T_{\mathbf{i}(n-m,1)}$), and the last level $-n$ interval contained in $J_\mathbf{i}$ is $J_{\mathbf{i}(n-m,Z)}$ (with left hand limit $T_{\mathbf{i}(n-m,Z)-}$). Let $B(x,r)$ denote an open ball centred in x, with radius r.

Lemma 4.5. *For $\epsilon > 0$ define*

$$\begin{aligned} G_4^\epsilon &= \{\mathbf{i} \in \partial\Upsilon_\emptyset \,:\, B(T_\mathbf{i}, \mu^{-\lfloor n(1+\epsilon)\rfloor}) \subset J_{\mathbf{i}|_n} \text{ eventually}\} \\ G_4 &= \lim_{\epsilon\downarrow 0} G_4^\epsilon \end{aligned}$$

then $\nu^{(\theta)}(G_4) = \nu^{(\theta)}(\partial\Upsilon_\emptyset)$ $\mathbb{P}$-a.s.

Throughout the next two proofs it will be understood that $n(1+\epsilon)$ and $n\epsilon$ are rounded down to the nearest integer.

Proof. Define $A_n = \{\mathbf{i} \in \partial\Upsilon_\emptyset \,:\, \mathbf{i}|_{n(1+\epsilon)} \neq \mathbf{i}|_n(n\epsilon, 1) \text{ nor } \mathbf{i}|_n(n\epsilon, Z)\}$. That is, A_n is constructed by taking all the level $-n$ intervals, then from each one removing the first and last level $-n(1+\epsilon)$ intervals. Let $A = \lim A_n$ then we claim that $A \subset G_4^\epsilon$ for all $\epsilon > 0$. The proof depends on the following result, which is just an application of Liu [15] Theorem 2.1 to the problem in hand: for any $\epsilon > 0$ we have $\mathbb{P}$-a.s. that

$$\min_{\mathbf{i}\in\partial\Upsilon_\emptyset} \mu^{-n} W_{\mathbf{i}|_n} \geq \mu^{-n(1+\epsilon)} \text{ eventually.}$$

From this we have immediately that, for n large enough,

$$\begin{aligned} B(T_{\mathbf{i}|_n(n\epsilon,1)}, \mu^{-n(1+\epsilon)}) &\subset J_{\mathbf{i}|_n} \text{ and} \\ B(T_{\mathbf{i}|_n(n\epsilon,Z)-}, \mu^{-n(1+\epsilon)}) &\subset J_{\mathbf{i}|_n} \end{aligned}$$

from which we see that for all $\mathbf{i} \in A_n$, $B(T_\mathbf{i}, \mu^{-n(1+\epsilon)}) \subset J_{\mathbf{i}|_n}$, so $A_n \subset G_4^\epsilon$ eventually, as required.

To finish we have

$$\begin{aligned} &\mathbb{E}\nu^{(\theta)}(\partial\Upsilon_\emptyset \backslash A_n) \\ &\quad= \mathbb{E} \sum_{\mathbf{i}\in\Upsilon_\emptyset,\, |\mathbf{i}|=n} \left(\nu^{(\theta)}(\partial\Upsilon_{\mathbf{i}(n\epsilon,1)}) + \nu^{(\theta)}(\partial\Upsilon_{\mathbf{i}(n\epsilon,Z)})\right) \\ &\quad= \mathbb{E} \sum_{\mathbf{i}\in\Upsilon_\emptyset,\, |\mathbf{i}|=n} \left(\rho^\theta_{\mathbf{i}(n\epsilon,1)} m(\theta)^{-n(1+\epsilon)} \mathcal{W}^{(\theta)}_{\mathbf{i}(n\epsilon,1)} + \rho^\theta_{\mathbf{i}(n\epsilon,Z)} m(\theta)^{-n(1+\epsilon)} \mathcal{W}^{(\theta)}_{\mathbf{i}(n\epsilon,Z)}\right) \\ &\quad= 2\mu^{-n\epsilon} \\ &\quad\to 0 \text{ as } n \to \infty. \end{aligned}$$

