Structural rules in FL: expressive power and cut elimination.

Nikolaos Galatos
Japan Advanced Institute of Science and Technology (until August)
University of Denver (from September)
galatos@jaist.ac.jp
Joint work with A. Ciabattoni and K. Terui
Overview

- FL and substructural logics
- Algebraic semantics: residuated lattices and FL-algebras
- Structural rules
- Cut elimination
- Expressive power
- Generating analytic calculi from FL + suitable axioms
**The system FL**

\[
\frac{\Pi \Rightarrow \alpha \quad \Gamma, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \quad \text{(cut)}
\]

\[
\frac{\alpha \Rightarrow \alpha}{\quad \text{(Id)}}
\]
The system FL

\[
\frac{\Pi \Rightarrow \alpha \quad \Gamma, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \quad \text{(cut)} \\
\frac{\alpha \Rightarrow \alpha}{\text{(Id)}}
\]

\[
\frac{\Gamma, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha \land \beta, \Delta \Rightarrow \Psi} \quad \text{(\land L)} \\
\frac{\Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, \alpha \land \beta, \Delta \Rightarrow \Psi} \quad \text{(\land R)} \\
\frac{\Pi \Rightarrow \alpha \quad \Pi \Rightarrow \beta}{\Pi \Rightarrow \alpha \land \beta} \quad \text{(\land R)}
\]

\[
\frac{\Gamma, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha \lor \beta, \Delta \Rightarrow \Psi} \quad \text{(\lor L)} \\
\frac{\Pi \Rightarrow \alpha}{\Pi \Rightarrow \alpha \lor \beta} \quad \text{(\lor R)} \\
\frac{\Pi \Rightarrow \beta}{\Pi \Rightarrow \alpha \lor \beta} \quad \text{(\lor R)}
\]
The system FL

\[
\frac{\Pi \Rightarrow \alpha \quad \Gamma, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \quad \text{(cut)} \quad \frac{\alpha \Rightarrow \alpha}{\Gamma} \quad \text{(Id)}
\]

\[
\frac{\Gamma, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha \land \beta, \Delta \Rightarrow \Psi} \quad \text{(^L)} \quad \frac{\Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, \alpha \land \beta, \Delta \Rightarrow \Psi} \quad \text{(^R)}
\]

\[
\frac{\Gamma, \alpha, \Delta \Rightarrow \Psi \quad \Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, \alpha \lor \beta, \Delta \Rightarrow \Psi} \quad \text{(vL)} \quad \frac{\Pi \Rightarrow \alpha \quad \Pi \Rightarrow \beta}{\Pi \Rightarrow \alpha \lor \beta} \quad \text{(vR)}
\]

\[
\frac{\Pi \Rightarrow \alpha \quad \Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, \Pi, (\alpha \backslash \beta), \Delta \Rightarrow \Psi} \quad \text{(\backslash L)} \quad \frac{\alpha, \Pi \Rightarrow \beta}{\Pi \Rightarrow \alpha \backslash \beta} \quad \text{(\backslash R)}
\]

\[
\frac{\Pi \Rightarrow \alpha \quad \Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, (\beta / \alpha), \Pi, \Delta \Rightarrow \Psi} \quad \text{(\slash L)} \quad \frac{\Pi, \alpha \Rightarrow \beta}{\Pi \Rightarrow \beta / \alpha} \quad \text{(\slash R)}
\]
The system FL

\[
\begin{align*}
\frac{\Pi \Rightarrow \alpha}{\Gamma, \Pi, \Delta \Rightarrow \Psi} & \quad \text{(cut)} \\
\frac{\Gamma, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha \land \beta, \Delta \Rightarrow \Psi} & \quad \text{($\land L$)} \\
\frac{\Gamma, \beta, \Delta \Rightarrow \Psi}{\Gamma, \alpha \land \beta, \Delta \Rightarrow \Psi} & \quad \text{($\land R$)} \\
\frac{\Pi \Rightarrow \alpha}{\Pi \Rightarrow \alpha \lor \beta} & \quad \text{($\lor L$)} \\
\frac{\Pi \Rightarrow \alpha}{\Pi \Rightarrow \alpha \lor \beta} & \quad \text{($\lor R$)} \\
\frac{\Pi \Rightarrow \alpha}{\Gamma, \Pi, (\alpha \setminus \beta), \Delta \Rightarrow \Psi} & \quad \text{($\setminus L$)} \\
\frac{\alpha, \Pi \Rightarrow \beta}{\Pi \Rightarrow \alpha \setminus \beta} & \quad \text{($\setminus R$)} \\
\frac{\Pi \Rightarrow \alpha}{\Gamma, (\beta / \alpha), \Pi, \Delta \Rightarrow \Psi} & \quad \text{(/L)} \\
\frac{\Pi \Rightarrow \beta}{\Pi \Rightarrow \beta / \alpha} & \quad \text{(/R)} \\
\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Psi}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \Psi} & \quad \text{($\cdot L$)} \\
\frac{\Pi \Rightarrow \alpha \ldots \beta}{\Pi, \Sigma \Rightarrow \alpha \cdot \beta} & \quad \text{($\cdot R$)} \\
\frac{\Gamma, \Delta \Rightarrow \Psi}{\Gamma, \Delta \Rightarrow \Psi} & \quad \text{(1L)} \\
\frac{\Gamma \Rightarrow 1}{\Pi \Rightarrow 1} & \quad \text{(1R)} \\
\frac{\Gamma \Rightarrow 0}{\Gamma \Rightarrow 0} & \quad \text{(0R)} \\
\frac{0 \Rightarrow 0}{\Rightarrow 0} & \quad \text{(0L)}
\end{align*}
\]
Letters $\alpha, \beta$ denote formulas in the language \{$\land, \lor, \setminus, /, \cdot, 1, 0$\}; $\Gamma, \Sigma, \Pi$ denote sequences of formulas, and $\Psi$ denotes either a formula or the empty set.
Basic structural rules

