Substructural Logics and Formal Linguistics: Back and Forth

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SUBSTRUCTURAL LOGICS

Associative Lambek Calculus **L***

$$(\text{Id})A \Rightarrow A$$

$$(\cdot\text{L})\ \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \cdot B, \Delta \Rightarrow C},\ (\cdot\text{R})\ \frac{\Gamma \Rightarrow A;\ \ \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \cdot B}$$

$$(\rightarrow\text{L})\ \frac{\Gamma, B, \Delta \Rightarrow C;\ \ \Phi \Rightarrow A}{\Gamma, \Phi, A \rightarrow B, \Delta \Rightarrow C}, (\rightarrow\text{R})\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$(\leftarrow\text{L})\ \frac{\Gamma, B, \Delta \Rightarrow C;\ \ \Phi \Rightarrow A}{\Gamma, B \leftarrow A, \Phi, \Delta \Rightarrow C},\ (\leftarrow\text{R})\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B \leftarrow A}$$

$$(\text{CUT})\ \frac{\Gamma, A, \Delta \Rightarrow B;\ \ \Phi \Rightarrow A}{\Gamma, \Phi, \Delta \Rightarrow B}$$

LAMBEK (1958) : **L**, $\Gamma \neq \epsilon$ in ($\rightarrow$R), ($\leftarrow$R)

(CUT) is admissible in **L**, **L***.

LAMBEK (1961): Nonassociative Lambek Calculus **NL**

Formula structures (trees): formulas, (X, Y); sequents: $X \Rightarrow A$

$$(\cdot\text{L})\ \frac{X[A, B] \Rightarrow C}{X[A \cdot B] \Rightarrow C},\ (\cdot\text{R})\ \frac{X \Rightarrow A;\ Y \Rightarrow B}{(X, Y) \Rightarrow A \cdot B}$$

$$(\rightarrow\text{L})\ \frac{X[B] \Rightarrow C;\ Y \Rightarrow A}{X[Y, A \rightarrow B] \Rightarrow C},\ (\rightarrow\text{R})\ \frac{(A, X) \Rightarrow B}{X \Rightarrow A \rightarrow B}$$

$$(\text{CUT})\ \frac{X[A] \Rightarrow B;\ Y \Rightarrow A}{X[Y] \Rightarrow B}$$

NL* (W.B., BULINSKA): the empty structure Λ

$(X, \Lambda) = (\Lambda, X) = X$

(CUT) is admissible in **NL** (LAMBEK 1961) and **NL***.

A residuated semigroup: $\mathcal{M} = (M, \leq, \cdot, \rightarrow, \leftarrow)$ s.t. $(M, \leq)$ is a poset, $(M, \cdot)$ is a semigroup, and $\rightarrow, \leftarrow$ are binary operations on M, satisfying *the residuation law*:

$$\text{(RES) } ab \leq c \text{ iff } b \leq a \rightarrow c \text{ iff } a \leq c \leftarrow b.$$

A residuated monoid: with identity 1, $a \cdot 1 = a = 1 \cdot a$.

A residuated groupoid: $\cdot$ need not be associative.

An assignment f of formulas in $\mathcal{M}$ (a homomorphism from the formula algebra into $\mathcal{M}$). Extended to sequences of formulas: $f(\epsilon) = 1$, $f(\Gamma, A) = f(\Gamma) \cdot f(A)$, and similarly for structures.

$(\mathcal{M}, f) \models \Gamma \Rightarrow A$ iff $f(\Gamma) \leq f(A)$.

L (resp. **L***) is strongly complete w.r.t. residuated semigroups (resp. monoids). **NL** (resp. **NL***) is strongly complete w.r.t. residuated groupoids (resp. with identity).

The rule EXCHANGE

$$(\mathrm{EXC})\ \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C}$$

L with (EXC) is strongly complete w.r.t. commutative residuated semigroups. Analogous facts hold for **L***, **NL**, **NL***. In these systems $A \to B \Leftrightarrow B \leftarrow A$ is provable.

The $(\to, \leftarrow)-$fragment of **L*** with (EXC) amounts to BCI. The analogous fragment of **L** is the Lambek–van Benthem calculus (of semantic types) (VAN BENTHEM 1986).

The identity constant 1

$$(1\mathrm{L})\ \frac{\Gamma, \Delta \Rightarrow A}{\Gamma, 1, \Delta \Rightarrow A},\ (1\mathrm{R})\ \Rightarrow 1$$

$$(1\mathrm{L})\ \frac{X[\Lambda] \Rightarrow A}{X[1] \Rightarrow A},\ (1\mathrm{R})\ \Lambda \Rightarrow A$$

Lattice operations $\wedge, \vee$ and constants $\top, \bot$; Full Lambek Calculus **FL**; (ONO 1993, JIPSEN 2004). **FL** is strongly complete w.r.t. residuated lattices, i.e. residuated monoids which are lattices.