$\square$

Theorem 4.6. *Suppose that Assumptions 2.1 and 4.1 hold, then for any α such that $\alpha = -\mathbb{E}R^\theta \log R/(\mathbb{E}R^\theta \log \mu)$ for some $\theta \in I_m$, we have*

$$\dim(F_\alpha) \geq \alpha\theta + 1 + \frac{\log \mathbb{E}R^\theta}{\log \mu}.$$

Proof. We show that $T(G) := T(G_1 \cap G_2 \cap G_3 \cap G_4) \subset F_\alpha$, that $\zeta^{(\theta)}(T(G)) = \nu^{(\theta)}(G) > 0$, and that for some d and all $\mathbf{i} \in G$

$$\limsup_{r \to 0} \frac{\zeta^{(\theta)}(B(T_\mathbf{i}, r))}{r^d} < \infty.$$

It then follows from [9] Propostion 4.9 that $\dim(F_\alpha) \geq \dim(T(G)) \geq d$.

That $\nu^{(\theta)}(G) > 0$ is immediate from Lemmas 4.3 to 4.5.

We next show that $T(G) \subset F_\alpha$. For any $\mathbf{i} \in G$ we have for any $\epsilon > 0$, $B(T_\mathbf{i}, \mu^{-n(1+\epsilon)}) \subset J_{\mathbf{i}|_n}$ eventually, whence for $\mu^{-(n+1)(1+\epsilon)} \leq r < \mu^{-n(1+\epsilon)}$,

$$\frac{\log \zeta(B(T_\mathbf{i}, r))}{\log r} \geq \frac{\log(\rho_{\mathbf{i}|_n} W_{\mathbf{i}|_n})}{-n(1+\epsilon)\log \mu},$$

and hence, as $\mathbf{i} \in G_1 \cap G_3$,

$$\lim_{r \to 0} \frac{\log \zeta(B(T_\mathbf{i}, r))}{\log r} \geq \lim_{n \to \infty} \frac{\log \rho_{\mathbf{i}|_n} + \log W_{\mathbf{i}|_n}}{-n(1+\epsilon)\log \mu} = \frac{\alpha}{1+\epsilon}. \tag{9}$$

As $\mathbf{i} \in G_4$ this is true for all $\epsilon > 0$ and we can replace the RHS by α.

Now, given r let l be such that $\mu^{-l} W_{\mathbf{i}|l} \leq r < \mu^{-(l-1)} W_{\mathbf{i}|_{l-1}}$. Then $J_{\mathbf{i}|l} \subset B(T_\mathbf{i}, r)$ and thus $\zeta(J_{\mathbf{i}|l}) \leq \zeta(B(T_\mathbf{i}, r))$. As $r \to 0$ we have $\log r/l \to -\log \mu$ and $\log \rho_{\mathbf{i}|_l}/l \to -\alpha \log \mu$, and so

$$\lim_{r \to 0} \frac{\log \zeta(B(T_\mathbf{i}, r))}{\log r} \leq \lim_{r \to 0} \frac{\log(\rho_{\mathbf{i}|_l} W_{\mathbf{i}|_l})}{\log r} = \alpha. \tag{10}$$

Combining (9) and (10) we have that $T(G) \subset F_\alpha$.

To finish, note again that for any $\mathbf{i} \in G$ we have for any $\epsilon > 0$, $B(T_\mathbf{i}, \mu^{-n(1+\epsilon)}) \subset J_{\mathbf{i}|_n}$ eventually, whence for $\mu^{-(n+1)(1+\epsilon)} \leq r < \mu^{-n(1+\epsilon)}$

$$\begin{aligned}
\frac{\zeta^{(\theta)}(B(T_\mathbf{i}, r))}{r^d} &\leq \frac{\rho^\theta_{\mathbf{i}|_n} m(\theta)^{-n} \mathcal{W}^{(\theta)}_{\mathbf{i}|_n}}{r^d} \\
&= \exp\left(\left(\frac{\theta \log \rho_{\mathbf{i}|_n} - n \log m(\theta) + \log \mathcal{W}^{(\theta)}_{\mathbf{i}|_n}}{\log r} - d\right) \log r\right)
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\frac{\theta \log \rho_{\mathbf{i}|_n} - n \log m(\theta) + \log \mathcal{W}^{(\theta)}_{\mathbf{i}|_n}}{\log r} &= \frac{\theta \log \rho_{\mathbf{i}|_n} - n \log m(\theta) + \log \mathcal{W}^{(\theta)}_{\mathbf{i}|_n}}{-n(1+\epsilon)\log \mu} \\
&\to \frac{\alpha\theta}{1+\epsilon} + \frac{\log m(\theta)}{(1+\epsilon)\log \mu} + 0 \\
&= \frac{\alpha\theta + 1 + (\log \mathbb{E}R^\theta)/\log \mu}{1+\epsilon}
\end{aligned}$$

