Notes on Complexity of Monoidal T-norm Based Logic and its Extensions

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**ML and MTL**

- **Monoidal Logic** (ML) is Full Lambek calculus with exchange and weakening (FL\textsubscript{ew}). Also known as IMALLW (multiplicative additive fragment of Intuitionistic Linear Logic with weakening).

- **Monoidal T-norm Based Logic** (MTL) is a schematic extension of ML by the following axiom schema:

  \[(\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)\]

- **C\textsubscript{n}ML** (resp. **C\textsubscript{n}MTL**) is an extension of ML (resp. MTL) by the following axiom schema:

  \[\varphi^{n-1} \rightarrow \varphi^{n}\]
ML-algebras and MTL-algebras

Definition
An **ML-algebra** is an algebra $A = (A, \ast, \rightarrow, \wedge, \vee, 0, 1)$ where the following conditions are satisfied:

- $(A, \ast, \rightarrow, \wedge, \vee, 1)$ is a commutative integral residuated lattice,
- $0$ is a bottom element.

Definition
An **MTL-algebra** is an ML-algebra $A = (A, \ast, \rightarrow, \wedge, \vee, 0, 1)$ such that

- $(x \rightarrow y) \vee (y \rightarrow x) = 1$ for all $x, y \in A$.

In other words, an MTL-algebra is a **representable** ML-algebra.
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Lower bound (hardness)?
Provability

- Monoidal logic ML is in **PSPACE** (it can be seen from its sequent calculus)

- Lower bound (hardness)?

- IMALL (ML without weakening) is known to be PSPACE-hard, hence **PSPACE-complete** (Lincoln, Mitchell, Scedrov, Shankar 94)
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- What is their complexity?
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- What is their complexity?

- IMALL (ML without weakening) is **undecidable** (as full ILL)
Motivation

- Hypersequent calculus for MTL is not suitable for proof search.
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- On the other hand, ML has a nice sequent calculus.
- Is it possible somehow to translate provability between MTL and ML?
Main result

Let $\varphi$ be a formula in the language of MTL and $S = \{\psi_1, \ldots, \psi_n\}$ a set of all subformulas of $\varphi$. 
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$$\psi_1 \leq \psi_2 \leq \cdots \leq \psi_n$$

can be coded as follows:

$$T = \{\psi_1 \rightarrow \psi_2, \psi_2 \rightarrow \psi_3, \ldots, \psi_{n-1} \rightarrow \psi_n\}$$
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Let $\mathcal{O}$ be the set of theories coding all possible linear orderings of elements of $S$. 
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Let $O$ be the set of theories coding all possible linear orderings of elements of $S$.

**Theorem**

$\models_{MTL} \varphi$ iff for all $T \in O$ we have $T \models_{ML} \varphi$. 
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Let $\mathcal{O}$ be the set of theories coding all possible linear orderings of elements of $S$.

Theorem

$\vdash_{C_n\text{MTL}} \varphi$ iff for all $T \in \mathcal{O}$ we have $T \vdash_{C_n\text{ML}} \varphi$. 
Sketch of the proof

- The right-to-left direction is easy since MTL is complete w.r.t. the class of all MTL-chains.

- Suppose that there is $T \in \mathcal{O}$ such that $T \not\models_{ML} \varphi$. 
Sketch of the proof

- The right-to-left direction is easy since MTL is complete w.r.t. the class of all MTL-chains.

- Suppose that there is $T \in \mathcal{O}$ such that $T \not\vdash_{\text{ML}} \varphi$.

- Since ML has FEP, there is a finite ML-algebra $A$ such that $T \not\models_A \varphi$.

- There is an $A$-evaluation $e$ such that $e(T) \subseteq \{1\}$ and $e(\varphi) < 1$.