Letters $\alpha, \beta$ denote formulas in the language \{\(\land, \lor, \setminus, /, \cdot, 1, 0\}\}; $\Gamma, \Sigma, \Pi$ denote \textit{sequences} of formulas, and $\Psi$ denotes either a formula or the empty set.

\[
\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \Psi}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \Psi} \quad (e) \quad \frac{\Gamma, \alpha, \alpha, \Sigma \Rightarrow \Psi}{\Gamma, \alpha, \Sigma \Rightarrow \Psi} \quad (c)
\]

\[
\frac{\Gamma, \Sigma \Rightarrow \Psi}{\Gamma, \alpha, \Sigma \Rightarrow \Psi} \quad (i) \quad \frac{\Gamma \Rightarrow \Psi}{\Gamma \Rightarrow \Psi} \quad (o) \quad (w) = (i) + (o)
\]

The rules exchange (e), contraction (c), left (i) and right (o) weakening are called \textit{structural}.

The system $\mathbf{FL}$ \textit{full Lambek calculus} is obtained from $\mathbf{LJ}$ by removing all structural rules and adding rules for $\cdot, \setminus, /, 1, 0$. 
We write \( \Phi \vdash_{FL} \psi \), if the sequent \( \Rightarrow \psi \) is provable in FL from the set of sequents \( \{( \Rightarrow \phi) | \phi \in \Phi \} \).
We write $\Phi \vdash_{FL} \psi$, if the sequent $\Rightarrow \psi$ is provable in $FL$ from the set of sequents $\{( \Rightarrow \phi) | \phi \in \Phi\}$.

A substructural logic (over $FL$) is a set of formulas closed under $\vdash_{FL}$ and substitution. (Equiv.: consequence relation).
We write $\Phi \vdash_{FL} \psi$, if the sequent $\Rightarrow \psi$ is provable in $FL$ from the set of sequents $\{(\Rightarrow \phi) | \phi \in \Phi\}$.

A *substructural logic* (over $FL$) is a set of formulas closed under $\vdash_{FL}$ and substitution. (Equiv.: consequence relation).

Examples:
- Classical,
- intuitionistic,
- many-valued ($\Lukasiewicz$),
- basic ($Hajek$),
- relevance ($Anderson$, $Belnap$),
- paraconsistent ($Johansson$),
- (the multiplicative additive fragment of) linear logic ($Girard$).
We write $\Phi \vdash_{FL} \psi$, if the sequent $\Rightarrow \psi$ is provable in $FL$ from the set of sequents $\{( \Rightarrow \phi) | \phi \in \Phi\}$.

A *substructural logic* (over $FL$) is a set of formulas closed under $\vdash_{FL}$ and substitution. (Equiv.: consequence relation).

Examples:
- Classical,
- intuitionistic,
- many-valued (Łukasiewicz),
- basic (Hajek),
- relevance (Anderson, Belnap),
- paraconsistent (Johansson),
- (the multiplicative additive fragment of) linear logic (Girard).

An equivalent Hilbert-style system has inference rules

\[
\frac{\phi}{\psi} \quad (mp) \quad \frac{\phi}{\psi \wedge \psi} \quad (adj) \quad \frac{\phi}{\psi \setminus \phi \psi} \quad (n) \quad \frac{\phi}{\psi \phi / \psi} \quad (n)
\]
Residuated lattices

A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra \( L = \langle L, \land, \lor, \cdot, \backslash, /, 1 \rangle \) such that
- \( \langle L, \land, \lor \rangle \) is a lattice,
- \( \langle L, \cdot, 1 \rangle \) is a monoid and
- for all \( a, b, c \in L \), \( ab \leq c \iff a \leq c/b \iff b \leq a\backslash c \).