So (11) is finite for all $d < \alpha\theta + 1 + \frac{\log \mathbb{E}R^\theta}{\log \mu}$, as required. $\square$

Note that if we put

$$\gamma(\theta) = -1 - \frac{\log \mathbb{E}R^\theta}{\log \mu} \tag{12}$$

then the minimum of $\alpha\theta - \gamma(\theta)$ occurs at $\alpha = -\mathbb{E}R^\theta \log R/(\mathbb{E}R^\theta \log \mu)$, so our lower bound $\alpha\theta + 1 + \frac{\log \mathbb{E}R^\theta}{\log \mu}$ can be expressed as the Legendre transform

$$\inf_\theta \alpha\theta - \gamma(\theta).$$

It is easy to check that for any t such that the local ζ-dimension exists and equals α, we have that the local Hölder exponent $h_\mathcal{M}(t)$ of $\mathcal{M}(t) = \zeta([0,t])$ is α. Thus if $G_\alpha = \{t \in [0, T_1^0] : h_\mathcal{M}(t) = \alpha\}$ then

$$\dim(G_\alpha) \geq \inf_\theta \alpha\theta - \gamma(\theta).$$

Note that the local dimension d_ζ of ζ is not-necessarily defined everywhere, while the Hölder exponent of $\mathcal{M}$ is. However, as we are after a lower bound on the multifractal spectrum of $h_\mathcal{M}$, this is not a problem.

4.2 Upper bound

To get an upper bound on the Hausdorff spectrum, we develop a multifractal formalism similar to [22]. However, instead of studying $\mathcal{M}$ on a regular grid, we study it on a grid adapted to its underlying random tree structure, which requires a change to the partition function we use.

Note that since $\mathcal{M}$ has non-decreasing continuous sample paths, we can write its local Hölder exponent as

$$\begin{aligned} h_\mathcal{M}(t) &= \liminf_{\epsilon\to 0} \frac{\log(\mathcal{M}(t+\epsilon) - \mathcal{M}(t-\epsilon))}{\log 2\epsilon} \\ &= \liminf_{\epsilon\to 0} \frac{\log \zeta([t-\epsilon, t+\epsilon])}{\log 2\epsilon}. \end{aligned}$$

Let $t = T_\mathbf{i} \in [0, T_1^0]$, for $\mathbf{i} \in \partial\Upsilon_\emptyset$. Our basic idea when bounding $h_\mathcal{M}(T_\mathbf{i})$ is to approximate the interval $[T_\mathbf{i} - \epsilon, T_\mathbf{i} + \epsilon]$ by the interval $J_{\mathbf{i}|_n}$, for a suitable n. Unfortunately this proves difficult when $T_\mathbf{i}$ is a crossing time, since $T_\mathbf{i}$ will then be one of the end points of $J_{\mathbf{i}|_n}$, for all n large enough. We deal with this problem by using intervals of the form $\bar{J}_{\mathbf{i}|_n} = J_{\mathbf{i}|_n-} \cup J_{\mathbf{i}|_n} \cup J_{\mathbf{i}|_n+}$, where $\mathbf{i}-$ and $\mathbf{i}+$ denote respectively the left and right neighbours of $\mathbf{i}$.

For $\mathbf{i} \in \Upsilon_\emptyset$, let $\bar{D}_\mathbf{i} = |\bar{J}_\mathbf{i}|$ be the length of $\bar{J}_\mathbf{i}$. Recall that $\mathcal{D}_\mathbf{i} = \rho_\mathbf{i}\mathcal{W}_\mathbf{i} = \zeta(J_\mathbf{i})$ is the duration of crossing $C_\mathbf{i}$, and define $\bar{\mathcal{D}}_\mathbf{i} = \zeta(\bar{J}_\mathbf{i}) = \mathcal{D}_{\mathbf{i}-} + \mathcal{D}_\mathbf{i} + \mathcal{D}_{\mathbf{i}+}$. Clearly if $\mathbf{i} = 11\cdots 1$ then we can omit $\mathcal{D}_{\mathbf{i}-}$.

