# Priestley Spaces of Free MV-algebras

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# Introduction


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# Introduction


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## Definition

A function

$$f : [0,1]^n \longrightarrow [0,1]$$

is called a **McNaughton function** of $n$ variables iff it satisfies the following conditions:

1. $f$ is continuous,
2. there are linear polynomials $p_1, \ldots, p_k$ such that for each $1 \leq i \leq k$. $b_i, m_i \in \mathbb{Z}$ and for each $x \in [0,1]^n$ there is and index $j$ with $f(x) = p_j(x)$

# Introduction

**Theorem**

*The $\mathrm{Free}_n(\mathcal{MV})$ is isomorphic to the MV-algebra of McNaughton functions of n variables.*

# Lattice prime filters of $Free_n(\mathcal{MV})$

Let $v \in [0,1]^n$ and $\mathbf{B} = (v_1, \ldots, v_n)$ an orthonormal base of $\mathbb{R}^n$ satisfying that there are $\delta_1, \ldots, \delta_n$ such that

$$\{v + \delta_1 v_1, \ldots, v + \delta_1 v_1 + \ldots + \delta_{n-1} v_{n-1} + \delta_n v_n\} \subseteq [0,1]^n .$$

Let us define

$$\varphi_{v,\mathbf{B}} : Free_n(\mathcal{MV}) \to \ [0,1] \times \mathbb{R}^n$$

$$\varphi_{v,\mathbf{B}}(f) = \left( f(v), \frac{\partial f}{\partial v_1}(v), \frac{\partial f}{\partial v_2}(v + \varepsilon_1 v_1), \ldots, \frac{\partial f}{\partial v_n}(v + \varepsilon_1 v_1 + \ldots + \varepsilon_{n-1} v_{n-1}) \right)$$

Where $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{R}_{>0}$ are such that $f$ is linear in the convex hull of

$$\{v, v + \varepsilon_1 v_1, \ldots, v + \varepsilon_1 v_1 + \ldots + \varepsilon_{n-1} v_{n-1} + \varepsilon_n v_n\}$$

# Lattice prime filters of $Free_n(\mathcal{MV})$

**Lemma**

*If we consider $[0,1] \times \mathbb{R}^n$ oredered lexicographically thus*

$$\varphi_{v,\mathbf{B}} : Free_n(\mathcal{MV}) \to \quad [0,1] \times \mathbb{R}^n$$

*is a lattice homorphism.*

Every lattice prime filter of $[0,1] \times \mathbb{R}^n$ has the form of

$$S_{(b_0,\ldots,b_m)} = \{(a_0, a_1, \ldots, a_n) : (b_0, \ldots, b_m) \leq (a_0, \ldots, a_m)\}$$

for some $0 \leq m \leq n$ and some $(b_0, \ldots, b_m) \in (0,1] \times \mathbb{R}^n$ or

$$S^+_{(b_0,\ldots,b_m)} = \{(a_0, a_1, \ldots, a_n) : (b_0, \ldots, b_m) < (a_0, \ldots, a_m)\}$$

for some $0 \leq m \leq n$ and some $(b_0, \ldots, b_m) \in [0,1] \times \mathbb{R}^n$.

# Lattice prime filters of $Free_n(\mathcal{MV})$

**Theorem**

*For every lattice prime filter $P$ of $Free_n(\mathcal{MV})$ there exist $v \in [0,1]^n$ and $\mathbf{B} = (v_1, \ldots, v_n)$ such that*

$$P = \varphi^{-1}_{v,\mathbf{B}}(S)$$

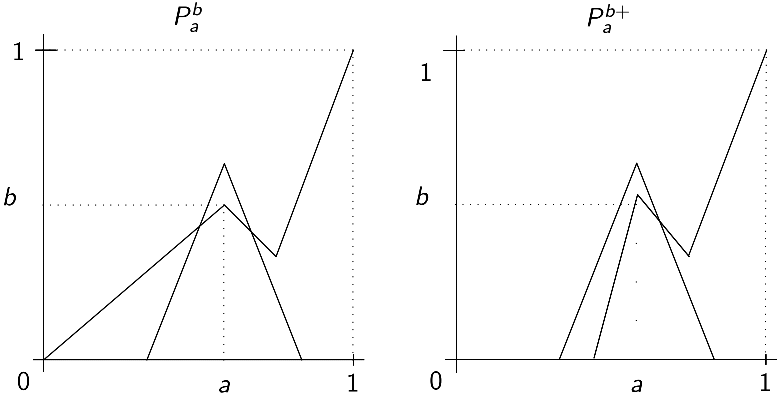
*for some lattice prime filter $S$ of $[0,1] \times \mathbb{R}^n$.*