An *FL-algebra* expands a residuated lattice by an extra constant 0. FL donotes the variety of FL-algebras.
A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra $\mathbf{L} = \langle L, \land, \lor, \cdot, \backslash, /, 1 \rangle$ such that

- $\langle L, \land, \lor \rangle$ is a lattice,
- $\langle L, \cdot, 1 \rangle$ is a monoid and
- for all $a, b, c \in L$, $ab \leq c \iff a \leq c/b \iff b \leq a\backslash c$.

An *FL-algebra* expands a residuated lattice by an extra constant $0$. FL donotes the variety of FL-algebras.

**Theorem.** FL is an *equivalent algebraic semantics* for it $\vdash_{\text{FL}}$. 
A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra $L = \langle L, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ such that

- $\langle L, \wedge, \vee \rangle$ is a lattice,
- $\langle L, \cdot, 1 \rangle$ is a monoid and
- for all $a, b, c \in L$, $ab \leq c \iff a \leq c/b \iff b \leq a \backslash c$.

An *FL-algebra* expands a residuated lattice by an extra constant $0$. FL donotes the variety of FL-algebras.

**Theorem.** FL is an *equivalent algebraic semantics* for it $\vdash_{FL}$.

Structural rules

\[ \frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha, \Delta \Rightarrow \Psi} \]  (c)  \quad \frac{\Gamma, \Pi, \Pi, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \]  (seq-c)
Structural rules

\[
\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha, \Delta \Rightarrow \Psi} \quad (c) \quad \frac{\Gamma, \Pi, \Pi, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \quad (seq-c)
\]

\[
\frac{\Pi \Rightarrow \alpha}{\Gamma, \alpha, \Delta \Rightarrow \Psi} \quad (c) \quad \frac{\Pi \Rightarrow \alpha \quad \Gamma, \alpha, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha, \Pi, \Delta \Rightarrow \Psi} \quad (cut)
\]

\[
\frac{\Pi \Rightarrow \alpha \quad \Gamma, \alpha, \Pi, \Delta \Rightarrow \Psi}{\Pi \Rightarrow \alpha \quad \Gamma, \alpha, \alpha, \Delta \Rightarrow \Psi} \quad (cut)
\]

\[
\frac{\Pi \Rightarrow \alpha}{\Gamma, \alpha, \Delta \Rightarrow \Psi} \quad (c) \quad \frac{\Gamma, \Pi, \Pi, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \quad (?)
\]
Structural rules

\[
\begin{align*}
\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha, \Delta \Rightarrow \Psi} \quad (c) \\
\frac{\Pi \Rightarrow \alpha}{\Gamma, \alpha, \Delta \Rightarrow \Psi} \quad (cut) \\
\frac{\Pi \Rightarrow \alpha, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \quad (c)
\end{align*}
\]

\[
\begin{align*}
\frac{\Gamma, \Pi, \Pi, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \quad (seq-c) \\
\frac{\Pi \Rightarrow \alpha}{\Gamma, \alpha, \Pi, \Delta \Rightarrow \Psi} \quad (cut) \\
\frac{\Pi \Rightarrow \alpha}{\Gamma, \alpha, \Pi, \Delta \Rightarrow \Psi} \quad (cut) \\
\end{align*}
\]

\[
\begin{align*}
\alpha \Rightarrow \delta \\
\alpha, \alpha \Rightarrow \delta \\
\alpha_1 \Rightarrow \delta, \alpha_2 \Rightarrow \delta \\
\alpha_1, \alpha_2 \Rightarrow \delta
\end{align*}
\]
Structural rules

\[
\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Psi}{\Gamma, \alpha, \Delta \Rightarrow \Psi} \quad (c) \quad \frac{\Gamma, \Pi, \Pi, \Delta \Rightarrow \Psi}{\Gamma, \Pi, \Delta \Rightarrow \Psi} \quad (seq-c)
\]

\[
\frac{\Pi \Rightarrow \alpha}{\Gamma, \alpha, \Delta \Rightarrow \Psi} \quad (c) \quad \frac{\Pi \Rightarrow \alpha}{\Gamma, \alpha, \Pi, \Delta \Rightarrow \Psi} \quad (cut)
\]

\[
\frac{\alpha \Rightarrow \delta}{\alpha, \alpha \Rightarrow \delta} \quad \frac{\alpha_1 \Rightarrow \delta \quad \alpha_2 \Rightarrow \delta}{\alpha_1, \alpha_2 \Rightarrow \delta}
\]