We use the $\bar{\mathcal{D}}_\mathbf{i}$ to define discretised versions of $h_\mathcal{M}$. For $\mathbf{i} \in \partial\Upsilon_\emptyset$, recall that $\mathbb{E}D_{\mathbf{i}|_n} = \mu^{-n}$ (under Assumption 2.1), then put

$$h^n_\mathcal{M}(T_\mathbf{i}) = \frac{\log \bar{\mathcal{D}}_{\mathbf{i}|_n}}{-n\log\mu} = \frac{\log \zeta(\bar{J}_{\mathbf{i}|_n})}{-n\log\mu}.$$

Including $\mathcal{D}_{\mathbf{i}-}$ and $\mathcal{D}_{\mathbf{i}+}$ in the definition of $\bar{\mathcal{D}}_\mathbf{i}$ helps us avoid boundary problems which can arise when $\mathbf{i} = \mathbf{j}(\infty, 1)$ or $\mathbf{j}(\infty, Z)$, for some $\mathbf{j} \in \Upsilon_\emptyset$ (that is, when $T_\mathbf{i}$ is a hitting time).

The following assumption, which is a strengthening of Assumption 2.1, will be used throughout this section. We will also need Assumption 3.1 to guarantee the existence of $\mathcal{M}$.

Assumption 4.7. *Suppose that $Z \in 2\mathbb{N}$, Z is non-trivial, and $\mathbb{E}Z^p < \infty$ for all $p > 1$.*

Lemma 4.8. *Suppose that Assumption 4.7 holds. Then, with probability 1,*

$$\log \max_{\mathbf{i}\in\Upsilon_n} W_\mathbf{i} = o(n) \text{ and } \log \min_{\mathbf{i}\in\Upsilon_n} W_\mathbf{i} = o(n).$$

Proof. This follows directly from Liu [15] Theorems 2.1 and 3.1. □

Lemma 4.9. *Suppose Assumptions 3.1 and 4.7 hold. Then, with probability 1, for all* $\mathbf{i} \in \partial\Upsilon_\emptyset$ *(equivalently for all* $t \in [0, T_1^0]$*),*

$$\liminf_{n\to\infty} h^n_{\mathcal{M}}(T_\mathbf{i}) = h_{\mathcal{M}}(T_\mathbf{i}).$$

Proof. For any $\epsilon > 0$ and $T_\mathbf{i}$ we can always find a positive $n = n(\epsilon, T_\mathbf{i})$ such that

$$\bar{\mathcal{D}}_{\mathbf{i}|_{n+1}} \leq \mathcal{M}(T_\mathbf{i}+\epsilon) - \mathcal{M}(T_\mathbf{i}-\epsilon) < \bar{\mathcal{D}}_{\mathbf{i}|_n}. \tag{13}$$

Given n and $t = T_\mathbf{i}$, let $I_n(t) = \{\epsilon \mid n(\epsilon, t) = n\}$. Now for any $\epsilon \in I_n(t)$ we see that if ϵ is larger than $D_{\mathbf{i}|_n} + D_{\mathbf{i}|_n-}$ and $D_{\mathbf{i}|_n} + D_{\mathbf{i}|_n+}$, then $\mathcal{M}(T_\mathbf{i}+\epsilon) - \mathcal{M}(T_\mathbf{i}-\epsilon) \geq \bar{\mathcal{D}}_{\mathbf{i}|_n}$, which contradicts (13). Thus $\epsilon < \max(D_{\mathbf{i}|_n} + D_{\mathbf{i}|_n-}, D_{\mathbf{i}|_n} + D_{\mathbf{i}|_n+})$. Similarly, if ϵ is strictly smaller than $D_{\mathbf{i}|_{n+1}-}$ and $D_{\mathbf{i}|_{n+1}+}$, then $\mathcal{M}(T_\mathbf{i}+\epsilon) - \mathcal{M}(T_\mathbf{i}-\epsilon) < \bar{\mathcal{D}}_{\mathbf{i}|_{n+1}}$, which also contradicts (13). Thus necessarily $\epsilon \geq \min(D_{\mathbf{i}|_{n+1}-}, D_{\mathbf{i}|_{n+1}+})$. Thus from Lemma 4.8 we have that for $\epsilon \in I_n(t)$,

$$-(n+1)\log\mu + \log 2 + o(n) \leq \log 2\epsilon < -n\log\mu + \log 4 + o(n). \tag{14}$$

