Free MV_n-algebras

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based on the joint work with Roberto Cignoli "Free MV_n-algebras" that will appear in Algebra Universalis.

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- R. S. Grigolia, *An algebraic analysis of Łukasiewicz - Tarski n-valued logical systems*. In: R. Wójcicki, G. Malinowski (Eds.), Selected papers on Łukasiewicz sentential calculus, Ossolineum, Wrocław (1977), 81-92.
- R. Cignoli, *Natural dualities for the algebras of Łukasiewicz finitely-valued logics*, Bull. Symb. Logic. **2** (1996), 218.
- K. Keimel and H. Werner, *Stone duality for varieties generated by a quasi-primal algebra*, Mem. Amer. Math. Soc. No. **148** (1974), 59 -85.
- Nied P. Niederkorn, *Natural dualities for varieties of MV-algebras*, J. Math. Anal. Appl. **225** (2001), 58-73.
- M. Busaniche, and R. Cignoli, *Free algebras in varieties of BL-algebras generated by a BL_n-chain*, Journal of the Australian Mathematical Society, **80** (2006) 419-439.

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Definitions

Let $n \in \mathbb{N}$. $\mathcal{MV}_n$ denotes the variety of MV-algebras generated by the chain $\mathbf{L}_n$

$$\left\{ \frac{0}{n-1}, \frac{1}{n-1}, \ldots, \frac{n-1}{n-1} \right\}.$$

Recall that $\mathbf{L}_{d+1}$ is a subalgebra of $\mathbf{L}_n$ iff d is a divisor of $n-1$.

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Recall that $\mathbf{L}_{d+1}$ is a subalgebra of $\mathbf{L}_n$ iff d is a divisor of $n-1$.

For each $\mathbf{A} \in \mathcal{MV}_n$ there is:

- a Boolean space $X(\mathbf{A})$
- a meet homomorphism $\rho_{\mathbf{A}}$ from $\mathrm{Div}(n-1)$ into the lattice of closed subsets of $X(\mathbf{A})$, satisfying $\rho_{\mathbf{A}}(n-1) = X(\mathbf{A})$

such that

$$\mathbf{A} \cong \mathcal{C}_n(X(\mathbf{A}), \rho_{\mathbf{A}})$$

where $\mathcal{C}_n(X(\mathbf{A}), \rho_{\mathbf{A}}) = \{f : X(\mathbf{A}) \to \mathbf{L}_n\}$ and $f(\rho_{\mathbf{A}}(d)) \subseteq L_{d+1}$.

Take $X(\mathbf{A}) = \{\chi : \mathbf{A} \to \mathbf{L}_n\}$ is isomorphic to the Stone space of the Boolean algebra $\mathbf{B}(\mathbf{A})$.

For each $d \in Div(n-1)$ take

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Moisil operators

For every $\mathbf{A} \in \mathcal{MV}_n$, we can define the *Moisil operators* $\sigma_i : \mathbf{A} \to \mathbf{B}(\mathbf{A})$ with $i = 1, \ldots n-1$.

In particular, when evaluated $\mathbf{L}_n$ they give:

$$\sigma_i\left(\frac{j}{(n-1)}\right) = \begin{cases} 1 & \text{if } i+j \geq n, \\ 0 & \text{if } i+j < n, \end{cases} \tag{1}$$

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for $i = 1, \ldots, n - 1$

$$
\begin{array}{cccccc}
 & 0 & \frac{1}{n-1} & \dots & \frac{n-2}{n-1} & 1 \\
\sigma_1 & 0 & 0 & \dots & 0 & 1 \\
\sigma_2 & 0 & 0 & \dots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sigma_{n-1} & 0 & 1 & \dots & 1 & 1
\end{array}
$$

The following properties hold in every $\mathbf{A} \in \mathcal{MV}_n$:

- $x \in B(A)$ if and only if $x = \sigma_i(x)$ for some $1 \leq i \leq n-1$ if and only if $x = \sigma_i(x)$ for all $1 \leq i \leq n-1$.
- $\sigma_1(x) \leq \sigma_2(x) \leq \ldots \leq \sigma_{n-1}(x)$.

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- $\sigma_1(x) \leq \sigma_2(x) \leq \ldots \leq \sigma_{n-1}(x)$.

$$\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(X))$$

is the free Boolean algebra over the poset

$$Y := \{\sigma_i(x) : x \in X, i = 1, \ldots, n-1\}.$$

The correspondence

$$S \subseteq Y \rightarrow U_S$$

where U_S is the boolean filter generated by the set $S \cup \{\neg y : y \in Y \setminus S\}$, defines a bijection from the set of upwards closed subsets of Y onto the ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(X))$.

