Structural rules in FL: expressive power and cut elimination.

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Overview

- FL and substructural logics
- Algebraic semantics: residuated lattices and FL-algebras
- Structural rules
- Cut elimination
- Expressive power
- Generating analytic calculi from FL + suitable axioms

The system FL

$$\frac{\Pi \Rightarrow \alpha \quad \Gamma, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \text{ (cut)} \qquad \frac{}{\alpha \Rightarrow \alpha} \text{ (Id)}$$

The system FL

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$$\frac{\Gamma, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \Psi} \text{ (}\wedge\text{L}\ell\text{)} \quad \frac{\Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, a \wedge \beta, \Delta \Rightarrow \Psi} \text{ (}\wedge\text{L}r\text{)} \quad \frac{\Pi \Rightarrow \alpha \quad \Pi \Rightarrow \beta}{\Pi \Rightarrow \alpha \wedge \beta} \text{ (}\wedge\text{R)}$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \Psi \quad \Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \Psi} \text{ (}\vee\text{L)} \quad \frac{\Pi \Rightarrow \alpha}{\Pi \Rightarrow \alpha \vee \beta} \text{ (}\vee\text{R}\ell\text{)} \quad \frac{\Pi \Rightarrow \beta}{\Pi \Rightarrow \alpha \vee \beta} \text{ (}\vee\text{R}r\text{)}$$

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$$\frac{\Gamma, \alpha, \Delta \Rightarrow \Psi \quad \Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \Psi} \ (\vee \mathrm{L}) \quad \frac{\Pi \Rightarrow \alpha}{\Pi \Rightarrow \alpha \vee \beta} \ (\vee \mathrm{R}\ell) \quad \frac{\Pi \Rightarrow \beta}{\Pi \Rightarrow \alpha \vee \beta} \ (\vee \mathrm{R}r)$$

$$\frac{\Pi \Rightarrow \alpha \quad \Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, \Pi, (\alpha \backslash \beta), \Delta \Rightarrow \Psi} \ (\backslash \mathrm{L}) \qquad \frac{\alpha, \Pi \Rightarrow \beta}{\Pi \Rightarrow \alpha \backslash \beta} \ (\backslash \mathrm{R})$$

$$\frac{\Pi \Rightarrow \alpha \quad \Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, (\beta / \alpha), \Pi, \Delta \Rightarrow \Psi} \ (/\mathrm{L}) \qquad \frac{\Pi, \alpha \Rightarrow \beta}{\Pi \Rightarrow \beta / \alpha} \ (/\mathrm{R})$$

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$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Psi}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \Psi}\ (\cdot \mathsf{L}) \qquad \frac{\Pi \Rightarrow \alpha \quad \Sigma \Rightarrow \beta}{\Pi, \Sigma \Rightarrow \alpha \cdot \beta}\ (\cdot \mathsf{R})$$

$$\frac{\Gamma, \Delta \Rightarrow \Psi}{\Gamma, 1, \Delta \Rightarrow \Psi}\ (1\mathsf{L}) \qquad \frac{}{\Rightarrow 1}\ (1\mathsf{R}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0\mathsf{R}) \qquad \frac{}{0 \Rightarrow}\ (0\mathsf{L})$$

Basic structural rules

Letters α, β denote formulas in the language $\{\wedge, \vee, \backslash, /, \cdot, 1, 0\}$; Γ, Σ, Π denote *sequences* of formulas, and Ψ denotes either a formula or the empty set.

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$$\frac{\Gamma, \Sigma \Rightarrow \Psi}{\Gamma, \alpha, \Sigma \Rightarrow \Psi} \text{ (i)} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \Psi} \text{ (o)} \qquad \text{(w)} = \text{(i)} + \text{(o)}$$

The rules exchange (e), contraction (c), left (i) and right (o) weakening are called *structural*.

The system $\mathbf{FL}$ *full Lambek calculus* is obtained from $\mathbf{LJ}$ by removing all structural rules and adding rules for $\cdot$, $\backslash$, $/$, 1, 0.

Substructural logics

We write $\Phi \vdash_{\mathbf{FL}} \psi$, if the sequent $\Rightarrow \psi$ is provable in $\mathbf{FL}$ from the set of sequents $\{(\Rightarrow \phi) | \phi \in \Phi\}$.