Modalities (in linguistics MOORTGAT 1995)

$$(\Diamond\mathrm{L})\ \frac{\Gamma, < A >, \Delta \Rightarrow B}{\Gamma, \Diamond A, \Delta \Rightarrow B},\ (\Diamond\mathrm{R})\ \frac{\Gamma \Rightarrow A}{< \Gamma > \Rightarrow \Diamond A}$$

$$(\Box\mathrm{L})\ \frac{\Gamma, A, \Delta \Rightarrow B}{\Gamma, < \Box A >, \Delta \Rightarrow B},\ (\Box\mathrm{R})\ \frac{< \Gamma > \Rightarrow A}{\Gamma \Rightarrow \Box A}$$

Generalized Lambek Calculus (W.B., M. KOLOWSKA-GAWIEJNOWICZ, M. KANDULSKI): f_i $n-$ary connective, $n \geq 1$, $0 \leq i \leq n$, f_i is the $i-$th residual of f_0. $f_0 = f$ (a multi-modal framework, also related to DUNN 1993).

Structures: formulas, $(X_1, \ldots, X_n)_f$

$$(f\mathrm{L})\ \frac{X[(A_1, \ldots, A_n)_f] \Rightarrow A}{X[f(A_1, \ldots, A_n)] \Rightarrow A},\ (f\mathrm{R})\ \frac{X_1 \Rightarrow A_1; \ldots; X_n \Rightarrow A_n}{(X_1, \ldots, X_n)_f \Rightarrow f(A_1, \ldots, A_n)}$$

$$(f_i\mathrm{L})\ \frac{X[A_i] \Rightarrow B;\ (Y_j \Rightarrow A_j)_{j \neq i}}{X[(Y_1, \ldots, f_i(A_1, \ldots, A_n), \ldots, Y_n)_f] \Rightarrow B},\ (f_i\mathrm{R})\ \frac{(A_1, \ldots, X, \ldots, A_n)_f \Rightarrow A_i}{X \Rightarrow f(A_1, \ldots, A_n)}$$

GRAMMARS

$\mathcal{L}$ a system of logic

$G = (\Sigma, I_G, S)$ s.t. Σ a finite alphabet, I_G a finite relations between symbols from Σ and formulas (types), S a designated type (the principal type).

$I_G(a) = \{A : (a, A) \in I_G\}$

G assigns type A to the string $a_1 \ldots a_n$, $a_i \in \Sigma$, if there exist types $A_i \in I_G(a_i)$, s.t. $A_1, \ldots, A_n \Rightarrow A$ is provable; the set of all such strings x is denoted by $L(G, A)$. $L(G) = L(G, S)$ is *the language* of G.

If antecedents of sequents are trees, then G assigns types to trees. $L_t(G, A)$ consists of all trees T which arise from structures X s.t. $X \Rightarrow A$ is provable by replacing each leave $A \in I_G(a)$ by a. $L_t(G) = L_t(G, S)$ is *the tree language* of G. $L(G)$ is the yield of $L_t(G)$.

One also considers trees determined by proof trees of sequents.

(1) Context-free grammars (CFG's)

Logic: sequents $p_1, \ldots, p_n \Rightarrow p$, where p_i, p atoms, axioms (Id) $p \Rightarrow p$, (CUT) the only rule.

A CFG ($\epsilon-$free) is based on a theory, axiomatized by finitely many assumptions (nonlogical axioms).

$n_1, v \Rightarrow s$; $v_t, n_1 \Rightarrow v$; $d, n \Rightarrow n_1$; $n_0 \Rightarrow n_1$, where n_1 is the type of noun phrase, n_0 of proper noun, n of common noun, d of determiner, v of verb phrase, v_t of transitive verb phrase, s of sentence. In linguistics, one usually writes $s \mapsto n_1, v$; $n_1 \mapsto d, n$ etc.

John passes every exam. $n_0, v_t, d, n \Rightarrow s$.
The tree (John, (passes, (every ,exam))).

(2) Context-sensitive grammars (CSG's)

Logic: the $(\cdot)-$fragment of **L** with (CUT). A CSG is based on a theory, axiomatized by finitely many assumptions $A \Rightarrow B$ s.t. $|A| \geq |B|$, where $|A|$ is the total number of atoms in A.