# Lattice prime filters of $Free_1(\mathcal{MV})$

Given $a, b \in [0,1]$ the following sets are prime filters of $Free_1(\mathcal{MV})$

a. $P_a^b = \varphi_{a,\mathbf{B}}^{-1}(S_b) = \{f \in Free_1(\mathcal{MV}) : f(a) \geq b\}$

b. $P_a^{b^+} = \varphi_{a,\mathbf{B}}^{-1}(S_b^+) = \{f \in Free_1(\mathcal{MV}) : f(a) > b\}$

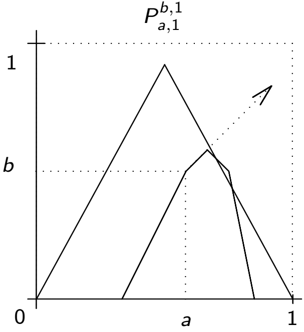


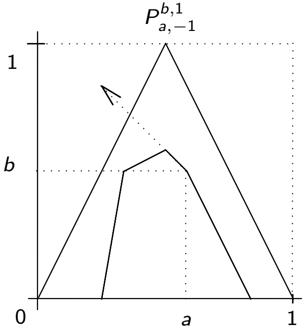
# Lattice prime filters of $Free_1(\mathcal{MV})$

Given $a, b \in [0,1]$ and $m \in \mathbb{Z}$, the following sets are prime filters of $Free_1(\mathcal{MV})$

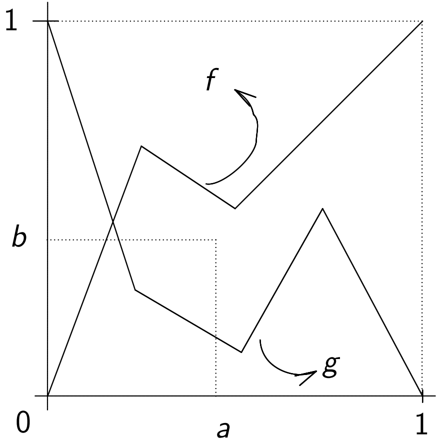
a. $P_{a,1}^{b,m} = \varphi_{a,(1)}^{-1}(S_{(b,m)}) =$
$\{f \in Free_1(\mathcal{MV}) : \exists \varepsilon > 0, \forall 0 < \partial < \varepsilon,\ f(a+\partial) \geq b + m\partial\}$

b. $P_{a,-1}^{b,m} = \varphi_{a,(-1)}^{-1}(S_{(b,m)}) =$
$\{f \in Free_1(\mathcal{MV}) : \exists \varepsilon > 0, \forall 0 < \partial < \varepsilon,\ f(a-\partial) \geq b + m\partial\}$



