Profinite Heyting algebras

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Profinite objects

Let $\mathcal{C}$ be a **Set**-based category.

An object A of $\mathcal{C}$ is called profinite if it is an inverse limit of an inverse system of finite $\mathcal{C}$-objects.

Examples.

1. In the category of compact Hausdorff spaces, an object is profinite iff it is a Stone space (that is, the dual of a Boolean algebra).

2. In the category of compact order-Hausdorff spaces, an object is profinite iff it is a Priestley space (that is, the dual of a bounded distributive lattice).

Profinite objects

3. In the category of Boolean algebras, an object is profinite iff it is complete and atomic.

4. In the category of bounded distributive lattices, an object is profinite iff it is complete and completely join-prime generated.

(An element $a \in A$ is completely join-prime if $a \leq \bigvee C$ implies there exists $c \in C$ such that $a \leq c$.

A lattice A is completely join-prime generated if every element of A is a join of completely join-prime elements.)

Question: What are profinite Heyting algebras?

Heyting algebras

A Heyting algebra is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ with a binary operation $\to: A^2 \to A$ such that for each $a, b, c \in A$ we have

$$a \wedge c \leq b \text{ iff } c \leq a \to b;$$

Algebraic characterization

An algebra A is finitely approximable if for every $a, b \in A$ with $a \neq b$, there exists a finite algebra B and a surjective homomorphism $h : A \to B$ such that $h(a) \neq h(b)$.

Theorem. A Heyting algebra A is profinite iff A is finitely approximable, complete and completely join-prime generated.

Duality

A pair $(X, \leq)$ is called a Priestley space if X is a Stone space (compact, Hausdorff, with a basis of clopens) and $\leq$ is a poset satisfying the Priestley separation axiom:

For every $x, y \in X$, $x \not\leq y$ implies there is a clopen upset U with $x \in U$ and $y \notin U$.

Esakia duality for Heyting algebras

A Priestley space $(X, \leq)$ is called an Esakia space if

$$\downarrow U \text{ is clopen for every clopen } U \subseteq X$$

Let $\mathrm{Up}_\tau(X)$ denote the Heyting algebra of clopen upsets of X, where

$$U \to V = X - \downarrow(U - V).$$

Theorem (Esakia 1974). For every Heyting algebra A, there exists an Esakia space $(X, \leq)$ such that A is isomorphic to $\mathrm{Up}_\tau(X)$.

Esakia duality

Let X be an Esakia space. We let

$$X_{\text{fin}} = \{x \in X : {\uparrow}x \text{ is finite}\}.$$

$$X_{\text{iso}} = \{x \in X : x \text{ is an isolated point of } X\}.$$

An Esakia space X is called extremally order-disconnected if $\overline{U}$ is clopen for every open upset $U \subseteq X$.

Topological characterization

Theorem. Let A be a Heyting algebra and let X be its dual space. Then the following conditions are equivalent.

1. A is profinite.
2. X is extremally order-disconnected, X_{iso} is a dense upset of X, and $X_{\mathrm{iso}} \subseteq X_{\mathrm{fin}}$.
3. A is complete, finitely approximable and completely join-prime generated.

Frame-theoretic characterization

A poset $(Y, \leq)$ is called image-finite if $\uparrow x$ is finite for every $x \in Y$.

For every poset $(Y, \leq)$ let $\mathrm{Up}(Y)$ denote the Heyting algebra of all upsets of $(Y, \leq)$.

Theorem. A Heyting algebra A is profinite iff there is an image-finite poset Y such that A is isomorphic to $\mathrm{Up}(Y)$.

Characterization of profinite Heyting algebras

Theorem. Let A be a Heyting algebra and let X be its dual space. Then the following conditions are equivalent.

1. A is profinite.

2. A is complete, finitely approximable, and completely join-prime generated.

3. X is extremally order-disconnected, X_{iso} is a dense upset of X, and $X_{\mathrm{iso}} \subseteq X_{\mathrm{fin}}$.

4. There is an image finite poset Y such that A is isomorphic to $\mathrm{Up}(Y)$.

Profinite Completions

For every Heyting algebra A its profinite completion is the inverse limit of the finite homomorphic images of A.

Theorem. Let A be a Heyting algebra and let X be its dual space. Then the following conditions are equivalent.

1. A is isomorphic to its profinite completion.
2. A is finitely approximable, complete, and the kernel of every finite homomorphic image of A is a principal filter of A.
3. X is extremally order-disconnected and $X_{\mathrm{iso}} = X_{\mathrm{fin}}$ is dense in X.

Profinite Completions

Corollary. Let A be a distributive lattice or Boolean algebra. Then A is isomorphic to its profinite completion iff A is finite.