\[
\frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \alpha \quad \alpha_2 \Rightarrow \alpha \Rightarrow \delta} \quad (cut) \quad \frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \alpha \Rightarrow \delta} \quad (cut) \quad \frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \alpha \quad \alpha_2 \Rightarrow \delta} \quad (cut)
\]

\[
\frac{\alpha_2 \Rightarrow \alpha}{\alpha_1 \Rightarrow \alpha \quad \alpha_2 \Rightarrow \alpha \Rightarrow \delta} \quad (cut) \quad \frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \delta} \quad (cut) \quad \frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \alpha \quad \alpha_2 \Rightarrow \delta} \quad (cut)
\]

\[
\frac{\alpha_2 \Rightarrow \alpha}{\alpha_1 \Rightarrow \alpha \quad \alpha_2 \Rightarrow \alpha \Rightarrow \delta} \quad (cut)
\]

\[
\frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \delta} \quad (cut)
\]

\[
\frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \delta} \quad (cut)
\]

\[
\frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \delta} \quad (cut)
\]

\[
\frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \delta} \quad (cut)
\]

\[
\frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \delta} \quad (cut)
\]

\[
\frac{\alpha \Rightarrow \delta}{\alpha_1 \Rightarrow \delta} \quad (cut)
\]
A structural rule of the form
\[ \gamma_1 \Rightarrow \ldots \gamma_k \Rightarrow \gamma_1' \Rightarrow \delta_1 \ldots \gamma_m' \Rightarrow \delta_m \quad \gamma_1'' \Rightarrow \Psi_1 \ldots \gamma_n'' \Rightarrow \Psi_n \]
\[ \gamma_0 \Rightarrow \Psi_0(\delta_0) \]
is called *separated*, if \( \gamma_0, \ldots, \gamma''_n \) are sequences of metavariables, \( \Psi, \Psi_1, \ldots, \Psi_n \) range over formulas and the empty set, and \( \delta_0, \ldots, \delta_m \) range over formulas that do not appear in \( \gamma_0, \ldots, \gamma''_n \).
A structural rule of the form

\[
\begin{align*}
\gamma_1 & \Rightarrow \ldots \Rightarrow \gamma_k \Rightarrow \gamma'_1 \Rightarrow \delta_1 \ldots \gamma'_m \Rightarrow \delta_m \\
\gamma''_1 & \Rightarrow \Psi_1 \ldots \gamma''_n \Rightarrow \Psi_n
\end{align*}
\]

\[
\gamma_0 \Rightarrow \Psi_0(\delta_0)
\]

is called **separated**, if \(\gamma_0, \ldots, \gamma''_n\) are sequences of metavariables, \(\Psi, \Psi_1, \ldots, \Psi_n\) range over formulas and the empty set, and \(\delta_0, \ldots, \delta_m\) range over formulas that do not appear in \(\gamma_0, \ldots, \gamma''_n\).

\[
I(\alpha_1, \ldots, \alpha_n \Rightarrow \delta) = (\alpha_1 \ldots \alpha_n \leq \delta)
\]

\[
I(\alpha_1, \ldots, \alpha_n \Rightarrow ) = (\alpha_1 \ldots \alpha_n \leq 0)
\]

\[
I\left(\frac{s_1 \ldots s_n}{s}\right) = (I(s_1) \& \ldots \& I(s_n) \Rightarrow I(s))
\]
A structural rule of the form
\[
\begin{align*}
\Gamma_1 & \Rightarrow \ldots \Rightarrow \Gamma_k \Rightarrow \Gamma'_1 \Rightarrow \delta_1 \ldots \Rightarrow \Gamma'_m \Rightarrow \delta_m \\
& \Rightarrow \Gamma''_1 \Rightarrow \Psi_1 \ldots \Rightarrow \Gamma''_n \Rightarrow \Psi_n \\
\end{align*}
\]
\[\Gamma_0 \Rightarrow \Psi_0(\delta_0)\]

is called \textit{separated}, if \(\Gamma_0, \ldots, \Gamma''_n\) are sequences of metavariables, \(\Psi, \Psi_1, \ldots, \Psi_n\) range over formulas and the empty set, and \(\delta_0, \ldots, \delta_m\) range over formulas that do not appear in \(\Gamma_0, \ldots, \Gamma''_n\).

\[
I(\alpha_1, \ldots, \alpha_n \Rightarrow \delta) = (\alpha_1 \cdot \ldots \cdot \alpha_n \leq \delta)
\]
\[
I(\alpha_1, \ldots, \alpha_n \Rightarrow \ ) = (\alpha_1 \cdot \ldots \cdot \alpha_n \leq 0)
\]
\[
I(\frac{s_1 \ldots s_n}{s}) = (I(s_1) \& \ldots \& I(s_n) \Rightarrow I(s))
\]