- Thus the set $e(S)$ is totally ordered, i.e.

$$e(S) = \{1 > a_1 > \cdots > a_n > 0\}$$
Let $M$ be the submonoid of $A$ generated by $e(S)$. 
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The partial order $\leq$ on $M$ inherited from $A$ induces a quasi-order on $\mathbb{N}^n$ defined by

$$x \preceq y \text{ iff } h(x) \leq h(y).$$
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$$x \preccurlyeq y \text{ iff } h(x) \leq h(y).$$

$\mathbf{M} \cong \mathbb{N}^n/\sim$ where $\sim$ is the equivalence corresponding to $\preccurlyeq$. 
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$$x \preceq y \text{ iff } h(x) \leq h(y).$$

$M \cong \mathbb{N}^n/\sim$ where $\sim$ is the equivalence corresponding to $\preceq$.

Note that the quasi-order $\preceq$ need not be total.
2 generators
2 generators
2 generators
2 generators
3 generators

\begin{align*}
a_1 & \quad a_2 & \quad a_3
\end{align*}
Results

3 generators

\[ a_1, a_2, a_3 \]
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Let $\leq_\ell$ be the component-wise partial order on $\mathbb{N}^n$ and $\leq_{\text{lex}}$ the lexicographic total order on $\mathbb{N}^n$. 
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**Lemma**

Let $x, y \in \mathbb{N}^n$ and

$$
R = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
$$

Then the relation $\leq_R$ defined by

$$x \leq_R y \text{ iff } R \cdot x \geq_\ell R \cdot y$$

is a partial order monotone w.r.t. $+$. Moreover, if $x \leq_R y$ then $x \preceq y$, i.e. $\leq_R$ is a sub-quasi-order of $\preceq$. 

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Definition
We define a relation $\preceq'$ by the following steps:

1. Extend $\preceq$ in such a way that if $y \not\preceq x$ and $x \not\preceq y$ then break ties according to $\leq R_{\text{lex}}$, where $x \leq R_{\text{lex}} y$ iff $R \cdot x \geq \text{lex} R \cdot y$.
2. Make the monotone closure.
3. Make the transitive closure.

Lemma
The relation $\preceq'$ is a monotone total quasi-order extending $\preceq$.

Lemma
Let $\sim'$ be the equivalence corresponding to $\preceq'$. Then $\frac{N}{\sim'}$ is an MTL-algebra into which the partial subalgebra $e(S)$ of $A$ can be embedded.
Definition
We define a relation \( \lesssim' \) by the following steps:

- Extend \( \lesssim \) in such a way that if \( y \not\lesssim x \) and \( x \not\lesssim y \) then break ties according to \( \leq_{\text{R}_{\text{lex}}} \), where

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\text{x \leq_{R_{\text{lex}}} y \text{ iff } R \cdot x \geq_{\text{lex}} R \cdot y.}
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Definition

We define a relation $\lesssim'$ by the following steps:

- Extend $\lesssim$ in such a way that if $y \nless x$ and $x \nless y$ then break ties according to $\leq_{\text{R}_{\text{lex}}}$, where

$$x \leq_{\text{R}_{\text{lex}}} y \iff \text{R} \cdot x \geq_{\text{lex}} \text{R} \cdot y.$$  

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- Extend $\preceq$ in such a way that if $y \not\preceq x$ and $x \not\preceq y$ then break ties according to $\leq_{R_{\text{lex}}}$, where

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**Lemma**

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- Extend $\lesssim$ in such a way that if $y \not\lesssim x$ and $x \not\lesssim y$ then break ties according to $\leq_{R_{\text{lex}}}$, where
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- Make the monotone closure.
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Lemma

The relation $\lesssim'$ is a monotone total quasi-order extending $\lesssim$.

Lemma

Let $\sim'$ be the equivalence corresponding to $\lesssim'$. Then $\mathbb{N}^n/\sim'$ is an MTL-algebra into which the partial subalgebra $e(S)$ of $A$ can be embedded.
Questions for audience

- Is there any bound on counter-models in ML?
- Is it known whether ML is PSPACE complete?
MTL and ML

⊢_{MTL} \varphi \text{ iff for all } T \in \mathcal{O} \text{ we have } T \vdash_{ML} \varphi.
MTL and ML

- $\vdash_{\text{MTL}} \varphi$ iff for all $T \in \mathcal{O}$ we have $T \vdash_{\text{ML}} \varphi$.