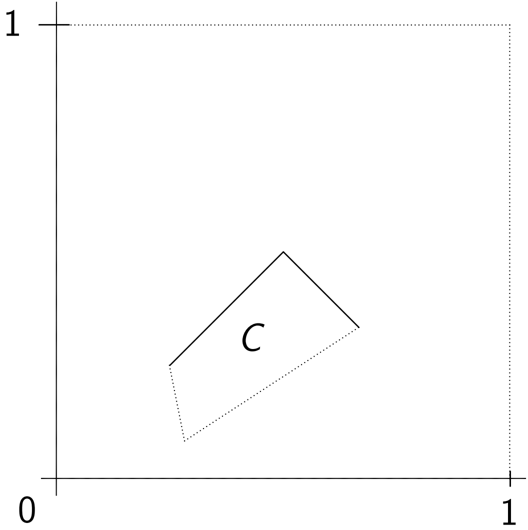



# Open sets of $\mathcal{X}\left(Free_1\left(\mathcal{MV}\right)\right)$




$$P_a^b \in \sigma\left(f\right) \cap \sigma\left(g\right)^c$$

# Open sets of $\mathcal{X}\left(Free_1\left(\mathcal{MV}\right)\right)$



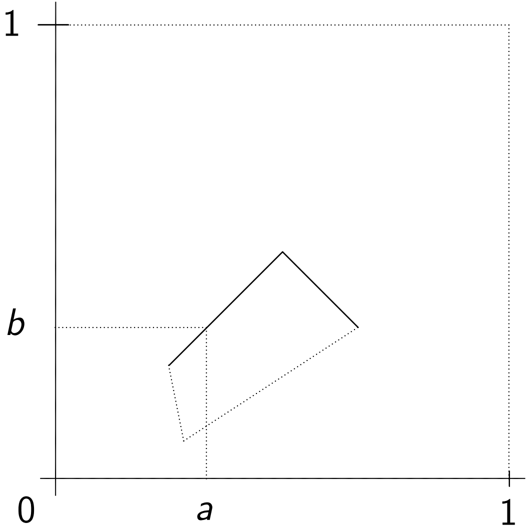
# Open sets of $\mathcal{X}(Free_1(\mathcal{MV}))$

If we define $C^* \subseteq \mathcal{X}(Free_1(\mathcal{MV}))$ by

1. $P_a^b \in C^*$ iff $(a, b) \in C$,
2. $P_a^{b^+} \in C^*$ iff $(a, b) \in int(C)$,
3. $P_{a,1}^{b,m} \in C^*$ iff $\exists\ \varepsilon > 0, \forall 0 < \delta < \varepsilon$, $(a + \delta, b + m\delta) \in C$,
4. $P_{a,-1}^{b,m} \in C^*$ iff $\exists \varepsilon > 0, \forall 0 < \delta < \varepsilon$, $(a - \delta, b + m\delta) \in C$,

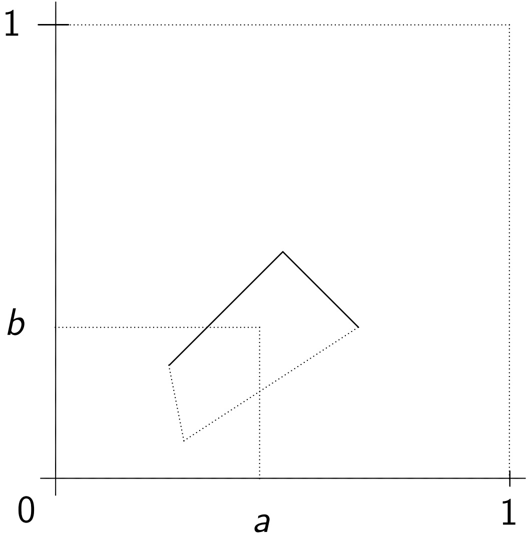
then the family of $C^*$ is a base for the topology of $\mathcal{X}(Free_1(\mathcal{MV}))$.