\textbf{Lemma} The interpretation of a separated structural rule is equivalent, over the theory of FL, to an equation.
Consider the separated structural rule

\[
\alpha, \gamma, \alpha \Rightarrow \beta, \gamma, \beta \Rightarrow \Gamma, \gamma, \alpha, \phi, \beta, \gamma, \Delta \Rightarrow \Psi
\]

Its interpretation is equivalent to the quasiequation

\[
aca \leq 0 \text{ and } bcb \leq 0 \text{ and } cafbc \leq d \implies cbfac \leq d
\]
Consider the separated structural rule

\[
\frac{\alpha, \gamma, \alpha \Rightarrow \beta, \gamma, \beta \Rightarrow \Gamma, \gamma, \alpha, \phi, \beta, \gamma, \Delta \Rightarrow \Psi}{\Rightarrow \Gamma, \gamma, \beta, \phi, \alpha, \gamma, \Delta \Rightarrow \Psi}
\]

Its interpretation is equivalent to the quasiequation

\[aca \leq 0 \text{ and } bcb \leq 0 \text{ and } cafbc \leq d \implies cbfac \leq d\]

For the choice of variables \(c\) for \(aca\), \(b\) for \(bcb\) and \(f\) for \(cafbc\), we obtain the equation

\[c' b' f' ac' \leq d\]

where \(c' = c \land a \not\land a\), \(b' = b \land 0 \not\land cb\) and \(f' = f \land ca \not\land d/bc\).
Consider the separated structural rule

\[
\alpha, \gamma, \alpha \Rightarrow \beta, \gamma, \beta \Rightarrow \Gamma, \gamma, \alpha, \phi, \beta, \gamma, \Delta \Rightarrow \Psi
\]

Its interpretation is equivalent to the quasiequation

\[
aca \leq 0 \text{ and } bcb \leq 0 \text{ and } cafbc \leq d \implies cbfac \leq d
\]

For the choice of variables \( c \) for \( aca \), \( b \) for \( bcb \) and \( f \) for \( cafbc \) we obtain the equation

\[
c'b'f'ac' \leq d
\]

where \( c' = c \wedge a \setminus 0 / a \), \( b' = b \wedge 0 / cb \) and \( f' = f \wedge ca \setminus d / bc \).

Alternatively, for the choice of variables \( c \) for \( aca \) and \( c \) for \( bcb \) we obtain the equation

\[
c'bf'ac' \leq d
\]

where \( c' = c \wedge a \setminus 0 / a \wedge b \setminus 0 / b \) and \( f' = f \wedge ca \setminus d / bc \).
Separated equations

For a set of variables $V$, we define the set of separated formulas (or terms) $sep(V)$ as the smallest set such that

1. $\{0, \top\} \cup V \subseteq sep(V)$, (if $\top$ is in the language),
2. if $t_1, t_2 \in sep(V)$, then $t_1 \land t_2 \in sep(V)$,
3. if $s$ is a $\{\cdot, \lor, 1\}$-term with no variable from $V$ and $t \in sep(V)$, then $s \backslash t, t/s \in sep(V)$. 

Nikolaos Galatos, TANCL, Oxford 2007

Structure rules
Basic structural rules
Substructural logics
Residuated lattices
Structural rules
Separated rules
Separated rules
Separated equations
Simple rules
Completing rules
Completing equations
Cut elimination
Rules without completion
Proof
Open Problems
Bibliography
Separated equations

For a set of variables $V$, we define the set of separated formulas (or terms) $sep(V)$ as the smallest set such that

1. $\{0, \top\} \cup V \subseteq sep(V)$, (if $\top$ is in the language),
2. if $t_1, t_2 \in sep(V)$, then $t_1 \land t_2 \in sep(V)$,
3. if $s$ is a $\{\cdot, \lor, 1\}$-term with no variable from $V$ and $t \in sep(V)$, then $s \setminus t, t/s \in sep(V)$.

A substitution $\sigma$ is called separated, relative to $V$, if there are variables $x_1, \ldots, x_n$ not in $V$ and terms $t_1, \ldots, t_n \in sep(V)$ such that $\sigma(x_i) = x_i \land t_i$, for all $i$, and $\sigma$ fixes all other variables.
Separated equations

For a set of variables $V$, we define the set of separated formulas (or terms) $sep(V)$ as the smallest set such that

1. $\{0, \top\} \cup V \subseteq sep(V)$, (if $\top$ is in the language),
2. if $t_1, t_2 \in sep(V)$, then $t_1 \land t_2 \in sep(V)$,
3. if $s$ is a $\{\cdot, \lor, 1\}$-term with no variable from $V$ and $t \in sep(V)$, then $s \setminus t, t/s \in sep(V)$.