(3) Classical categorial grammars (CCG's)

Logic: $(\rightarrow, \leftarrow)$−sequents of **L**. (Id), ($\rightarrow$L), ($\leftarrow$L).

Equivalently: (Id),and $A, A \rightarrow B \Rightarrow B$ and $B \leftarrow A, A \Rightarrow B$, (CUT).

The latter is, essentially, the reduction system **AB** of AJDUKIEWICZ (1935) and BAR-HILLEL, GAIFMAN, SHAMIR (1960).

I_G: John: n_1, passes: $(n_1 \rightarrow s) \leftarrow n_1$, every: $n_1 \leftarrow n$, exam: n

John passes every exam.

$n_1, (n_1 \rightarrow s) \leftarrow n_1, n_1 \leftarrow n, n \Rightarrow n_1, (n_1 \rightarrow s) \leftarrow n_1, n_1 \Rightarrow$

$\Rightarrow n_1, n_1 \rightarrow s \Rightarrow s$

Tree: (John (passes (every exam))).
Functor-argument (FA) structure: (John (passes (every exam)$_1$)$_1$)$_2$.

CCG's (like most categorial grammars) are *lexical*: logic is common for all languages, the language specification is given by the lexical type assignment I_G only.

(4) Lambek categorial grammars (**L**-grammars)

Logic: **L** (often its (→, ←)−fragment)

Define: $n_1 = s \leftarrow (n_0 \rightarrow s)$

John: n_0. From $A, A \rightarrow B \Rightarrow B$ one derives $A \Rightarrow B \leftarrow (A \rightarrow B)$. So, $n_0 \Rightarrow n_1$ is provable.

Also $(A \rightarrow B) \leftarrow C \Leftrightarrow A \rightarrow (B \leftarrow C)$ is provable.
$(n_1 \rightarrow s) \leftarrow n_1 \Rightarrow n_1 \rightarrow (s \leftarrow n_1)$ is provable.
((John passes) (every exam))

Generating trees: I_G provides $a_i : A_i$

$A_i \Rightarrow B_i$ provable, $B_1, \ldots, B_n \Rightarrow A$ by **AB**. This determines an FA-structure with yield $a_1 \ldots a_n$.

One can generate all possible FA-structures, whence all possible trees, on the generated strings (W.B. 1988).

CURRY–HOWARD CORRESPONDENCE

Natural Deduction

$$(\to\mathrm{E})\ \frac{\Gamma \Rightarrow A \to B;\ \Delta \Rightarrow A}{\Gamma, \Delta \Rightarrow B},\ (\to\mathrm{I})\ \frac{\Gamma, A, \Delta \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

The proof of $A_1, \ldots, A_n \Rightarrow A$ represented as $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$, where M a (linear) lambda-term with free variables $x_1, \ldots, x_n$.

ND-proofs determine denotations in a type-theoretic semantics of Montague style.

$e \Rightarrow (e \to t) \to t.\quad x : e \vdash (\lambda y : e \to t).(yx) : (e \to t) \to t.$
The lambda-term denotes the (characteristic function of) family of all properties of the individual assigned to x.

VAN BENTHEM 1986: every sequent provable in the $(\to, \leftarrow)$−fragment of **L** with (EXC) admits only finitely many different readings (different normal ND-proofs).

WANSING 1992: Lambda-calculus for noncommutative systems (Curry–Howard, normalization).

For Linear Logic: DE GROOTE 1996, ABRUSCI, RUET 1999

STANDARD FRAMES

$\mathcal{M} = (M, \cdot)$ a semigroup. $P(M) = \{X : X \subseteq M\}$.

$X \cdot Y = \{xy : x \in X, y \in Y\}$. $X \to Y = \{z \in M : X \cdot \{z\} \subseteq Y\}$.
$Y \leftarrow X = \{z \in M : \{z\} \cdot X \subseteq Y\}$.

$P(\mathcal{M}) = (P(M), \subseteq, \cdot, \to, \leftarrow)$ a residuated semigroup. If $\mathcal{M}$ a monoid, then $P(\mathcal{M})$ a residuated monoid with identity $\{1\}$. $P(\Sigma^+)$ the algebra of $\epsilon-$free languages on Σ. $P(\Sigma^*)$ the algebra of languages on Σ.

L (also with $\wedge$) is strongly complete w.r.t. powerset frames over semigroups, and similarly for **L*** and powerset frames over monoids (W.B. 1986). The $(\to, \leftarrow, \wedge)-$fragments are strongly complete with respect to powerset frames over free semigroups and monoids, respectively.