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Examples:

- Classical,
- intuitionistic,
- many-valued (Łukasiewicz),
- basic (Hajek),
- relevance (Anderson, Belnap),
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An equivalent Hilbert-style system has inference rules

$$\frac{\phi \quad \phi\backslash\psi}{\psi}\ \text{(mp)} \qquad \frac{\phi \quad \psi}{\phi \wedge \psi}\ \text{(adj)} \qquad \frac{\phi}{\psi\backslash\phi\psi}\ \text{(n)} \qquad \frac{\phi}{\psi\phi/\psi}\ \text{(n)}$$

Residuated lattices

A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that

- $\langle L, \wedge, \vee\rangle$ is a lattice,
- $\langle L, \cdot, 1\rangle$ is a monoid and
- for all $a, b, c \in L$, $ab \leq c \Leftrightarrow a \leq c/b \Leftrightarrow b \leq a\backslash c$.

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N. Galatos, P. Jipsen, T. Kowalski and H. Ono. Residuated Lattices: an algebraic glimpse at substructural logics, Studies in Logics and the Foundations of Mathematics, Elsevier, 2007.

Structural rules

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha, \Delta \Rightarrow \Psi}\ (c) \qquad \frac{\Gamma, \Pi, \Pi, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi}\ (seq\text{-}c)$$

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A structural rule of the form

$$\frac{\Upsilon_1 \Rightarrow \quad \ldots \quad \Upsilon_k \Rightarrow \quad \Upsilon'_1 \Rightarrow \delta_1 \quad \ldots \quad \Upsilon'_m \Rightarrow \delta_m \quad \Upsilon''_1 \Rightarrow \Psi_1 \quad \ldots \quad \Upsilon''_n \Rightarrow \Psi_n}{\Upsilon_0 \Rightarrow \Psi_0(\delta_0)}$$

is called *separated*, if $\Upsilon_0, \ldots, \Upsilon''_n$ are sequences of metavariables, $\Psi, \Psi_1, \ldots, \Psi_n$ range over formulas and the empty set, and $\delta_0, \ldots, \delta_m$ range over formulas that do not appear in $\Upsilon_0, \ldots, \Upsilon''_n$.

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$$I(\alpha_1, \dots, \alpha_n \Rightarrow \delta) = (\alpha_1 \cdot \dots \cdot \alpha_n \leq \delta)$$

$$I(\alpha_1, \dots, \alpha_n \Rightarrow \ ) = (\alpha_1 \cdot \dots \cdot \alpha_n \leq 0)$$

$$I(\tfrac{s_1 \ \dots \ s_n}{s}) = (I(s_1) \& \dots \& I(s_n) \Longrightarrow I(s))$$

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$$I(\frac{s_1 \ \ldots \ s_n}{s}) = (I(s_1) \& \ldots \& I(s_n) \Longrightarrow I(s))$$

Lemma The interpretation of a separated structural rule is equivalent, over the theory of FL, to an equation.

Separated rules

Consider the separated structural rule

$$\frac{\alpha, \gamma, \alpha \Rightarrow \quad \beta, \gamma, \beta \Rightarrow \quad \Gamma, \gamma, \alpha, \phi, \beta, \gamma, \Delta \Rightarrow \Psi}{\Gamma, \gamma, \beta, \phi, \alpha, \gamma, \Delta \Rightarrow \Psi}$$

Its interpretation is equivalent to the quasiequation

$$aca \leq 0 \text{ and } bcb \leq 0 \text{ and } cafbc \leq d \Longrightarrow cbfac \leq d$$

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For the choice of variables c for aca, b for bcb and f for $cafbc$ we obtain the equation

$$c'b'f'ac' \leq d$$

where $c' = c \wedge a\backslash 0/a$, $b' = b \wedge 0/cb$ and $f' = f \wedge ca\backslash d/bc$.

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For the choice of variables c for aca, b for bcb and f for $cafbc$ we obtain the equation

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where $c' = c \wedge a\backslash 0/a$, $b' = b \wedge 0/cb$ and $f' = f \wedge ca\backslash d/bc$.
Alternatively, for the choice of variables c for aca and c for bcb we obtain the equation

$$c'bf'ac' \leq d$$

where $c' = c \wedge a\backslash 0/a \wedge b\backslash 0/b$ and $f' = f \wedge ca\backslash d/bc$.