A substitution $\sigma$ is called separated, relative to $V$, if there are variables $x_1, \ldots, x_n$ not in $V$ and terms $t_1, \ldots, t_n \in sep(V)$ such that $\sigma(x_i) = x_i \land t_i$, for all $i$, and $\sigma$ fixes all other variables.

An equation is called separated, if it is of the form $\sigma(t) \leq z$, where $\sigma$ is a separated substitution, $t \in sep(V)$ and $z \in V$. 
Separated equations

For a set of variables $V$, we define the set of *separated* formulas (or terms) $\text{sep}(V)$ as the smallest set such that

1. $\{0, \top\} \cup V \subseteq \text{sep}(V)$, (if $\top$ is in the language),
2. if $t_1, t_2 \in \text{sep}(V)$, then $t_1 \land t_2 \in \text{sep}(V)$,
3. if $s$ is a $\{\cdot, \lor, 1\}$-term with no variable from $V$ and $t \in \text{sep}(V)$, then $s \backslash t, t/s \in \text{sep}(V)$.

A substitution $\sigma$ is called *separated*, relative to $V$, if there are variables $x_1, \ldots, x_n$ not in $V$ and terms $t_1, \ldots, t_n \in \text{sep}(V)$ such that $\sigma(x_i) = x_i \land t_i$, for all $i$, and $\sigma$ fixes all other variables.

An equation is called *separated*, if it is of the form $\sigma(t) \leq z$, where $\sigma$ is a separated substitution, $t \in \text{sep}(V)$ and $z \in V$.

**Theorem.** (Sets of) separated structural rules correspond to (Sets of) separated equations.
A substructural rule is called *simple* if it is of one of the forms

\[
\begin{align*}
\Gamma, \gamma_1, \Delta &\Rightarrow \psi \\
\Gamma, \gamma_0, \Delta &\Rightarrow \psi
\end{align*}
\]

\[
\begin{align*}
\gamma_1 \Rightarrow \ldots \gamma_n \Rightarrow \psi
\end{align*}
\]

where \(\psi\) is a metavariable for formulas or the empty set, \(\Gamma, \Delta\) are metavariables for *sequences* of formulas and \(\gamma_0, \gamma_1, \ldots, \gamma_m\) are specific sequences of metavariables for *sequences* of formulas, and \(\gamma_0\) is *linear*. 
A substructural rule is called *simple* if it is of one of the forms

\[
\begin{align*}
\gamma_1' & \Rightarrow \cdots \gamma_n' \Rightarrow \Gamma, \gamma_1, \Delta \Rightarrow \psi \cdots \Gamma, \gamma_m, \Delta \Rightarrow \psi \\
\Gamma, \gamma_0, \Delta & \Rightarrow \psi \\
\gamma_1' & \Rightarrow \cdots \gamma_n' \Rightarrow \\
\gamma_0' & \Rightarrow
\end{align*}
\]

where \( \psi \) is a metavariable for formulas or the empty set, \( \Gamma, \Delta \) are metavariables for *sequences* of formulas and \( \gamma_0', \gamma_1', \ldots, \gamma_m \) are specific sequences of metavariables for *sequences* of formulas, and \( \gamma_0 \) is *linear*.

**Lemma.** The interpretation of a simple structural rule is equivalent, over the theory of FL, to an equation of the form

\[
\sigma(t_0) \leq \sigma(t_1 \lor \cdots \lor t_m),
\]

where \( t_i \) is a product of variables, for all \( i \), \( t_0 \) is *linear*, and \( \sigma \) is a *simple* \( (V = \emptyset) \) substitution.
Theorem. [CGT] (cf. [Ter]) Every separated rule is equivalent, over $\mathbf{FL}$, to a simple rule.
Completing rules

Theorem. [CGT] (cf. [Ter]) Every separated rule is equivalent, over FL, to a simple rule.

Redundand premises: Remove premises that involve variables not occurring in the conclusion.

Sequencing: Replace lower-case letters by upper-case ones.

\[
\frac{\Gamma, \alpha, \alpha \Rightarrow \Psi}{\Gamma, \alpha \Rightarrow \Psi} \quad \sim \quad \frac{\Gamma, \Pi, \Pi \Rightarrow \Psi}{\Gamma, \Pi \Rightarrow \Psi}
\]

Linearizarion: Make sure all occurrences of the variables are distinct.

\[
\frac{\alpha \Rightarrow \delta}{\alpha, \alpha \Rightarrow \delta} \quad \sim \quad \frac{\alpha_1 \Rightarrow \delta \quad \alpha_2 \Rightarrow \delta}{\alpha_1, \alpha_2 \Rightarrow \delta}
\]

Contexting: Uniformly enter a context \( \Gamma, \_, \Delta \Rightarrow \Psi \).