- $T \vdash_{\text{ML}} \varphi$ iff $\exists n_1 \ldots n_k \vdash_{\text{ML}} \alpha_{n_1} \rightarrow (\ldots \alpha_{n_k} \rightarrow \varphi) \ldots$
MTL and ML

- $\vdash_{\text{MTL}} \varphi$ iff for all $T \in \mathcal{O}$ we have $T \vdash_{\text{ML}} \varphi$.

- $T \vdash_{\text{ML}} \varphi$ iff $\exists n_1 \ldots n_k \vdash_{\text{ML}} \alpha_1^{n_1} \rightarrow (\ldots \alpha_k^{n_k} \rightarrow \varphi) \ldots$

- iff $\emptyset \Rightarrow \alpha_1, \ldots, \emptyset \Rightarrow \alpha_k \vdash_{\text{GML}} \emptyset \Rightarrow \varphi$ by a directed proof.
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- iff $\vdash_{\text{GILLW}} !\alpha_1, \ldots, !\alpha_k \Rightarrow \varphi$ by a cut-free proof
MTL and ML

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- iff $\vdash_{\text{GILLW}} !\alpha_1, \ldots, !\alpha_k \Rightarrow \varphi$ by a cut-free proof

- a standard argument used e.g. in (Lincoln, Mitchell, Scedrov, Shankar 94)

- What is complexity of ILLW?

- We need less than full GILLW:
GML! = GML + !-left rule:

\[
\frac{\Gamma, \varphi, !\varphi \Rightarrow \delta}{\Gamma, !\varphi \Rightarrow \delta} \quad !-l
\]
GML! = GML + !-left rule:

$$\Gamma, \varphi, !\varphi \Rightarrow \delta \quad \Gamma, !\varphi \Rightarrow \delta$$

Contraction rule for !\varphi admissible

$$\Gamma, !\varphi, !\varphi \Rightarrow \delta \quad \Gamma, !\varphi \Rightarrow \delta$$

Cut rule can be eliminated

How to create proof search in GML! and what is its complexity?
$C_n$MTL and $C_n$ML

$\vdash_{C_n\text{MTL}} \varphi$ iff for all $T \in \mathcal{O}$ we have $T \vdash_{C_n\text{ML}} \varphi$. 
$C_n\text{MTL and } C_n\text{ML}$

- $\vdash_{C_n\text{MTL}} \varphi$ iff for all $T \in \mathcal{O}$ we have $T \vdash_{C_n\text{ML}} \varphi$.

- $T \vdash_{C_n\text{ML}} \varphi$ iff $\vdash_{C_n\text{ML}} \alpha_1^{n-1} \rightarrow (\ldots \alpha_k^{n-1} \rightarrow \varphi) \ldots$ by a cut free proof.
$\vdash_{C_nMTL} \varphi$ iff for all $T \in \mathcal{O}$ we have $T \vdash_{C_nML} \varphi$.

$T \vdash_{C_nML} \varphi$ iff $\vdash_{C_nML} \alpha_{1}^{n-1} \rightarrow (\ldots \alpha_{k}^{n-1} \rightarrow \varphi) \ldots$)

iff $\vdash_{GC_nML} \alpha_{1}^{n-1}, \ldots, \alpha_{k}^{n-1} \Rightarrow \varphi$ by a cut free proof
$C_nMTL$ and $C_nML$

- $\vdash_{C_nMTL} \varphi$ iff for all $T \in \mathcal{O}$ we have $T \vdash_{C_nML} \varphi$.

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- iff $\vdash_{GC_nML} \alpha^n_{n-1}, \ldots, \alpha^n_{k} \Rightarrow \varphi$ by a cut free proof

- Complexity of proof search in $GC_nML$?