SOME NOTES ON TREES AND PATHS

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ABSTRACT. These notes cover background material on trees which are used in the paper [1].

1. TREES AND PATHS - BACKGROUND INFORMATION

In the paper [1] it is shown that trees have an important role as the negligible sets of control theory, quite analogous to the null sets of Lebesgue integration. The trees considered are *analytic* objects in flavour, and not the finite combinatorial objects of undergraduate courses. In this note we collect together a few related ways of looking at them, and prove a basic characterisation generalising the concept of height function.

We first recall that

(1). Graphs (E,V) that are acyclic and connected are generally called *trees*. If such a tree is non-empty and has a distinguished vertex $\mathbf{v}$ it is called a *rooted tree*.

(2). A rooted tree induces and is characterised by a *partial order on V with least element* $\mathbf{v}$. The partial order is defined as follows

$$a \preceq b \quad \text{if the circuit free path from the root } \mathbf{v} \to b \text{ goes through } a.$$

This order has the property that for each fixed b the set $\{a \preceq b\}$ *is totally ordered* by $\preceq$.

Conversely any partial order on a finite set V with a least element v and the property that for each b the set $\{a \preceq b\}$ is totally ordered defines a unique rooted tree on V. One of the simplest ways to construct a tree is to consider a (finite) collection Ω of paths in a graph sharing a fixed initial or starting vertex, and with the partial order that $\omega \preceq \omega'$ iff ω is an initial segment of ω'.

(3). Alternatively, let (E,V) be a graph extended into a continuum by assigning a length to each edge. Let $d(a,b)$ be the infimum of the lengths of paths[1] between the two vertices a, b in the graph. Then g is a geodesic metric on V. Trees are exactly *the graphs that give rise to 0-hyperbolic metrics* in the sense of Gromov (see for example [2]).

(4). There are many ways to enumerate the edges and nodes of a finite rooted tree. One way is to think of a family tree recording the descendants of a single individual (the root). Start with the root. At the root, if all children have been visited stop, at any other node, if all the children have been visited, move up to the parent. If there are children who have not been visited, then visit the oldest unvisited child. At each time n the enumeration either moves up an edge or down an edge - each edge is visited exactly twice. Let $h(n)$ denote the distance from the top of the family tree after n steps in this enumeration with the convention that $h(0) = 0$,

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[1]the sum of the lengths of the edges

then h is similar to the path of a random walk, moving up or down one unit at each step, except that it is positive and returns to zero exactly as many times as there are edges coming from the root. Hence $h(2|E|) = 0$.

The function h completely describes the rooted tree. The function h directly yields the nearest neighbour metric on the tree. If h is a function such that $h(0) = 0$, it moves up or down one unit at each step, is positive and $h(2|E|) = 0$, then d defined by

$$d(m, n) = h(m) + h(n) - 2 \inf_{u \in [m,n]} h(u),$$

is a pseudo-metric on $[0, 2|V|]$. If we identify points in $[0, 2|V|]$ that are zero distance apart and join by edges the equivalence classes of points that are distance one apart, then one recovers an equivalent rooted tree.

Put less pedantically, let the enumeration be a at step n and b at step m and define

$$d(a, b) = h(m) + h(n) - 2 \inf_{u \in [m,n]} h(u),$$

then it is simple to check that d is well defined and is a metric on vertices making the set of vertices a tree.

Thus excursions of simple (random) walks are a convenient (and well studied) way to describe abstract graphical trees. This particular choice for *coding a tree with a positive function on the interval* can be extended to describe continuous trees. This approach was used by Le Gall [3] in his development of the Brownian snake associated to the measure valued Dawson-Watanabe process.

2. **R**-trees are coded by continuous functions

One of the early examples of a continuous tree is the evolution of a continuous time stochastic process, where, as is customary in probability theory, one identifies the evolution of two trajectories until the first time they separate. (This idea dates back at least to Kolmogorov and his introduction of filtrations). Another popular and equivalent approach to continuous trees is through **R**-trees ([4] p425 and the references there).

Interestingly, analysts and probabilists have generally rejected the abstract tree as too wild an object, and usually add extra structure, essentially a second topology or Borel structure on the tree that comes from thinking of the tree as a family of paths in a space which also has some topology. This approach is critical to the arguments used in [1] where tree-like paths are approximated by with simpler tree-like paths in 1-variation. (They would never converge in the 'hyperbolic' metric). In contrast, group theorists and low dimensional topologists have made a great deal of progress by studying specific symmetry groups of these trees and do not seem to find their hugeness too problematic.