Without $\cdot$, one uses $f(A) = \{\Gamma : \Gamma \Rightarrow A$ provable $\}$. With $\cdot$, one employs a Labeled Deductive System; from $\Gamma \Rightarrow A \cdot B$ one infers $(\Gamma)_1^{A \cdot B} \Rightarrow A$ and $(\Gamma)_2^{A \cdot B} \Rightarrow B$.

L is weakly complete w.r.t. algebras of ϵ−free languages, and similarly for **L*** and algebras of languages (PENTUS 1993, 1996).

Proofs are involved; they employ the completeness w.r.t. special relation frames (see below) and some complexity measures of formulas.

NL is not weakly complete w.r.t. powerset frames over free groupoids (tree models) (DOSEN 1994). Soundness and completeness holds for some extensions of **NL** w.r.t. special classes of tree models (VENEMA 1994, 1996).

The $(\rightarrow, \leftarrow)$−fragment of **NL** is (strongly) complete with respect powerset frames over free groupoids (KANDULSKI 1988).

NL is (strongly) complete w.r.t. powerset frames over groupoids; this naturally extends to Generalized Lambek Calculus and powerset frames over abstract algebras (KOLOWSKA-GAWIEJNOWICZ 1996).

RELATION FRAMES

$P(U^2)$ (a square relation algebra) is a residuated monoid with:

$R \circ S = \{(x,y) \in U^2 : \exists z((x,z) \in R \text{ and } (z,y) \in S)\}$.
$R \to S = \{(x,y) \in U^2 : R \circ \{(x,y)\} \subseteq S\}$.
$S \leftarrow R = \{(x,y) \in U^2 : \{(x,y)\} \circ R \subseteq S\}$. $I_U = \{(x,x) : x \in U\}$.

Relativized frames $P(T)$, where T a transitive relation. In definitions of $R \to S$, $S \leftarrow R$ write $(x,y) \in T$. They are residuated semigroups.

L (also with $\wedge$) is strongly complete w.r.t. frames $P(T)$, where T is an irreflexive, transitive relation, and similarly for **L*** and frames $P(U^2)$. (ANDREKA and MIKULAS 1994).

Related results, e.g. for **NL**, total orderings T, and others (KURTONINA 1995, W.B. and KOLOWSKA-GAWIEJNOWICZ 1997, SZCZERBA 1998, W.B. 2003).

Some proofs employ Labeled Deductive Systems of GABBAY (1991); labeled formulas $(x,y) : A$.

REFINEMENTS

Strong completeness proofs, using a definition of a model $(\mathcal{M}, f)$ s.t. the sequents true in $(\mathcal{M}, f)$ are precisely the sequents provable in the logic, yield representation theorems: embedding abstract algebras in concrete algebras.

The strong completeness of **L** w.r.t. powerset frames over semigroups yields: every residuated semigroup can be embedded into $P(\mathcal{M})$, for some semigroup $\mathcal{M}$ (W.B. 1986, 1997). Similar facts hold for models of **L***, **NL** also with $\wedge$ and for the representation of residuated semigroups (monoids) in relation frames (the embedding weakly preserves identity: $1 \leq a$ iff $I_U \subseteq h(a)$) (ANDREKA and MIKULAS 1994, W.B. and KOLOWSKA-GAWIEJNOWICZ 1997).

The assignment $f(A) = \{\Gamma :\vdash \Gamma \Rightarrow A\}$ can be modified. Replace $\vdash \Gamma \Rightarrow A$ by $\vdash_T \Gamma \Rightarrow A$, which means that $\Gamma \Rightarrow A$ has a proof in which every formula belongs to T: a finite set of formulas, closed under subformulas. This yields: if $\vdash \Gamma \Rightarrow A$ is valid, then $\vdash_T \Gamma \Rightarrow A$.

In this way *the subformula property* can be proved for systems with nonlogical axioms and (CUT).

Another modification. $f(p) = \{\Gamma \in T^+ :\vdash_T \Gamma \Rightarrow p \text{ or } c(\Gamma) > n\}$, where $c(\Gamma)$ is a natural measure of complexity of Γ. One proves that for $(\to, \leftarrow, \wedge)-$formulas A, $f(A)$ is a co-finite language. This yields: (1) FMP for product-free **L**, **L***, BCI with $\wedge$, (2) the completeness of these systems w.r.t. regular languages (languages accepted by finite state automata). (W.B. 1982, 1997, 2002, 2006).

