Quasi-p-morphisms and small varieties of KTB-algebras

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- And so is the next, I'm happy to add.

KTB-algebras and KTB-frames

An algebra $\mathbf{A} = \langle A; \vee, \wedge, \neg, f, 0, 1 \rangle$ is a *KTB-algebra* if $\mathbf{A}$ is a modal algebra and f satisfies:

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One conclusion (most important for this talk): if all graphs from $\mathcal{V}$ are of finite diameter, then they are of a bounded finite diameter.

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The bottom of Λ^{KTB} is a three element chain:
trivial $\prec V(K_1) \prec V(K_2)$.

where $V(K_1)$ is the variety generated by the algebra of the complete graph on one element and $V(K_2)$ is the variety generated by the algebra of the complete graph on two elements.

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Theorem (T.K., Stevens 2006)

There are at least $\aleph_0$ covers of $V(K_2)$ in Λ^{KTB}.

Lame spiders

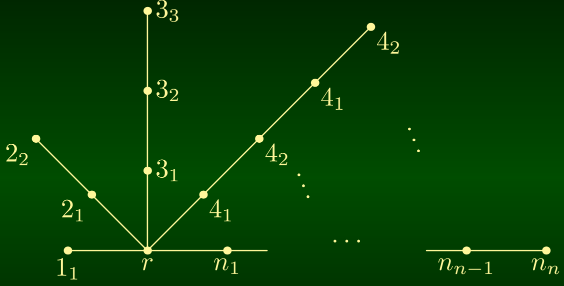

Figure: The graph S_n

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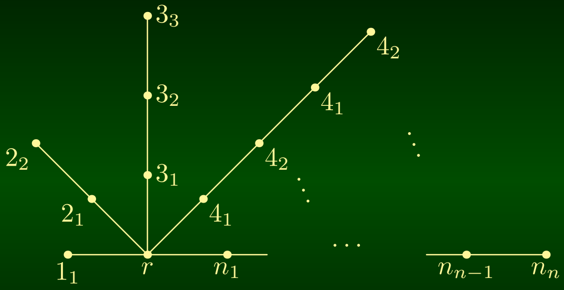

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For each n, the variety $V(S_n)$ is a cover of $V(K_2)$ in Λ^{KTB}.

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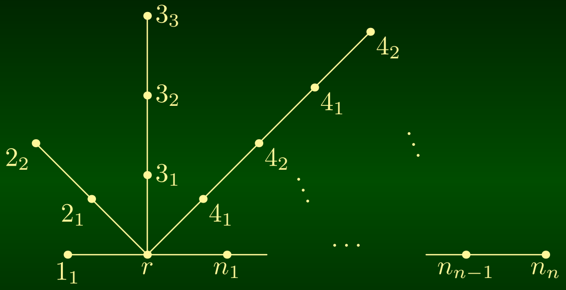

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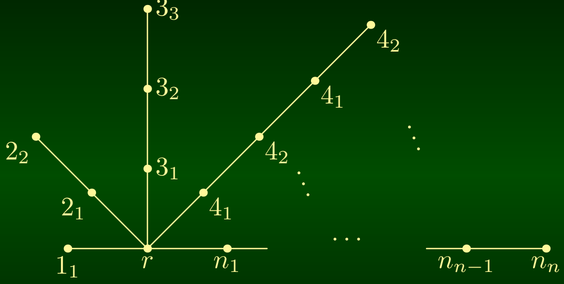

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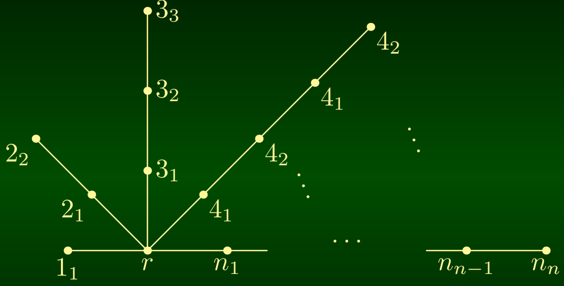

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For each n, the variety $V(S_n)$ is a cover of $V(K_2)$ in Λ^{KTB}. So there should be an algebra of subsets of S_ω with the same property. . . one would hope. Alas, that cannot work.

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- $\forall a \in W_S \exists K \in \omega \forall n \in \omega$ the distance from a to $\bigcup_{u \in U} \bigcup_{i=0}^n X_i^u$ is not greater than K.

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Theorem (T.K., Miyazaki, 2007)

Let $\mathfrak{F}$ and $\mathfrak{G}$ be frames, and $\mathfrak{G}$ be finite. Let $\mathfrak{F}^*$ and $\mathfrak{G}^*$ be the respective dual algebras of $\mathfrak{F}$ and $\mathfrak{G}$. Let $f : \mathfrak{F} \to \mathfrak{G}$ be a quasi-p-morphism. Then $\mathfrak{G}^* \in SHP(\mathfrak{F}^*)$.