# Open sets of $\mathcal{X}\left(Free_1\left(\mathcal{MV}\right)\right)$



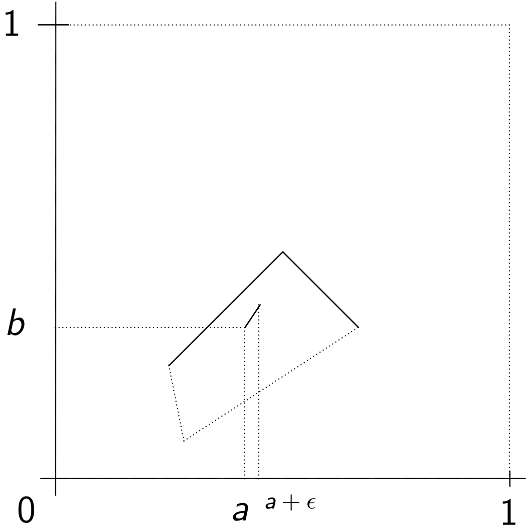

$P_a^b \in C^*$

## Open sets of $\mathcal{X}\left(Free_1\left(\mathcal{MV}\right)\right)$



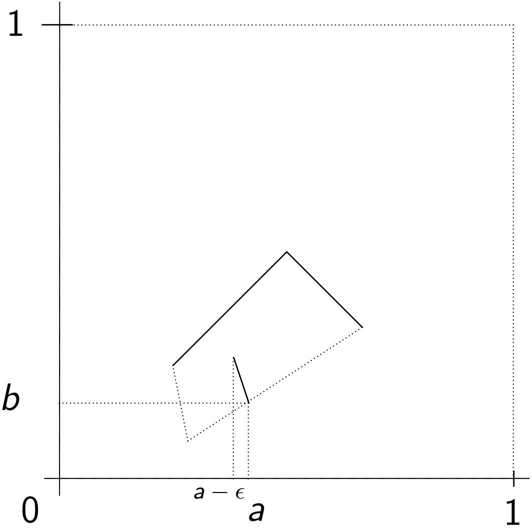

$P_a^{b^+} \in C^*$

# Open sets of $\mathcal{X}(Free_1(\mathcal{MV}))$




$P_{a,1}^{b,m} \in C^*$

# Open sets of $\mathcal{X}\left(Free_1\left(\mathcal{MV}\right)\right)$




$$P^{b,m}_{a,-1} \in C^*$$

# Different dualities

| **DLI**-algebras | Implicative Lattices |
|---|---|
| $a \rightarrow (b \wedge c) \approx (a \rightarrow b) \wedge (a \rightarrow c)$ | $a \rightarrow (b \wedge c) \approx (a \rightarrow b) \wedge (a \rightarrow c)$ |
| $(a \vee b) \rightarrow c \approx (a \rightarrow c) \wedge (b \rightarrow c)$ | $(a \vee b) \rightarrow c \approx (a \rightarrow c) \wedge (b \rightarrow c)$ |
| $a \rightarrow 1 \approx 1$ | $a \rightarrow (b \vee c) \approx (a \rightarrow b) \vee (b \rightarrow c)$ |
| $0 \rightarrow a \approx 1$ | $(a \wedge b) \rightarrow c \approx (a \rightarrow c) \vee (b \rightarrow c)$ |

# Different dualities

| **DLI**-algebras | Implicative Lattices | |
|---|---|---|
| Celani $T_{\mathbf{A}} \subseteq \mathcal{X}(\mathbf{A})^3$ | Martinez | $\Phi : \mathcal{S}(\mathbf{A}) \times \mathcal{S}(\mathbf{A}) \to \mathcal{S}(\mathbf{A})$ |
| | Martinez-Priestley | $\mu : \mathcal{D}(\mathcal{S}(\mathbf{A})) \times \mathcal{S}(\mathbf{A}) \to \mathcal{S}(\mathbf{A})$ |

Where $\mathcal{S}(\mathbf{A}) = \mathcal{X}(\mathbf{A}) \cup \{\emptyset, A\}$.

$(P, Q, D) \in T_{\mathbf{A}}$ Iff $\{a \in A : \exists (f, g) \in P \times Q,\ f \leq g \to a\} \subseteq D$

$$\Phi(P, Q) = \bigcup_{x \in P} \{y : x \to y \in Q\}$$

$$\mu(\beta(a), P) = \{x \in A : x \to a \notin P\}$$

If **A** is a **DLI**-algebra

$$\mathbf{A} \models a \to (b \vee c) \approx (a \to b) \vee (b \to c)$$
iff
$$(P, Q, D), (P, Z, W) \in T_{\mathbf{A}} \text{ implies}$$
$$\exists K,\ K \subseteq D \cap W \text{ and } ((P, Q, K) \in T_{\mathbf{A}} \text{ or } (P, Z, K) \in T_{\mathbf{A}})$$

$$\mathbf{A} \models (a \wedge b) \to c \approx (a \to c) \vee (b \to c)$$
iff
$$(P, Q, D), (P, Z, W) \in T_{\mathbf{A}} \text{ implies}$$
$$\exists H,\ Q, Z \subseteq H \text{ and } ((P, H, D) \in T_{\mathbf{A}} \text{ or } (P, H, W) \in T_{\mathbf{A}}).$$

# Different dualities

If **A** is a bounded Implicative Lattice

$$\mathbf{A} \vDash a \to 1 = 1 \text{ iff } \quad \begin{array}{l} \forall P, Q \in \mathcal{X}(\mathbf{A}),\ \Phi(P, Q) \neq \emptyset\ . \\ \forall P \in \mathcal{S}(\mathbf{A}) \setminus \{\emptyset\},\ \mu(\beta(1), P) = \emptyset. \end{array}$$

$$\mathbf{A} \vDash 0 \to a = 1 \text{ iff } \quad \begin{array}{l} \forall Q \in \mathcal{X}(\mathbf{A}),\ \Phi(A, Q) = A. \\ \forall P \in \mathcal{S}(\mathbf{A}) \setminus \{\emptyset\}\ ,\ \mu(\beta(0), P) \neq A. \end{array}$$