MALL is strongly complete w.r.t. phase-space models, which are powerset frames over commutative monoids with a designated subset $0 \subseteq M$. All operations of **MALL** are definable in terms of $\to, \wedge, 0$. This yields a faithful interpretation of **MALL** in BCI with $\wedge$ (0 is a designated variable). $I(p) = p \to 0$. Similarly, Cyclic **MALL** is faithfully interpretable in the product-free **L*** with $\wedge$, the constant 0 and the rule: from $\Gamma, \Delta \Rightarrow 0$ infer $\Delta, \Gamma \Rightarrow 0$. So for consequence relations.

From FMP of BCI with $\wedge$, proved by means of powerset frames, one obtains FMP of **MALL**, and so for Cyclic Noncommutative Linear Logic (W.B. 1996, 2002).

A direct proof of FMP of MALL: LAFONT (1996). For **FL** and other intuitionistic systems with $\cdot, \vee$: OKADA and TERUI (1999), BELARDINELLI, JIPSEN, ONO (2004).

GENERATIVE POWER AND COMPLEXITY

(1) CCGs are weakly equivalent to $\epsilon-$free CFGs (BGS 1960).

Every CCG is equivalent to a CFG. Straightforward: since **AB** always reduces types, then rules $A, A \to B \Rightarrow B$, $B \leftarrow A, A \Rightarrow B$ can be restricted to subtypes of types appearing in I_G.

Every $\epsilon-$free CFG is equivalent to a CCG; furthermore, the latter employs types of the form p, $p \leftarrow q$, $(p \leftarrow q) \leftarrow r$ only.

This is nontrivial. Actually, it is equivalent to the Greibach Normal Form theorem for CFGs: every $\epsilon-$free CFG is equivalent to a CFG with production rules of the form: $a \mapsto p$, $aq \mapsto p$, $arq \mapsto p$, where $a \in \Sigma$, p, q, r are variables (proved directly by S. Greibach 1967).

The proof in (BGS 1960) is combinatorial. A logical analysis is given in (W.B. 1988, 1996). **L** derives the types assigned by the CCG G' from the rules of the CFG G. $L(G') \subseteq L(G)$ follows from the strong soundness of **L** w.r.t. frames $P(\Sigma^+)$.

(2) **NL**-grammars are weakly equivalent to ϵ−free CFGs. (without product W.B. 1986, with product KANDULSKI 1988).

Every ϵ−free CFG is equivalent to some **NL**-grammar. Now it is an easy consequence of (1) and the fact that, for types restricted as above, a sequent $\Gamma \Rightarrow p$ is provable in **AB** iff it is provable in **NL** (also **L**, **L***, **FL** and so on); use cut elimination.

Every **NL**-grammar is equivalent to a CFG. Now it is nontrivial, since **NL** can expand types, e.g. $A \Rightarrow (B \leftarrow A) \rightarrow B$, $A \Rightarrow B \rightarrow (B \cdot A)$.

NL restricted to simple sequents $A \Rightarrow B$ can be axiomatized as a term rewriting system, based on rewriting rules, e.g. rewrite A on a positive position in C into $(B \leftarrow A) \rightarrow B$, and conversely for negative positions.

KEY LEMMA: $A_1, \ldots, A_n \Rightarrow B$ is provable in **NL** iff there exist $C_1, \ldots, C_n, C$ s.t. $A_i \Rightarrow C_i$ by reducing rules only, $C_1, \ldots, C_n \Rightarrow C$ by **AB**, and $C \Rightarrow B$ by expanding rules only.

This shows that every **NL**-grammar is equivalent to a CCG (which generates the same trees).

(3) **L**-grammars are weakly equivalent to $\epsilon-$free CFGs (PENTUS 1993).

Every **L**-grammar is equivalent to a CFG.

P a finite set of variables. $|A|$ the total number of variables in A. $T(P, m)$ the set of all types A on P s.t. $|A| \leq m$.

Binary Reduction Lemma: Let $A_1, \ldots, A_n \Rightarrow A_{n+1}$ be provable in **L**, $n \geq 2$, $A_i \in T(P, m)$, for all $i = 1, \ldots, n+1$. Then, there exist $k < n$ and $B \in T(P, m)$ s.t.:
(i) $A_k, A_{k+1} \Rightarrow B$ is provable in **L**,
(ii) $A_1, \ldots, A_{k-1}, B, A_{k+2}, \ldots, A_n \Rightarrow A_{n+1}$ is provable in **L**.

Consequently, if G is an **L**-grammar on P and m is the maximal A, for A appearing in I_G, then G is equivalent to a CFG whose production rules are all **L**-provable sequents $A \Rightarrow B$ and $A, B \Rightarrow C$, for $A, B, C \in T(P, m)$.