\[
\frac{\Gamma, \alpha_1 \Rightarrow \delta \quad \Gamma, \alpha_2 \Rightarrow \delta}{\Gamma, \alpha_1, \alpha_2 \Rightarrow \delta} \quad \sim \quad \frac{\Gamma, \alpha_1, \Delta \Rightarrow \Psi \quad \Gamma, \alpha_2, \Delta \Rightarrow \Psi}{\Gamma, \alpha_1, \alpha_2, \Delta \Rightarrow \Psi}
\]
Completing equations

\[
\frac{\alpha \Rightarrow \delta}{\alpha, \alpha \Rightarrow \delta}
\]

\[
a \leq d \implies a^2 \leq d
\]

\[
a^2 \leq a
\]

\[
(a_1 \vee a_2)^2 \leq a_1 \vee a_2
\]

\[
a_1^2 \vee a_1 a_2 \vee a_2 a_1 \vee a_2^2 \leq a_1 \vee a_2
\]

\[
a_1 a_2 \leq a_1 \vee a_2
\]

\[
a_1 \vee a_2 \leq G \setminus p/D \implies a_1 a_2 \leq G \setminus p/D
\]

\[
a_1 \leq G \setminus p/D \& a_2 \leq G \setminus p/D \implies a_1 a_2 \leq G \setminus p/D
\]

\[
G a_1 D \leq p \& G a_2 D \leq p \implies G a_1 a_2 D \leq p
\]

\[
\frac{\Gamma, \alpha_1, \Delta \Rightarrow \Psi \quad \Gamma, \alpha_2, \Delta \Rightarrow \Psi}{\Gamma, \alpha_1, \alpha_2, \Delta \Rightarrow \Psi} \quad \text{(min)}
\]
Theorem. [CGT] (cf [Ter], [GO]) Simple rules admit cut elimination.

Proof: 1. Using syntactic arguments presented in [CT].
2. Using semantial arguments presented in [GJ].
**Theorem.** [CGT] (cf [Ter], [GO]) Simple rules admit cut elimination.

Proof: 1. Using syntactic arguments presented in [CT].
2. Using semantical arguments presented in [GJ].

*Gentzen frames* \((W, B)\) are defined in [GJ].

To an FL-algebra \(L\), we associate a Gentzen frame \((W^L, L)\).

Also, to a Gentzen frame \((W, B)\), we associate its dual algebra \(R(W)\), which is an FL-algebra.
Theorem. [CGT] (cf [Ter], [GO]) Simple rules admit cut elimination.

Proof: 1. Using syntactic arguments presented in [CT].
2. Using semantic arguments presented in [GJ].

Gentzen frames \((W, B)\) are defined in [GJ].

To an FL-algebra \(L\), we associate a Gentzen frame \((W_L, L)\).

Also, to a Gentzen frame \((W, B)\), we associate its dual algebra \(R(W)\), which is an FL-algebra.

Lemma. If \(L\) is an FL-algebra, then \(R(W_L)\) is the Dedekind-MacNeille completion of \(L\).
**Theorem.** [CGT] (cf [Ter], [GO]) Simple rules admit cut elimination.

Proof: 1. Using syntactic arguments presented in [CT].
2. Using semantial arguments presented in [GJ].

*Gentzen frames* $(W, B)$ are defined in [GJ].
To an FL-algebra $L$, we associate a Gentzen frame $(W_L, L)$.
Also, to a Gentzen frame $(W, B)$, we associate its dual algebra $R(W)$, which is an FL-algebra.

**Lemma.** If $L$ is an FL-algebra, then $R(W_L)$ is the Dedekind-MacNeille competion of $L$.

**Theorem.** [GJ] If $(W, B)$ is a (cut-free) Gentzen frame, then every sequent *valid in* $R(W)$ is also *valid in* $(W, B)$.
Theorem. [CGT] (cf [Ter], [GO]) Simple rules admit cut elimination.

Proof: 1. Using syntactic arguments presented in [CT].
2. Using semantical arguments presented in [GJ].

*Gentzen frames* $(\mathbf{W}, \mathbf{B})$ are defined in [GJ].
To an FL-algebra $\mathbf{L}$, we associate a Gentzen frame $(\mathbf{W}_{\mathbf{L}}, \mathbf{L})$.
Also, to a Gentzen frame $(\mathbf{W}, \mathbf{B})$, we associate its dual algebra $\mathbf{R}(\mathbf{W})$, which is an FL-algebra.

Lemma. If $\mathbf{L}$ is an FL-algebra, then $\mathbf{R}(\mathbf{W}_{\mathbf{L}})$ is the Dedekind-MacNeille completion of $\mathbf{L}$.