Our goal in this subsection of the appendix is to prove the simple representation: that the general **R**-tree arises from identifying the contours of a continuous function on a locally connected and connected space. The height functions we considered on $[0, T]$ are a special case.

Definition 2.1. An **R**-tree is a uniquely arcwise connected metric space, in which the arc between two points is isometric to an interval.

Such a space is locally connected, for let B_x be the set of points a distance at most $1/n$ from x. If $z \in B_x$, then the arc connecting x with z is isometrically

embedded, and hence is contained in B_x. Hence B_x is the union of connected sets with non-empty common intersection (they contain x) and is connected. The sets B_x form a basis for the topology induced by the metric. Observe that if two arcs meet at two points, then the uniqueness assertion ensures that they coincide on the interval in between.

Fix some point v as the 'root' and let x and y be two points in the **R**-tree. The arcs from x and y to v have a maximal interval in common starting at v and terminating at some v_1, after that time they never meet again. One arc between them is the join of the arcs from x to v_1 to y (and hence it is the arc and a geodesic between them). Hence

$$d(x, y) = d(x, v) + d(y, v) - 2d(v, v_1).$$

Example 2.2. Consider the space Ω of continuous paths $X_t \in E$ where each path is defined on an interval $[0, \xi(\omega))$ and has a left limit at $[0, \xi(\omega))$. Suppose that if $X \in \Omega$ is defined on $[0, \xi)$, then $X|_{[0,s)} \in \Omega$ for every s less than ξ. Define

$$d(\omega, \omega') = \xi(\omega) + \xi(\omega') - 2\sup\{t < \min(\xi(\omega), \xi(\omega')) \mid \omega(s) = \omega'(s) \ \forall s \leq t\}.$$

Then (Ω, d) is an R-tree.

We now give a way of constructing **R**-trees. The basic idea for this is quite easy, but the core of the argument lies in the detail so we proceed carefully in stages.

Let I be a connected and locally connected topological space, and $h : I \to \mathbb{R}$ be a positive continuous function that attains its lower bound at a point $v \in I$.

Definition 2.3. For each $x \in I$ and $\lambda \leq h(x)$ define $C_{x,\lambda}$ to be the maximal connected subset of $\{y \mid h(y) \geq \lambda\}$ containing x.

Lemma 2.4. *The sets $C_{x,\lambda}$ exist, and are closed. Moreover, if $C_{x,\lambda} \cap C_{x',\lambda'} \neq \phi$ and $\lambda \leq \lambda'$, then*

$$C_{x',\lambda'} \subset C_{x,\lambda}.$$

Proof. An arbitrary union of connected sets with non-empty intersection is connected, taking the union of all connected subsets of $\{y \mid h(y) \geq \lambda\}$ containing x constructs the unique maximal connected subset. Since h is continuous the closure $D_{x,\lambda}$ of $C_{x,\lambda}$ is also a subset of $\{y \mid h(y) \geq \lambda\}$. The closure of a connected set is always connected hence $D_{x,\lambda}$ is also connected. It follows from the fact that $C_{x,\lambda}$ is maximal that $C_{x,\lambda} = D_{x,\lambda}$ and so is a closed set.

If $C_{x,\lambda} \cap C_{x',\lambda'} \neq \phi$ and $\lambda \leq \lambda'$, then

$$x \in C_{x,\lambda} \cup C_{x',\lambda'} \subset \{y \mid h(y) \geq \lambda\},$$

and since $C_{x,\lambda} \cap C_{x',\lambda'} \neq \phi$, the set $C_{x,\lambda} \cup C_{x',\lambda'}$ is connected. Hence maximality ensures $C_{x,\lambda} = C_{x,\lambda} \cup C_{x',\lambda'}$ and hence $C_{x',\lambda'} \subset C_{x,\lambda}$. $\square$

Corollary 2.5. *Either $C_{x,\lambda}$ equals $C_{x',\lambda}$ or it is disjoint from it.*

Proof. If they are not disjoint, then the previous Lemma can be applied twice to prove that $C_{x',\lambda} \subset C_{x,\lambda}$ and $C_{x,\lambda} \subset C_{x',\lambda}$. $\square$

Corollary 2.6. *If $C_{x,\lambda} = C_{x',\lambda}$, then $C_{x,\lambda''} = C_{x',\lambda''}$ for all $\lambda'' < \lambda$.*

Proof. The set $C_{x,\lambda}, C_{x',\lambda}$ are nonempty and have nontrivial intersection. $C_{x,\lambda} \subset C_{x,\lambda''}$ and $C_{x',\lambda} \subset C_{x',\lambda''}$ hence $C_{x,\lambda''}$ and $C_{x',\lambda''}$ have nontrivial intersection. Hence they are equal. $\square$

Corollary 2.7. $y \in C_{x,\lambda}$ *if and only if* $C_{y,h(y)} \subset C_{x,\lambda}$.