The proof of Binary Reduction Lemma is based on certain proof-theoretic properties of **L**.

$|\Gamma|_p$ the number of occurrences of p in Γ

(I) Interpolation Lemma (ROORDA 1991): Let $\Gamma, \Phi, \Delta \Rightarrow A$ be provable in **L**. Then, there exists a type B s.t.:
(i) $\Phi \Rightarrow B$ is provable in **L**,
(ii) $\Gamma, B, \Delta \Rightarrow A$ is provable in **L**,
(iii) for any variable p, $|B|_p \leq \min(|\Phi|_p, |\Gamma, \Delta, A|_p)$.

A type is *thin*, if each variable occurs at most once in it. A sequent is *thin*, if it is provable in **L**, each type in this sequent is thin, and each variable occurring in the sequent occurs twice in it.

(II) Every **L**-provable sequent is a substitution instance of an **L**-provable sequent in which each variable occurs twice.

PENTUS proves Binary Reduction Lemma for thin sequents, using free group models. By(I) and (II), it holds for arbitrary sequents.

The method is applicable for other multiplicative systems. PENTUS applied it to **L***, Cyclic **MLL** and Noncommutative **MLL**.

BULINSKA (2005) extended it to theories on **L** with finitely many assumptions of the form $p \Rightarrow q$.

OPEN PROBLEM: What about assumptions $p_1, \ldots, p_n \Rightarrow p$?

W.B. (1982) shows that even the $(\rightarrow)-$fragment of **L** with assumptions of the form $p, q \Rightarrow r$ and $p \rightarrow q \Rightarrow r$ is Σ_1^0-complete, and the corresponding grammars generate all $\epsilon-$free r.e. languages. This also holds for **L***.

The cardinality of $T(P, m)$ is exponential in the size of P and m. Accordingly, PENTUS's transformation of an **L**-grammar G into an equivalent CFG is exptime in the size of G.

By reducing SAT to the provability problems for **L**, **L***, Cyclic and Noncommutative **MLL**, PENTUS (2006) proves the NP-completeness of these problems.

The universal membership problem for CFGs is polytime. So, if one provided a polytime transformation of any **L**-grammar into an equivalent CFG, then she would prove P=NP.

(4) Again **NL**.

Interpolation Lemma (JÄGER 2004): Let $X[Y] \Rightarrow A$ be provable in **NL**. Then, there exists a type B s.t.:
(i) $Y \Rightarrow B$ is provable in **NL**,
(ii) $X[B] \Rightarrow A$ is provable in **NL**,
(iii) B is a subtype of a type occurring in $X[Y] \Rightarrow A$.

This yields a new proof of the context-freeness of **NL**-grammars.

The provability problem for **NL** is polytime (AARTS 1995 without product, DE GROOTE 2002 with product). So, we get a polytime transformation of any **NL**-grammar into an equivalent CFG (but not into a strongly equivalent CCG).

(W.B. 2005) uses this kind of interpolation to prove that: (i) the consequence relation for **NL** is polytime, (ii) grammars based on finite theories on **NL** are context-free. This holds for **NL** with (EXC), with modalities, and Generalized Lambek Calculus. BULINSKA (2006) extends these results for **NL** with 1.

BACK TO LOGIC

FARULEWSKI (2006) uses interpolation (of the second kind) to prove Finite Embeddability Property (FEP) for residuated groupoids (an open problem in BLOK and VAN ALTEN 2005).

Residuated groupoids are closed under products, whence FEP is equivalent to Strong FMP (FMP for Horn formulas). The Horn theory of residuated groupoids is represented by the consequence relation for **NL**.

T a finite set of formulas, closed under subformulas, which contains all formulas appearing in assumptions. We only consider formulas from T and sequents formed out of these formulas.

$X \sim Y$ iff, for all $A \in T$, $X \Rightarrow A$ is provable iff $Y \Rightarrow A$ is provable. This equivalence relation has a finite index.

LEMMA: If $X \sim Y$, then, for any context $Z[\cdot] \Rightarrow A$ on T, $Z[X] \Rightarrow A$ is provable iff $Z[Y] \Rightarrow A$ is provable. (The proof uses interpolation).

M is the set of all trees X on T (a free groupoid).
$S(Z[\cdot] \Rightarrow A) = \{X \in M :\vdash Z[X] \Rightarrow A\}$. A closure operator on $P(M)$:
$C(U) = \bigcap\{S(Z[\cdot] \Rightarrow A) : U \subseteq S(Z[\cdot] \Rightarrow A)\}$. Every closed set is a union of some equivalence classes of $\sim$, whence the family of closed sets is finite.