Dividing each member of the double inequality (13) by $\log 2\epsilon < 0$ and using the bounds (14), we obtain

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log \bar{\mathcal{D}}_{\mathbf{i}|_n}}{-(n+1)\log\mu + o(n)} &\leq \liminf_{\epsilon\to 0} \frac{\log(\mathcal{M}(T_\mathbf{i}+\epsilon) - \mathcal{M}(T_\mathbf{i}-\epsilon))}{\log 2\epsilon} \\ &\leq \liminf_{n\to\infty} \frac{\log \bar{\mathcal{D}}_{\mathbf{i}|_{n+1}}}{-n\log\mu + o(n)},\end{aligned}$$

and the result follows. □

We use $h^n_{\mathcal{M}}$ to define the coarse spectrum of $\mathcal{M}$. Let

$$S^{(n)}(\alpha,\epsilon) = \{\mathbf{i} \in \Upsilon_\emptyset(-n) \mid \alpha - \epsilon \leq h^n_{\mathcal{M}}(T_\mathbf{i}) \leq \alpha + \epsilon\},$$

then define

$$f(\alpha) := \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{\log |S^{(n)}(\alpha,\epsilon)|}{n\log\mu}. \tag{15}$$

This differs from the usual definition of the coarse spectrum in that an irregular grid is used to partition the real line rather than a regular grid.

Lemma 4.10. *Suppose Assumptions 3.1 and 4.7 hold. Then, with probability 1, for all* α

$$\dim(G_\alpha) \leq f(\alpha),$$

where $G_\alpha = \{t \in [0, T_1^0] : h_{\mathcal{M}}(t) = \alpha\}$*.*

Proof. Let $\alpha \in \mathbb{R}$ and $g > f(\alpha)$ be arbitrary. We find a covering for G_α and show that its g-Hausdorff measure is zero for all $g > f(\alpha)$. The result then follows by sending g to $f(\alpha)$.

Consider $t = T_\mathbf{i} \in [0, T_1^0]$. If $t \in G_\alpha$, then for any $0 < \epsilon$, there exists m such that for all $n \geq m$, $\alpha - \epsilon \leq h^n_{\mathcal{M}}(T_\mathbf{i}) < \alpha + \epsilon$. Since $t \in J_{\mathbf{i}|_n}$ for all n, for any m,

$$\bigcup_{n\geq m} \bigcup_{\mathbf{j}\in S^{(n)}(\alpha,\epsilon)} J_\mathbf{j} \quad \text{is a covering of } G_\alpha.$$

Let $\eta > 0$ be such that $g > f(\alpha) + 2\eta$. By definition of $f(\alpha)$, there exists $\epsilon_0 > 0$ and m_0 such that for all $n \geq m_0$ and $0 < \epsilon \leq \epsilon_0$,

$$|S^{(n)}(\alpha, \epsilon)| \leq \mu^{n[f(\alpha)+\eta]} . \tag{16}$$

It follows that if $m \geq m_0$ and $0 < \epsilon \leq \epsilon_0$, then

$$\begin{aligned}\sum_{n\geq m} \sum_{\mathbf{j}\in S^{(n)}(\alpha,\epsilon)} |J_{\mathbf{j}}|^g &\leq \sum_{n\geq m} \left| \max_{\mathbf{j}\in\Upsilon_\emptyset(-n)} \mu^{-n} W_{\mathbf{j}} \right|^g \mu^{n[f(\alpha)+\eta]} \\ &= \sum_{n\geq m} \left| \max_{\mathbf{j}\in\Upsilon_\emptyset(-n)} W_{\mathbf{j}} \right|^g \mu^{-n[g-f(\alpha)-\eta]}.\end{aligned}$$