Proof. Suppose that $y \in C_{x,\lambda}$, then $C_{y,h(y)}$ and $C_{x,\lambda}$ are not disjoint. It follows from the definition of $C_{x,\lambda}$ and $y \in C_{x,\lambda}$ that $h(y) \geq \lambda$. By Lemma 2.4 $C_{y,h(y)} \subset C_{x,\lambda}$. Suppose that $C_{y,h(y)} \subset C_{x,\lambda}$, since $y \in C_{y,h(y)}$ it is obvious that $y \in C_{x,\lambda}$. $\square$

Definition 2.8. The set $C_x := C_{x,h(x)}$ is commonly referred to as the *contour* of h through x.

The map $x \rightarrow C_x$ induces a partial order on I with $x \preceq y$ if $C_x \supseteq C_y$. If h attains its lower bound at x, then $C_x = I$ since $\{y \mid h(y) \geq h(x)\} = I$ and I is connected by hypothesis. Hence the root $v \preceq y$ for all $y \in I$.

Lemma 2.9. *Suppose that* $\lambda \in [h(v), h(x)]$, *then there is a* y *in* $C_{x,\lambda}$ *such that* $h(y) = \lambda$ *and, in particular, there is always a contour* $(C_{x,\lambda})$ *at height* λ *through* y *that contains* x.

Proof. By the definition of $C_{x,\lambda}$ it is the maximal connected subset of $h \geq \lambda$ containing x; assume the hypothesis that there is no y in $C_{x,\lambda}$ with $h(y) = \lambda$ so that it is contained in $h > \lambda$, hence $C_{x,\lambda}$ is a maximal connected subset of $h > \lambda$. Now $h > \lambda$ is open and locally connected, hence its maximal connected subsets of $h > \lambda$ are open and $C_{x,\lambda}$ is open. However it is also closed, which contradicts the connectedness of the I. Thus we have established the existence of the point y. $\square$

The contour is obviously unique, although y is in general not. If we consider the equivalence classes $x\tilde{~}y$ if $x \preceq y$ and $y \preceq x$, then we see that the equivalence classes $[y]_-$ of $y \preceq x$ are totally ordered and in one to one correspondence with points in the interval $[h(v), h(x)]$.

Lemma 2.10. *If* $z \in C_{y,\lambda}$ *and* $h(z) > \lambda$, *then* z *is in the interior of* $C_{y,\lambda}$. *If* $C_{x',\lambda'} \subset C_{x,\lambda}$ *with* $\lambda' > \lambda$, *then* $C_{x,\lambda}$ *is a neighbourhood of* $C_{x',\lambda'}$.

Proof. I is locally connected, and h is continuous, hence there is a connected neighbourhood U of z such that $h(z) \geq \lambda$. By maximality $U \subset C_{z,\lambda}$. Since $C_{z,\lambda} \cap C_{y,\lambda} \neq \phi$ we have $C_{z,\lambda} = C_{y,\lambda}$ and thus $U \subset C_{y,\lambda}$. Hence $C_{y,\lambda}$ is a neighbourhood of z. The last part follows trivially once by noting that for all $z \in C_{x',\lambda'}$ we have $h(z) \geq \lambda' > \lambda$ and hence $C_{y,\lambda}$ is a neighbourhood of z. $\square$

We now define a pseudo-metric on I. Lemma 2.10 (the only place we will use local connectedness) is critical to showing that the map from I to the resulting quotient space is continuous.

Definition 2.11. If y and z are points in I, define $\lambda(y,z) \leq \min(h(y), h(z))$ such that $C_{y,\lambda} = C_{z,\lambda}$

$$\lambda(y,z) = \sup\{\lambda \mid C_{y,\lambda} = C_{z,\lambda}, \lambda \leq h(y), \lambda \leq h(z)\}.$$

The set

$$\{\lambda \mid C_{y,\lambda} = C_{z,\lambda}, \lambda \leq h(y), \lambda \leq h(z)\}$$

is a non-empty interval $[h(v), \lambda(y,z)]$ or $[h(v), \lambda(y,z))$ where $\lambda(y,z)$ satisfies

$$h(v) \leq \lambda(y,z) \leq \min(h(y), h(z)).$$

Clearly $\lambda(x,x) = h(x)$.