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Consider the poset $Y = \{\sigma_i(x) : x \in X, i = 1, \ldots, n-1\}$. If S is an upwards closed subset of Y then for each $x \in X$ we have a chain C_x of the form

$$C_x = \sigma_j(x) \leq \ldots \leq \sigma_{n-1}(x)$$

for some $1 \leq j \leq n-1$ and $S = \cup_{x \in X} C_x$.

Let R_{n-1} be the set of upwards closed subsets of Y.
For each $d \in Div^*(n-1)$, let $R_d \subseteq R_{n-1}$ be such that

$$S = \cup_{x \in X} C_x \in R_d$$

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Free$_{\mathcal{MV}_n}(X)$ is isomorphic to the algebra of continuous functions f from the Stone space of the free Boolean algebra over the poset $Y = \{\sigma_i(x) : x \in X, i = 1, \ldots, n-1\}$ into $\mathbf{L}_n$ such that for each $d \in \mathrm{Div}^*(n-1)$ and each $S \in R_d$, $f(U_S) \subseteq L_{d+1}$.

Recall that

$$\mathbf{Free}_{\mathcal{MV}_n}(X) \cong \mathcal{C}_n(X(\mathbf{Free}_{\mathcal{MV}_n}(X)), \rho)$$

where $X(\mathbf{Free}_{\mathcal{MV}_n}(X)) = \{\chi : \mathbf{Free}_{\mathcal{MV}_n}(X) \to \mathbf{L}_n\}$
and for each $d \in Div(n-1)$

$$\rho(d) = \{U = \chi^{-1}\{1\} \cap \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(X)) : \chi(\mathbf{Free}_{\mathcal{MV}_n}(X)) \subseteq L_{d+1}\}.$$

We only need to prove that for each $d \in \mathrm{Div}^*(n-1)$,

$$U_S \in \rho(d) \text{ iff } S \in R_d$$

Let $U_S \in \rho(d)$. Then there is a homomorphism $\chi : \mathbf{Free}_{\mathcal{MV}_n}(X) \to L_{d+1}$ such that

$$U_S = \chi^{-1}(\{1\}) \cap B(\mathbf{Free}_{\mathcal{MV}_n}(X))$$

Fix $x \in X$ and let $\chi(x) = \frac{j}{n-1} \in L_{d+1}$. Thus

$$\sigma_i(x) \in S \text{ iff } \chi(\sigma_i(x)) = 1 \text{ iff } \sigma_i(\chi(x)) = 1$$

iff $i + j \geq n$.

The chain $C_x \subseteq S$ has cardinality j, thus $\frac{\#C_x}{n-1} = \frac{j}{n-1} \in L_{d+1}$.

This happens for all $x \in X$. Hence if $U_S \in \rho(d)$, then $S \in R_d$.

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This happens for all $x \in X$. Hence if $U_S \in \rho(d)$, then $S \in R_d$.

Now let $S \in R_d$ and $x \in X$, i.e., $\frac{\#C_x}{n-1} \in L_{d+1}$ for each $C_x \subseteq S$.

Let

$$\chi : \mathbf{Free}_{\mathcal{MV}_n}(X) \to \mathbf{L}_n$$

by

$$\chi(x) = \frac{\#C_x}{n-1}, \quad \text{for each } x \in X.$$

Then

$$\chi(\mathbf{Free}_{\mathcal{MV}_n}(X)) \subseteq L_{d+1}.$$

We also have $U_S = \chi^{-1}(\{1\}) \cap B(\mathbf{Free}_{\mathcal{MV}_n}(X))$. Therefore $U_S \in \rho(d)$.

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Examples

Let X be a finite set of cardinality $r \in \mathbb{N}$.

Then the Stone space of $B(\mathbf{Free}_{\mathcal{MV}_n}(X))$ is a discrete space with n^r elements (i.e, the cardinality of R_{n-1}).

For each $S \in R_{n-1}$ let

$$r_S = \{d : S \in R_d \text{ and } S \notin R_j \text{ for any } j \in Div^*(d)\}.$$

To every $f \in \mathbf{Free}_{\mathcal{MV}_n}(X)$ we can assign an element

$$\overline{x} \in \prod_{S \in R_{n-1}} L_{r_S+1}$$

such that $x_S = f(U_S)$.
It is not hard to check that the assignment is an bijective homomorphism.

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Thus we conclude

$$\mathbf{Free}_{\mathcal{MV}_n}(X) \cong \prod_{d \in Div(n-1)} \mathbf{L}_{d+1}^{\alpha_d},$$

where for each $d \in Div(n-1)$, α_d is the cardinality of the set $(R_d \setminus \cup_{k \in Div^*(d)} R_k)$.

Thank you