Since $g > f(\alpha) + 2\eta$ we have $\mu^{-n[g-f(\alpha)-\eta]} < \mu^{-n\eta}$ and thus

$$\sum_{n\geq m} \sum_{\mathbf{j}\in S^{(n)}(\alpha,\epsilon)} |J_{\mathbf{j}}|^g \leq \sum_{n\geq m} \left| \max_{\mathbf{j}\in\Upsilon_\emptyset(-n)} W_{\mathbf{j}} \right|^g \mu^{-n\eta}.$$

By Lemma 4.8, for any $\epsilon > 0$ there exists m_1 such that $\mathbb{P}$-a.s., for all $n \geq m_1$,

$$\max_{\mathbf{j}\in\Upsilon_\emptyset(-n)} W_{\mathbf{j}} \leq e^{n\varepsilon}.$$

It follows for all $m > \max(m_0, m_1)$:

$$\sum_{n\geq m} \sum_{\mathbf{j}\in S^{(n)}(\alpha,\epsilon)} |J_{\mathbf{j}}|^g \leq \sum_{n\geq m} e^{n\varepsilon g} \mu^{-n\eta} = \sum_{n\geq m} e^{n(\varepsilon g - \eta \log \mu)} .$$

For $\varepsilon < \eta(\log \mu)/g$, the exponent is strictly negative and the sum is finite. Hence, sending m to infinity, we obtain for all $0 < \epsilon < \epsilon_0$

$$\sum_{n\geq m} \sum_{\mathbf{j}\in S^{(n)}(\alpha,\epsilon)} |J_{\mathbf{j}}|^g \to 0 \text{ as } m \to \infty$$

which concludes the proof. $\square$

Define

$$\begin{aligned}\tau(\theta) &= \liminf_{n\to\infty} \frac{\log\left(\sum_{\mathbf{i}\in\Upsilon_\emptyset(-n)} \mathcal{D}_{\mathbf{i}}^\theta\right)}{-n\log\mu} \\ \bar{\tau}(\theta) &= \liminf_{n\to\infty} \frac{\log\left(\sum_{\mathbf{j}\in\Upsilon_\emptyset(-n)} \bar{\mathcal{D}}_{\mathbf{j}}^\theta\right)}{-n\log\mu}.\end{aligned}$$

Also let $\tau^*(\alpha) = \inf_\theta \alpha\theta - \tau(\theta)$ and $\bar{\tau}^*(\alpha) = \inf_\theta \alpha\theta - \bar{\tau}(\theta)$ be the Legendre transforms of τ and $\bar{\tau}$.

Lemma 4.11. *Suppose Assumptions 3.1 and 4.7 hold. Then $\mathbb{P}$-a.s.*

$$f \leq \bar{\tau}^*.$$

Proof. The sum $\sum_{\mathbf{j}\in\Upsilon_\emptyset(-n)} \bar{\mathcal{D}}_{\mathbf{j}}^\theta$ can be bounded below by considering only singularity coefficients which are roughly equal to α:

$$\sum_{\mathbf{j}\in\Upsilon_\emptyset(-n)} \bar{\mathcal{D}}_{\mathbf{j}}^\theta \geq \sum_{\mathbf{j}\in S^{(n)}(\alpha,\epsilon)} \bar{\mathcal{D}}_{\mathbf{j}}^\theta.$$

Moreover if $\mathbf{j} \in \Upsilon_\emptyset(-n)$ then $\bar{\mathcal{D}}_\mathbf{j} = \mu^{-nh^n_{\mathcal{M}}(T_\mathbf{j})}$, whence for $\mathbf{j} \in S^{(n)}(\alpha, \epsilon)$ we have

$$\begin{aligned}
&\text{for } \theta \geq 0, \quad \bar{\mathcal{D}}_\mathbf{j}^\theta \geq \mu^{-n\theta(\alpha+\epsilon)} \\
&\text{for } \theta < 0, \quad \bar{\mathcal{D}}_\mathbf{j}^\theta \geq \mu^{-n\theta(\alpha-\epsilon)},
\end{aligned}$$

so that $\bar{\mathcal{D}}_\mathbf{j}^\theta \geq \mu^{-n(\alpha\theta+|\theta|\epsilon)}$ for all θ. Thus $\sum_{\mathbf{j}\in S^{(n)}(\alpha,\epsilon)} \bar{\mathcal{D}}_\mathbf{j}^\theta \geq |S^{(n)}(\alpha,\epsilon)|\mu^{-n(\alpha\theta+|\theta|\epsilon)}$.