Separated equations

For a set of variables V, we define the set of *separated* formulas (or terms) $sep(V)$ as the smallest set such that

1. $\{0, \top\} \cup V \subseteq sep(V)$, (if $\top$ is in the language),
2. if $t_1, t_2 \in sep(V)$, then $t_1 \wedge t_2 \in sep(V)$,
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A substitution σ is called *separated*, relative to V, if there are variables $x_1, \ldots, x_n$ not in V and terms $t_1, \ldots, t_n \in sep(V)$ such that $\sigma(x_i) = x_i \wedge t_i$, for all i, and σ fixes all other variables.

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An equation is called *separated*, if it is of the form $\sigma(t) \leq z$, where σ is a separated substitution, $t \in sep(V)$ and $z \in V$.

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Theorem. (Sets of) separated structural rules correspond to (Sets of) separated equations.

Simple rules

A substructural rule is called *simple* if it is of one of the forms

$$\frac{\Upsilon_1' \Rightarrow \quad \dots \quad \Upsilon_n' \Rightarrow \quad \Gamma, \Upsilon_1, \Delta \Rightarrow \Psi \quad \dots \quad \Gamma, \Upsilon_m, \Delta \Rightarrow \Psi}{\Gamma, \Upsilon_0, \Delta \Rightarrow \Psi}$$

$$\frac{\Upsilon_1' \Rightarrow \quad \dots \quad \Upsilon_n' \Rightarrow}{\Upsilon_0' \Rightarrow}$$

where Ψ is a metavariable for formulas or the empty set, Γ, Δ are metavariables for *sequences* of formulas and $\Upsilon_0', \Upsilon_1', \dots, \Upsilon_m$ are specific sequences of metavariables for *sequences* of formulas, and Υ_0 is *linear*.

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$$\frac{\Upsilon'_1 \Rightarrow \quad \ldots \quad \Upsilon'_n \Rightarrow}{\Upsilon'_0 \Rightarrow}$$

where Ψ is a metavariable for formulas or the empty set, Γ, Δ are metavariables for *sequences* of formulas and $\Upsilon'_0, \Upsilon'_1, \ldots, \Upsilon_m$ are specific sequences of metavariables for *sequences* of formulas, and Υ_0 is *linear*.

Lemma. The interpretation of a simple structural rule is equivalent, over the theory of FL, to an equation of the form

$$\sigma(t_0) \leq \sigma(t_1 \vee \ldots \vee t_m),$$

where t_i is a product of variables, for all i, t_0 is *linear*, and σ is a *simple* ($V = \emptyset$) substitution.

Completing rules

Theorem. [CGT] (cf. [Ter]) Every separated rule is equivalent, over $\mathbf{FL}$, to a simple rule.

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Redundand premises: Remove premises that involve variables not occuring in the conclusion.

Sequencing: Replace lower-case letters by upper-case ones.

$$\frac{\Gamma, \alpha, \alpha \Rightarrow \Psi}{\Gamma, \alpha \Rightarrow \Psi} \quad \rightsquigarrow \quad \frac{\Gamma, \Pi, \Pi \Rightarrow \Psi}{\Gamma, \Pi \Rightarrow \Psi}$$

Linearizarion: Make sure all occurences of the variables are distinct.

$$\frac{\alpha \Rightarrow \delta}{\alpha, \alpha \Rightarrow \delta} \quad \rightsquigarrow \quad \frac{\alpha_1 \Rightarrow \delta \quad \alpha_2 \Rightarrow \delta}{\alpha_1, \alpha_2 \Rightarrow \delta}$$

Contexting: Uniformly enter a context $\Gamma, \_, \Delta \Rightarrow \Psi$.

$$\frac{\Gamma, \alpha_1 \Rightarrow \delta \quad \Gamma, \alpha_2 \Rightarrow \delta}{\Gamma, \alpha_1, \alpha_2 \Rightarrow \delta} \quad \rightsquigarrow \quad \frac{\Gamma, \alpha_1, \Delta \Rightarrow \Psi \quad \Gamma, \alpha_2, \Delta \Rightarrow \Psi}{\Gamma, \alpha_1, \alpha_2, \Delta \Rightarrow \Psi}$$