Lemma 2.12. *The function λ is lower semi-continuous*

$$\liminf_{z \to z_0} \lambda\left(y, z\right) \geq \lambda\left(y, z_0\right).$$

Proof. Fix y, z_0 and choose some $\lambda' < \lambda\left(y, z_0\right)$. By the definition of $\lambda(y, z_0)$ we have that $C_{y,\lambda'} = C_{z_0,\lambda'}$. Since $h\left(z_0\right) \geq \lambda'$ there is a neighbourhood U of z_0 so that $U \subset C_{z_0,\lambda'}$. For any $z \in U$ one has $z \in C_{z,\lambda'} \cap C_{z_0,\lambda'}$. Hence $C_{z_0,\lambda'} = C_{z,\lambda'}$ and $C_{y,\lambda'} = C_{z,\lambda'}$. Thus $\lambda\left(y, z\right) \geq \lambda'$ for $z \in U$ and hence

$$\liminf_{z \to z_0} \lambda\left(y, z\right) \geq \lambda'.$$

Since $\lambda' < \lambda\left(y, z_0\right)$ was arbitrary

$$\liminf_{z \to z_0} \lambda\left(y, z\right) \geq \lambda(y, z_0)$$

and the result is proved. $\square$

Lemma 2.13. *The following inequality holds*

$$\min\left\{\lambda\left(x, z\right), \lambda\left(y, z\right)\right\} \leq \lambda\left(x, y\right).$$

Proof. If $\min\left\{\lambda\left(x, z\right), \lambda\left(y, z\right)\right\} = h\left(v\right)$, then there is nothing to prove. Recall that

$$\left\{\lambda \mid C_{y,\lambda} = C_{z,\lambda}, \; \lambda \leq h\left(y\right), \; \lambda \leq h\left(x\right)\right\}$$

is connected and contains $h\left(v\right)$. Suppose $h\left(v\right) \leq \lambda < \min\left\{\lambda\left(x, z\right), \lambda\left(y, z\right)\right\}$, then it follows that the identity $C_{x,\lambda} = C_{z,\lambda}$ holds for λ. Similarly $C_{y,\lambda} = C_{z,\lambda}$. As a result $C_{x,\lambda} = C_{y,\lambda}$ and $\lambda\left(x, y\right) \geq \lambda$. $\square$

Definition 2.14. Define d on $I \times I$ by

$$d\left(x, y\right) = h\left(x\right) + h\left(y\right) - 2\lambda\left(x, y\right).$$

Lemma 2.15. *The function d is a pseudo-metric on I. If $\left(\tilde{I}, d\right)$ is the resulting quotient metric space, then the projection $I \to \tilde{I}$ from the topological space I to the metric space is continuous.*

Proof. Clearly d is positive, symmetric and we have remarked that for all x, $\lambda\left(x, x\right) = h\left(x\right)$ hence it is zero on the diagonal. To see the triangle inequality, assume

$$\lambda\left(x, z\right) = \min\left\{\lambda\left(x, z\right), \lambda\left(y, z\right)\right\}$$

and then observe

$$\begin{aligned}
d\left(x, y\right) &= h\left(x\right) + h\left(y\right) - 2\lambda\left(x, y\right) \\
&\leq h\left(x\right) + h\left(y\right) - 2\lambda\left(x, z\right) \\
&= h\left(x\right) + h\left(z\right) - 2\lambda\left(x, z\right) + h\left(y\right) - h\left(z\right) \\
&\leq d\left(x, z\right) + \left|h\left(y\right) - h\left(z\right)\right|
\end{aligned}$$

but $\lambda\left(y, z\right) \leq \min\left(h\left(y\right), h\left(z\right)\right)$ and hence

$$\begin{aligned}
\left|h\left(y\right) - h\left(z\right)\right| &= h\left(y\right) + h\left(z\right) - 2\min\left(h\left(y\right), h\left(z\right)\right) \\
&\leq h\left(y\right) + h\left(z\right) - 2\lambda\left(y, z\right) \\
&= d\left(y, z\right)
\end{aligned}$$

hence

$$d\left(x, y\right) \leq d\left(x, z\right) + d\left(y, z\right)$$

as required. $\square$

We can now introduce the equivalence relation $x\tilde{\ }y$ if $d(x,y)=0$ and the quotient space $I/\tilde{\ }$. We write $I/\tilde{\ } = \tilde{I}$ and $i : I \to \tilde{I}$ for the canonical projection. The function d projects onto $\tilde{I} \times \tilde{I}$ and is a metric there.