Let $\alpha \in \mathbb{R}$ be such that $f(\alpha) > -\infty$, and take any $g < f(\alpha)$. Then there exist $\epsilon_0 > 0$ and m_0 such that for all $\epsilon \in (0, \epsilon_0)$ and $n \geq m_0$ we have

$$g \leq \frac{\log |S^{(n)}(\alpha, \epsilon)|}{n \log \mu}.$$

It follows that for such ϵ and n,

$$\sum_{\mathbf{j}\in\Upsilon_\emptyset(-n)} \bar{\mathcal{D}}_\mathbf{j}^\theta \geq |S^{(n)}(\alpha,\epsilon)|\mu^{-n(\alpha\theta+|\theta|\epsilon)} \geq \mu^{-n(\alpha\theta-g+|\theta|\epsilon)}.$$

Thus

$$\bar{\tau}(\theta) = \liminf_{n\to\infty} \frac{\log\left(\sum_{\mathbf{i}\in\Upsilon_\emptyset(-n)} \bar{\mathcal{D}}_\mathbf{i}^\theta\right)}{-n\log\mu} \leq \alpha\theta - g + |\theta|\epsilon.$$

Now let $\epsilon \to 0$ and $\gamma \to f(\alpha)$ to get $\bar{\tau}(\theta) \leq \alpha\theta - f(\alpha)$, or $f(\alpha) \leq \alpha\theta - \bar{\tau}(\theta)$. This inequality is obviously true if $f(\alpha) = -\infty$, so minimising over θ we get, for all α, $f(\alpha) \leq \bar{\tau}^*(\alpha)$. $\square$

Lemma 4.12. *Suppose Assumptions 3.1 and 4.7 hold. Then $\mathbb{P}$-a.s.*

$$\tau = \bar{\tau}. \tag{17}$$

Proof. We take cases depending on θ.

Suppose $\theta \geq 0$. In this case it is clear from the definitions that $\bar{\tau}(\theta) \leq \tau(\theta)$. The reverse inequality follows directly from

$$\begin{aligned}
\sum_{\mathbf{i}\in\Upsilon_\emptyset(-n)} |\mathcal{D}_\mathbf{i}|^\theta &\leq 3^\theta \sum_{\mathbf{i}\in\Upsilon_\emptyset(-n)} \max(\mathcal{D}_{\mathbf{i}-}, \mathcal{D}_\mathbf{i}, \mathcal{D}_{\mathbf{i}+})^\theta \\
&\leq 3^{\theta+1} \sum_{\mathbf{i}\in\Upsilon_\emptyset(-n)} \mathcal{D}_\mathbf{i}^\theta.
\end{aligned}$$

Suppose now $\theta < 0$. In this case we get immediately from the definitions that $\bar{\tau}(\theta) \geq \tau(\theta)$. For the reverse inequalities, note first that the crossing tree has at least two children per node. Thus, for any given realisation, given $\mathbf{i} \in \Upsilon_\emptyset(-n)$ we can always find a node $\mathbf{j} \in \Upsilon_\emptyset(-(n+2))$ such that $\bar{J}_\mathbf{j} \subset J_\mathbf{i}$, whence

$$\sum_{\mathbf{i}\in\Upsilon_\emptyset(-n)} \mathcal{D}_\mathbf{i}^\theta \leq \sum_{\mathbf{j}\in\Upsilon_\emptyset(-(n+2))} \bar{\mathcal{D}}_\mathbf{j}^\theta.$$

Taking logs, dividing by $n\log\mu$, and sending n to infinity, the result follows. $\square$

Define the partition function

$$\hat{\gamma}(\theta) = \liminf_{n\to\infty} \frac{\log\left(\mathbb{E}\sum_{\mathbf{i}\in\Upsilon_\emptyset(-n)} \mathcal{D}_\mathbf{i}^\theta\right)}{-n\log\mu}.$$

In Lemma 4.15 below we give conditions under which $\hat{\gamma}(\theta) = \gamma(\theta)$.