Completing equations

$$\frac{\alpha \Rightarrow \delta}{\alpha, \alpha \Rightarrow \delta}$$

$$a \leq d \Longrightarrow a^2 \leq d$$

$$a^2 \leq a$$

$$(a_1 \vee a_2)^2 \leq a_1 \vee a_2$$

$$a_1^2 \vee a_1 a_2 \vee a_2 a_1 \vee a_2^2 \leq a_1 \vee a_2$$

$$a_1 a_2 \leq a_1 \vee a_2$$

$$a_1 \vee a_2 \leq G\backslash p/D \Longrightarrow a_1 a_2 \leq G\backslash p/D$$

$$a_1 \leq G\backslash p/D \;\&\; a_2 \leq G\backslash p/D \Longrightarrow a_1 a_2 \leq G\backslash p/D$$

$$G a_1 D \leq p \;\&\; G a_2 D \leq p \Longrightarrow G a_1 a_2 D \leq p$$

$$\frac{\Gamma, \alpha_1, \Delta \Rightarrow \Psi \quad \Gamma, \alpha_2, \Delta \Rightarrow \Psi}{\Gamma, \alpha_1, \alpha_2, \Delta \Rightarrow \Psi} \text{ (min)}$$

Cut elimination

Theorem. [CGT] (cf [Ter], [GO]) Simple rules admit cut elimination.

Proof: 1. Using syntactic arguments presented in [CT].
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Gentzen frames $(\mathbf{W}, \mathbf{B})$ are defined in [GJ].
To an FL-algebra $\mathbf{L}$, we associate a Gentzen frame $(\mathbf{W_L}, \mathbf{L})$.
Also, to a Gentzen frame $(\mathbf{W}, \mathbf{B})$, we associate its dual algebra $\mathbf{R}(\mathbf{W})$, which is an FL-algebra.

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To an FL-algebra $\mathbf{L}$, we associate a Gentzen frame $(\mathbf{W_L}, \mathbf{L})$.
Also, to a Gentzen frame $(\mathbf{W}, \mathbf{B})$, we associate its dual algebra $\mathbf{R}(\mathbf{W})$, which is an FL-algebra.

Lemma. If $\mathbf{L}$ is an FL-algebra, then $\mathbf{R}(\mathbf{W_L})$ is the Dedekind-MacNeille competion of $\mathbf{L}$.

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Theorem. [CGT] Separated equations are preserved under the Dedekind-MacNeille completion. (cf. [TV])

Rules without completion

Theorem. The rule

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Proof (Sketch) Assume that there is a set of rules R that is equivalent to (we) and admits cut elimination. So, there is a proof of $q, p \Rightarrow q$ from $p, q \Rightarrow q$ in $\mathbf{FL} + R$, where p, q are propositional variables.

Fact (using [CT]) There is a cut free proof of $q, p \Rightarrow v$ from assumptions $q \Rightarrow v$; $p, q \Rightarrow v$; $\ldots$; $p, p, \ldots, p, q \Rightarrow v \ldots$ in $\mathbf{FL} + R$, where v is a new propositional variable.
So, we have

$$\{p^n q \leq v : n \in \omega\} \models_{\mathsf{FL_R}} qp \leq v.$$

To disprove this, we will construct an algebra $\mathbf{A}$ in $\mathsf{FL_r}$ and elements $a, b, c \in A$ such that $a^n b \leq c$ for all $n \in \omega$, but $ba \not\leq c$.

Proof

We take $\mathbf{A}$ to be the totally ordered ℓ-group based on the free group on two generators.

Fact [Ber] $\mathbf{A}$ satisfies: if $1 \leq x^m \leq y$, for all $m \in \omega$, then $x^m \leq y^{-1}xy$, for all $m \in \omega$.

Since $\mathbf{A}$ is based on the free group it is not abelian, hence not archimedian (it is totally ordered). So, there exist elements $g, h \in A$ with $1 < g, h$ and $g^m < h$, for all $m \in \omega$.

By the property of the constructed ℓ-group, we get $g^m \leq h^{-1}gh$, namely $g^m h^{-1} \leq h^{-1}g$, for all $m \in \omega$. Now, let

$$a = g^2,\ b = h^{-1}, \text{ and } c = h^{-1}g.$$

We have $a^n b = g^{2n}h^{-1} \leq h^{-1}g = c$, for all $n \in \omega$; but $c = h^{-1}g < h^{-1}g^2 = ba$, because $1 < g$, so $ba \not\leq c$.

Open Problems

- Characterize all structural rules that cannot be completed.
- Characterize all structural rules that are equivalent to equations.
  [Separated rules and rules over a single variable are.]
- Find all equations that are preserved under the Dedekind-MacNeille completion.
  [Simple equations and prelinearity are preserved.]
- Characterize the equations that correspond to rules that admit cut elimination.
- Develop more general framework, like hypersequents, and study the expressive power and cut elimination.
  [We can prove standard completeness for all logics of the form $\mathbf{FL}_e+$ linearity $+$ simple rules.]

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