The consequence relations for BCI with $\wedge$ and BCI with $\vee$ are undecidable (W.B. 2006).

Propositional Linear Logic (with exponentials) is undecidable. This also holds for its fragment restricted to sequents with only negative occurrences of formulas $!A$, where A contains no exponential (no occurrences of ?, par, and lattice bounds) (LINCOLN et al. 1992, KANOVICH 1995).

Negative occurrences of $!A$ can be replaced by assumptions:
$p_A \Rightarrow A$; $p_A \Rightarrow 1$; $p_A \Rightarrow p_A \otimes p_A$, corresponding to:
$!A \Rightarrow A$; $!A \Rightarrow 1$; $!A \Rightarrow !A \otimes !A$, provable in **PLL**. So, the consequence relation for **MALL** is undecidable. By the interpretation of **MALL** in BCI with $\wedge$, mentioned above, our claim holds (for $\vee$, we use the particular form of KANOVICH sequents).

The problem for BCI remains open (like the undecidability of **MELL**). There is some natural connection with linguistics.

The emptiness problem for CFGs: $L(G) \neq \emptyset$, for a CFG G.

This problem is decidable. As a consequence, we obtain the decidability of the following problem: for a finite set of formulas T and a formula A, decide whether there exists $\Gamma \in T^*$ s.t. $\Gamma \Rightarrow A$ is provable in **L***. It follows from the PENTUS theorem for **L***.

The order of a $(\rightarrow)$−formula: $o(p) = 0$, $o(A \rightarrow B) = \max(o(A) + 1, o(B))$. For instance, $o(p \rightarrow q) = 1$, $o(p \rightarrow (q \rightarrow r)) = 1$, $o((p \rightarrow q) \rightarrow r) = 2$.

BCI-grammars provide languages closed under permutations. Grammars whose all types are of order at most 1 and the designated type is a variable generate all permutation-closures of CF-languages (W.B. 1983, VAN BENTHEM 1983); they are not CF, in general. Nonetheless, the emptiness problem for BCI-grammars of order at most 1 is decidable.

$\Rightarrow A$ is a consequence of assumptions $\Rightarrow A_1, \ldots, \Rightarrow A_n$ in BCI iff there exists a sequence Γ on $\{A_1, \ldots, A_n\}$, s.t. $\Gamma \Rightarrow A$ is provable in BCI (the deduction theorem).

So, $\Rightarrow A$ is a consequence of these assumptions iff $L(G) \neq \emptyset$, where G is an appropriate BCI-grammar. Assumptions can be reduced to formulas of order at most 2. Then, the entailment problem for BCI is equivalent to the emptiness problem for BCI-grammars of order at most 2.

KLEENE STAR

Kleene algebra: $\mathcal{M} = (M, \vee, \cdot, ^*, 0, 1)$ s.t.

$(M, \vee, 0)$ a join semilattice with the lower bound 0:

$(a \vee b) \vee c = a \vee (b \vee c)$, $a \vee b = b \vee a$, $a \vee a = a$, $a \vee 0 = a$,

$(M, \cdot, 1)$ is a monoid: $(ab)c = a(bc)$, $1a = a = a1$,

product $\cdot$ distributes over join $\vee$, and 0 is an annihilator:

$a(b \vee c) = ab \vee ac$, $(a \vee b)c = ac \vee bc$, $a0 = 0 = 0a$,

and * satisfies the following conditions:

(K1) $1 \vee aa^* \leq a^*$, $\quad 1 \vee a^*a \leq a^*$,

(K2) if $ab \leq b$ then $a^*b \leq b$; $\quad$ if $ba \leq b$ then $ba^* \leq b$,

where: $a \leq b$ iff $a \vee b = b$.

KOZEN (1990, 1994)

An action algebra is a residuated Kleene algebra (PRATT 1991). With $\wedge$: action lattice.

KA the class of Kleene algebras, ACTA the class of action algebras, ACTL the class of action lattices.

Standard models: powerset frames over monoids, in particular $P(\Sigma^*)$, relation frames $P(U^2)$.

They are *-continuous: $xa^*y = \bigvee\{xa^n y : n \in \omega\}$.

KA* the class of *-continuous Kleene algebras, and similarly for other classes. LAN_K the class of Kleene algebras $P(\Sigma^*)$, REG_K the class of their subalgebras consisting of regular languages, REL_K the class of Kleene algebras $P(U^2)$.