# Different dualities

If $\mathbf{A}$ is a **DLI**-algebra and an Implicative Lattice, $\Phi$ can be described using the ternary relation $T_{\mathbf{A}}$ by:

$$\Phi(P,Q) = \begin{cases} A & \text{if} \quad P = A \text{ and } Q \neq \emptyset, \\ A & \text{if} \quad Q = A \text{ and } P \neq \emptyset, \\ \emptyset & \text{if} \quad P = \emptyset \text{ or } Q = \emptyset, \\ D & \text{if} \quad P, Q \in \mathcal{X}(\mathbf{A}) \text{ and } T_{\mathbf{A}}(P,Q) = [D). \end{cases}$$

If **A** is a **DLI**-algebra and an Implicative Lattice, $\mu$ can be described using the ternary relation $T_{\mathbf{A}}$ by:

$$\mu\left(\beta\left(a\right), P\right) = \begin{cases} \emptyset & \text{if} \quad P = A, \\ A & \text{if} \quad P = \emptyset, \\ D & \text{if} \quad P \in \mathcal{X}\left(A\right) \text{ and } . \\ & \quad\ \ (D] = \left\{Q \in \mathcal{X}\left(\mathbf{A}\right) : T\left(P, Q\right) \cap \beta^{c}\left(a\right) \neq \emptyset\right\} \end{cases}$$

# Different dualities

If **A** is a **DLI**-algebra and an Implicative Lattice, $T_{\mathbf{A}}$ can be obtained by:

$$(P, Q, D) \in T_{\mathbf{A}} \text{ iff } \quad \Phi(Q, P) \subseteq D.$$
$$D \in \bigcap \{\beta_A(a) : a \in A \text{ and } \mu(\beta(a), P)\}.$$

# Ternary relation $T_{Free_1(\mathcal{MV})}$

$(P, Q, D) \in T_{Free_1(\mathcal{MV})}$ if one of the following propositions holds

| $P$ | $Q$ | Conditions | | $D$ |
|---|---|---|---|---|
| $P_a^b$ | $P_a^c$ | $1 < b + c$ | | $P_a^{b \odot c} \subseteq D$ |
| $P_a^b$ | $P_a^{c^+}$ | $1 \leq b + c$ | | $P_a^{b \odot c^+} \subseteq D$ |
| $P_a^b$ | $P_{a,1}^{c,m}$ | $1 < b + c$ | | $P_a^{b \odot c} \subseteq D$ |
| $P_a^b$ | $P_{a,-1}^{c,m}$ | $1 < b + c$ | | $P_a^{b \odot c} \subseteq D$ |
| $P_a^{b^+}$ | $P_a^{c^+}$ | $1 \leq b + c$ | | $P_a^{b \odot c^+} \subseteq D$ |
| $P_a^{b^+}$ | $P_{a,1}^{c,m}$ | $1 \leq b + c$ | | $P_a^{b \odot c^+} \subseteq D$ |
| $P_a^{b^+}$ | $P_{a,-1}^{c,m}$ | $1 \leq b + c$ | | $P_a^{b \odot c^+} \subseteq D$ |
| $P_{a,1}^{b,m}$ | $P_{a,1}^{c,k}$, | $1 < b + c$ | | $P_{a,1}^{b \odot c, m+k} \subseteq D$ |
| $P_{a,1}^{b,m}$ | $P_{a,1}^{c,k}$ | $1 = b + c$ | $0 < m + k$ | $P_{a,1}^{b \odot c, m+k} \subseteq D$ |
| $P_{a,1}^{b,m}$ | $P_{a,-1}^{c,k}$ | $1 < b + c$ | | $P_a^{b \odot c} \subseteq D$ |

# Summarizing

- $\mathcal{X}\left(Free_n\left(\mathcal{MV}\right)\right)$
- The family of $C^*$ is a basis for the topology of $\mathcal{X}\left(Free_1\left(\mathcal{MV}\right)\right)$.
- We have compared the Priestley dualities for Implicative Lattices and **DLI**-algebras.
- We have decribed the ternary relation $T_{Free_1(\mathcal{MV})} \subseteq \mathcal{X}\left(Free_1\left(\mathcal{MV}\right)\right)^3$.

# The end

THANK YOU !

Any questions?