Lemma 4.13. *Suppose Assumptions 3.1 and 4.7 hold. Then, $\mathbb{P}$-a.s., for all $\alpha \in \mathbb{R}$,*

$$\tau^*(\alpha) \leq \hat{\gamma}^*(\alpha) = \inf_\theta \alpha\theta - \hat{\gamma}(\theta).$$

Proof. This can be proved in the same way as [22] Lemma 3.9. □

Assumption 4.14. *Assume that $R \in (0,1]$, $\mathbb{P}(\sum_{k=1}^{Z_\emptyset} R_k = 1) < 1$ and for all $p > 1$, $\mathbb{E}\left(\sum_{k=1}^{Z_\emptyset} R_k\right)^p < \infty$.*

Under Assumptions 3.1, 4.7 and 4.14 we have from [13] that $\mathbb{E}\mathcal{W}^p < \infty$ for all p.

Lemma 4.15. *Suppose Assumptions 3.1, 4.7 and 4.14 hold. Then*

$$\hat{\gamma}(\theta) = \gamma(\theta) = -1 - \frac{\log \mathbb{E}R^\theta}{\log \mu}.$$

Proof. Since $\rho_{\mathbf{i}} = \prod_{k=1}^{|\mathbf{i}|} R_{\mathbf{i}|_k}$ and $\mathcal{W}_{\mathbf{i}}$ are independent, we have

$$\mathbb{E} \sum_{\mathbf{i} \in \Upsilon_\emptyset(-n)} \mathcal{D}_{\mathbf{i}}^\theta = \mathbb{E} \sum_{\mathbf{i} \in \Upsilon_\emptyset(-n)} \rho_{\mathbf{i}}^\theta \mathcal{W}_{\mathbf{i}}^\theta = \mu^n (\mathbb{E}R^\theta)^n \mathbb{E}\mathcal{W}^\theta.$$

We note that the RHS is finite under the conditions of the lemma. Taking logs, dividing by $-n \log \mu$, and sending $n \to \infty$ gives the result. □

5 Hausdorff spectrum of an MEBP

Combining the results of Sections 4.1 and 4.2 we have the following.

Theorem 5.1. *Suppose Assumptions 4.1, 4.7 and 4.14 are satisfied. Then the Hausdorff spectrum of $\mathcal{M}$ is*

$$D_{\mathcal{M}}(\alpha) = \dim(G_\alpha) = \inf_\theta \alpha\theta - \gamma(\theta),$$

where $\gamma(\theta) = -1 - \log \mathbb{E}R^\theta / \log \mu$.

Remark 5.2. *Since γ is equal to the partition function $\hat{\gamma}$, we have a multifractal formalism for $\mathcal{M}$.*

Combining Theorem 5.1 with Theorem 2.8 we get the Hausdorff spectrum of an MEBP process.

Theorem 5.3. *Suppose Assumptions 2.5, 4.1, 4.7 and 4.14 hold. Let Y be an MEBP process, with offspring distribution Z and $\mathcal{M}$ the time change. Recall that $G_\alpha = \{t \in [0, T_1^0] : h_{\mathcal{M}}(t) = \alpha\}$, and let $H_\alpha = \{t \in [0, \mathcal{T}_1^0] : h_Y(t) = \alpha\}$. Then for all $\alpha > 0$*

$$D_Y(\alpha) = \dim(H_\alpha) = \frac{\alpha}{H} \dim(G_{H/\alpha}) = \frac{\alpha}{H} D_{\mathcal{M}}(H/\alpha),$$

where $H = \log 2 / \log \mathbb{E}Z$.

Proof. Let $\tilde{G}_\alpha = \{t \in [0, \mathcal{T}_1^0] : h_{\mathcal{M}^{-1}}(t) = \alpha\}$ then (see [18, 23]),

$$\dim(\tilde{G}_\alpha) = \alpha \dim(G_{1/\alpha}).$$

Finally, composition with a process of constant modulus of continuity H transforms the spectrum as follows:

$$\dim(H_\alpha) = \dim(\tilde{G}_{\alpha/H}) = \frac{\alpha}{H} \dim(G_{H/\alpha}).$$

□

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