The Kozen completeness theorem: $L(\alpha) = L(\beta)$ iff $\alpha = \beta$ is valid in KA (α, β regular expressions).

Consequences: Eq(KA)=Eq(KA*)=Eq(LAN_K)=Eq(REG_K)=Eq(REL_K). This theorey is decidable (PS-complete).

KA is a qusi-variety, not a variety. Eq(KA) is not finitely axiomatizable.

PRATT (1991) shows that ACTA and ACTL are finitely based varieties. For regular expressions α, β, $\alpha = \beta$ is valid in KA iff it is valid in ACTA (ACTL).

JIPSEN (1994) axiomatizes Eq(ACTL) in the form of a sequent system: **FL** with 0 (the lower bound) and 1 plus certain axioms and rules for *. This system does not admit cut elimination. The decidability of Eq(ACTA) and Eq(ACTL) remains open.

W.B. (2007) axiomatizes Eq(ACTL*) as an extension of **FL** by the following rules for *:

$$(*L)\ \frac{(\Gamma, A^n, \Delta \Rightarrow B)_{n\in\omega}}{\Gamma, A^*, \Delta \Rightarrow B},\ (*R)\ \frac{\Gamma_1 \Rightarrow A; \ldots ; \Gamma_n \Rightarrow A}{\Gamma_1, \ldots, \Gamma_n \Rightarrow A^*}$$

(*L) is an $\omega-$rule. This system is denoted ACTω.

The total language problem $L(G) = \Sigma^*$ for CFGs is Π^0_1-complete. Using the theorem from (BGS 1960) we show that the problem $L(G) = \Sigma^+$ for CCGs is also Π^0_1-complete. The latter is reducible to the provability problem for ACTω, whence Eq(ACTL*) is Π^0_1-hard, and the same holds for Eq(ACTA*). Consequently, Eq(ACTA*) is strictly contained in Eq(ACTA), and similarly for Eq(ACTL*) and Eq(ACTL).

PALKA (2007) proves cut elimination for ACTω and a theorem on the elimination of negative occurrences of *, which shows that Eq(ACTL*) is Π^0_1.

OTHER TOPICS

(1) Proof nets

A graph-theoretic representation of proofs in multiplicative fragments of substructural logics, introduced by GIRARD (1987) for **MLL**. For noncommutative logics, they are planar graphs. Linguists use them to represent semantic structures of expressions (C. RETORE, P. DE GROOTE, G. PENN, G. MORRILL, R. MOOT). BECHET (2007) gives a new proof of the PENTUS theorem for **L*** (context-freeness), applying proof nets. PENTUS (2006) uses proof nets in his proof of NP-completeness of **L** and related systems.

(2) Bilinear Logic and pregroups

L* is a conservative fragment of Bilinear Logic **BL**; the latter is the multiplicative fragment of both Noncommutative and Cyclic **MLL**. Some authors use **BL** rather than **L*** or **L** as a logic for grammars (V.M. ABRUSCI, C. CASADIO). LAMBEK (1999) introduces a simplified formalism, called Compact Bilinear Logic **CBL**, which arises from **BL** by identifying $\otimes$ and its dual 'par'. Models of **CBL** are called pregroups. This system is essentially stronger than **BL** and incompatible with intuitionistic and classical logics. (W.B., D. BECHET, C. CASADIO, A. PRELLER, N. FRANCEZ, M. KAMINSKI, A. KISLAK-MALINOWSKA)

(3) Unification-based learning

W.B. and PENN (1990) apply the method of unification to design some learning algorithms for categorial grammars (earlier W.B. 1987, VAN BENTHEM 1987). KANAZAWA (1996, 1998) develops and studies these algorithms in direction of Gold's paradigm: learning from positive data. Several authors have obtained interesting results, e.g. D. BECHET, A. FORET, J. MARCINIEC, C. RETORE, C. COSTA FLORENCIO, B. DZIEMIDOWICZ.

(4) Type-theoretic approaches

Many authors prefer to use richer formalisms, e.g. different versions of typed lambda-calculus or higher-order intensional logic. STEEDMAN (1988), following H.B. CURRY, applies certain systems of combinators. RANTA (1994) develops the 'formulas-as-types' paradigm as a type-theoretic grammar. DE GROOTE (2001) introduces Abstract Categorial Grammars, in which both syntactic and semantic structures are represented by lambda-terms, and the two levels are linked by a homomorphism.

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Handbook of Logic and Language, J. van Benthem and A. ter Meulen, eds., Elsevier, The MIT Press, 1997.