It is tempting to think that $x\tilde{\ }y$ if and only if $C_x = C_y$ and this is true if I is compact Hausdorff. However the definitions imply a slightly different criteria: $x\tilde{\ }y$ iff

$$h(x) = h(y) = \lambda \text{ and } C_{x,\lambda''} = C_{y,\lambda''} \text{ for all } \lambda'' < \lambda.$$

The stronger statement $x\tilde{\ }y$ if and only if $C_x = C_y$ is not true for all continuous functions h on $\mathbb{R}^2$ as it is easy to find a decreasing family of closed connected sets there whose limit is a closed set that is not connected.

Consider again the new metric space $\tilde{I}$ that has as its points the equivalence classes of points indistinguishable under d. We now prove that the projection i taking I to $\tilde{I}$ is continuous. Fix $y \in I$ and $\varepsilon > 0$. Since $\lambda(y,.)$ is lower semi-continuous and h is (upper semi)continuous there is a neighbourhood U of y so that for $z \in U$ one has $\lambda(y,z) > \lambda(y,y) - \varepsilon/4$ and $h(z) < h(y) + \varepsilon/2$. Thus $d(y,z) < \varepsilon$ for $z \in U$. Hence $\tilde{d}(i(y),i(z)) < \varepsilon$ if $z \in U$. The function i is continuous and as continuous images of compact sets are compact we have the following.

Corollary 2.16. *If I is compact, then $\tilde{I}$ is a compact metric space.*

To complete this section we will show $\tilde{I}$ is a uniquely arcwise connected metric space, in which the arc between two points is isometric to an interval and give a characterisation of compact trees.

Proposition 2.17. *If I is a connected and locally connected topological space, and $h : I \to \mathbb{R}$ is a positive continuous function that attains its lower bound, then its "contour tree" the metric space $\left(\tilde{I}, \tilde{d}\right)$ is an $\mathbf{R}$-tree. Every $\mathbf{R}$-tree can be constructed in this way.*

Proof. It is enough to prove that the metric space $\tilde{I}$ we have constructed is really an $\mathbf{R}$-tree and that every $\mathbf{R}$-tree can be constructed in this way. Let $\tilde{x}$ any point in $\tilde{I}$ and $x \in I$ satisfy $i(x) = \tilde{x}$. Then $h(x)$ does not depend on the choice of x. Fix $h(v) < \lambda < h(x)$. We have seen that there is a y such that $h(y)=\lambda$ and $y \prec x$ moreover any two choices have the same contour through them and hence the same $\tilde{y}(\lambda)$. In this way we see that there is a map from $[h(v),h(x)]$ into $\tilde{I}$ that is injective. Moreover, it is immediate from the definition of d that it is an isometry and that $\tilde{I}$ is uniquely arc connected.

Suppose that Ω is an $\mathbf{R}$-tree, then we may fix a base point, and for each point in the tree consider the distance from V it is clear that this continuous function is just appropriate to ensure that the contour tree is the original tree. $\square$

Remark 2.18. 1. In the case where I is compact, obviously $\tilde{I}$ is both complete and totally bounded as it is compact.

2. An R-tree is a metric space; it is therefore possible to complete it. Indeed the completion consists of those paths, all of whose initial segments are in the tree[2]; we have not identified a simple sufficient condition on the continuous function and topological space Ω to ensure this. An R-tree is totally bounded if it is bounded and for each $\varepsilon > 0$ there is an N so that for each t the paths that extend a distance

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[2]We fix a root and identify the tree with the geodesic arc from the root to the point in the tree.

t from the root have at most N ancestral paths between them at time $t - \varepsilon$. In this way we see that the R-tree that comes out of studying the historical process for the Fleming-Viot or the Dawson Watanabe measure-valued processes is, with probability one, a compact R-tree for each finite time.

Lemma 2.19. *Given a compact R-tree, there is always a height function on a closed interval that yields the same tree as its quotient.*

Proof. As the tree is compact, path connected and locally path connected, there is always as based loop mapping $[0, 1]$ onto the tree. Let h denote the distance from the root. Its pullback onto the interval $[0, 1]$ is a height function and the natural quotient is the original tree. In this way we see that there is always a version of Le Gall's snake [3] traversing a compact tree. $\square$

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