Theorem. [GJ] If $(\mathbf{W}, \mathbf{B})$ is a (cut-free) Gentzen frame, then every sequent *valid in* $\mathbf{R}(\mathbf{W})$ is also *valid in* $(\mathbf{W}, \mathbf{B})$.

Theorem. [CGT] Let $(\mathbf{W}, \mathbf{B})$ be a cut free Gentzen frame and let $\varepsilon$ be a simple equation. Then, $(\mathbf{W}, \mathbf{B})$ satisfies $\mathbf{R}(\varepsilon)$ iff $\mathbf{R}(\mathbf{W})$ satisfies $\varepsilon$. 
**Theorem.** [CGT] (cf [Ter], [GO]) Simple rules admit cut elimination.

Proof: 1. Using syntactic arguments presented in [CT].
2. Using semantical arguments presented in [GJ].

*Gentzen frames* \((W, B)\) are defined in [GJ].

To an FL-algebra \(L\), we associate a Gentzen frame \((W_L, L)\).

Also, to a Gentzen frame \((W, B)\), we associate its dual algebra \(R(W)\), which is an FL-algebra.

**Lemma.** If \(L\) is an FL-algebra, then \(R(W_L)\) is the Dedekind-MacNeille completion of \(L\).

**Theorem.** [GJ] If \((W, B)\) is a (cut-free) Gentzen frame, then every sequent *valid in* \(R(W)\) is also *valid in* \((W, B)\).

**Theorem.** [CGT] Let \((W, B)\) be a cut free Gentzen frame and let \(\varepsilon\) be a simple equation. Then, \((W, B)\) satisfies \(R(\varepsilon)\) iff \(R(W)\) satisfies \(\varepsilon\).

**Theorem.** [CGT] Separated equations are preserved under the Dedekind-MacNeille completion. (cf. [TV])
Theorem. The rule
\[
\frac{\alpha, \beta \Rightarrow \beta}{\beta, \alpha \Rightarrow \beta} \quad (we)
\]
is not equivalent to a rule that admits cut elimination.
Theorem. The rule

$$\frac{\alpha, \beta \Rightarrow \beta}{\beta, \alpha \Rightarrow \beta} \quad (we)$$

is not equivalent to a rule that admits cut elimination.

Proof (Sketch) Assume that there is a set of rules $R$ that is equivalent to $(we)$ and admits cut elimination. So, there is a proof of $q, p \Rightarrow q$ from $p, q \Rightarrow q$ in $\text{FL} + R$, where $p, q$ are propositional variables.

Fact (using [CT]) There is a cut free proof of $q, p \Rightarrow v$ from assumptions $q \Rightarrow v; p, q \Rightarrow v; \ldots; p, p, \ldots, p, q \Rightarrow v \ldots$ in $\text{FL} + R$, where $v$ is a new propositional variable.

So, we have

$$\{p^n q \leq v : n \in \omega\} \models_{\text{FL}_R}qp \leq v.$$  

To disprove this, we will construct an algebra $A$ in $\text{FL}_r$ and elements $a, b, c \in A$ such that $a^n b \leq c$ for all $n \in \omega$, but $ba \nleq c$. 
We take $A$ to be the totally ordered $\ell$-group based on the free group on two generators.

**Fact [Ber]** $A$ satisfies: if $1 \leq x^m \leq y$, for all $m \in \omega$, then $x^m \leq y^{-1}xy$, for all $m \in \omega$.

Since $A$ is based on the free group it is not abelian, hence not archimedean (it is totally ordered). So, there exist elements $g, h \in A$ with $1 < g, h$ and $g^m < h$, for all $m \in \omega$.

By the property of the constructed $\ell$-group, we get $g^m \leq h^{-1}gh$, namely $g^m h^{-1} \leq h^{-1}g$, for all $m \in \omega$. Now, let $a = g^2$, $b = h^{-1}$, and $c = h^{-1}g$.

We have $a^nb = g^{2n}h^{-1} \leq h^{-1}g = c$, for all $n \in \omega$; but $c = h^{-1}g < h^{-1}g^2 = ba$, because $1 < g$, so $ba \nleq c$. 
Open Problems

- Characterize all structural rules that cannot be completed.

- Characterize all structural rules that are equivalent to equations.
  [Separated rules and rules over a single variable are.]

- Find all equations that are preserved under the Dedekind-MacNeille completion.
  [Simple equations and prelinearity are preserved.]

- Characterize the equations that correspond to rules that admit cut elimination.

- Develop more general framework, like hypersequents, and study the expressive power and cut elimination.
  [We can prove standard completeness for all logics of the form $\text{FL}_e + \text{linearity} + \text{simple rules}.]
Bibliography


[GJ] N. Galatos and P. Jipsen. Residuated frames and applications to decidability, manuscript.