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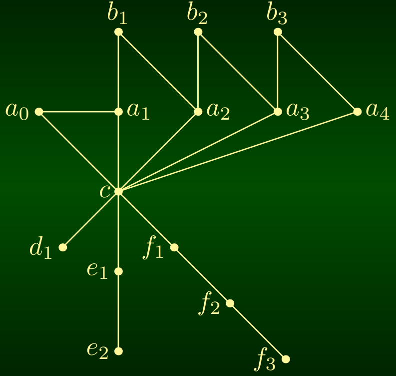
Proof.

Let $U = \{1, \ldots, m\}$ (for notational convenience). Idea: for $\ell \in U$ put $Z_n^\ell = \bigcup_{i=0}^{\ell}$. Then let $Z^\ell = (Z_n^\ell \colon n \in \omega)$. This is an element of $(\mathfrak{F}^*)^\omega$. Consider the congruence $\Theta = Cg(\bigvee_{\ell=1}^{m}, 1)$ on $(\mathfrak{F}^*)^\omega$ and show that $(\mathfrak{F}^*)^\omega/\Theta$ has a subalgebra isomorphic to $\mathfrak{G}^*$. □

Finite saws



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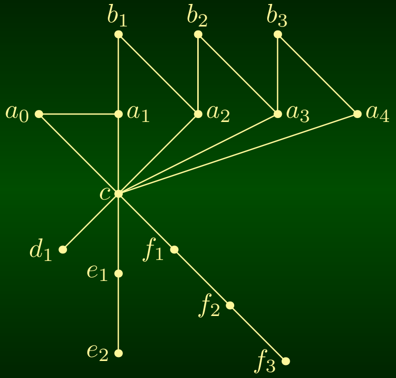

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Any such thing generates a cover of $V(K_2)$.

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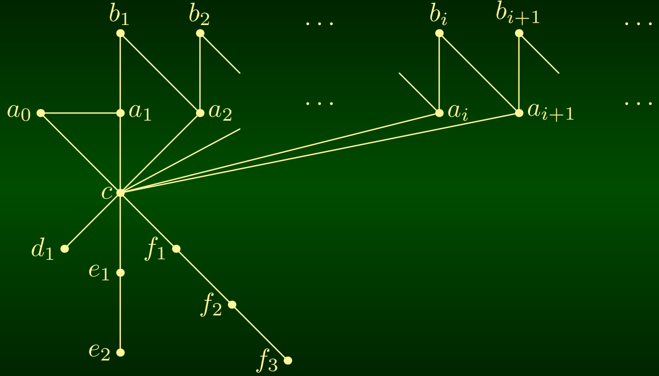

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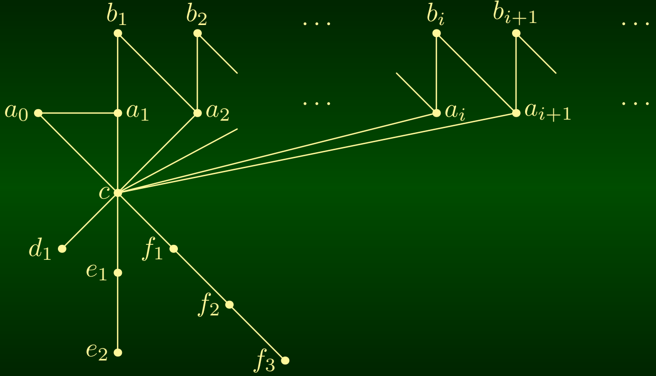

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That generates a cover of $V(K_2)$, too.

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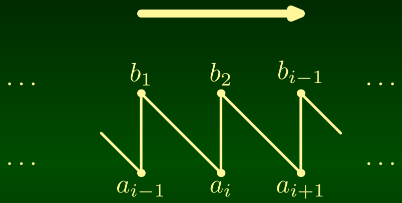

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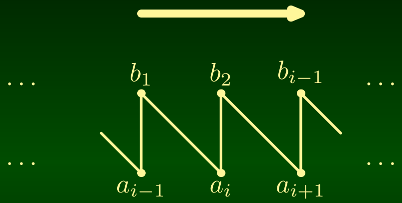

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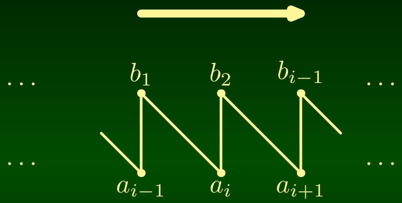

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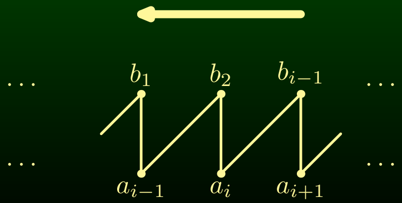
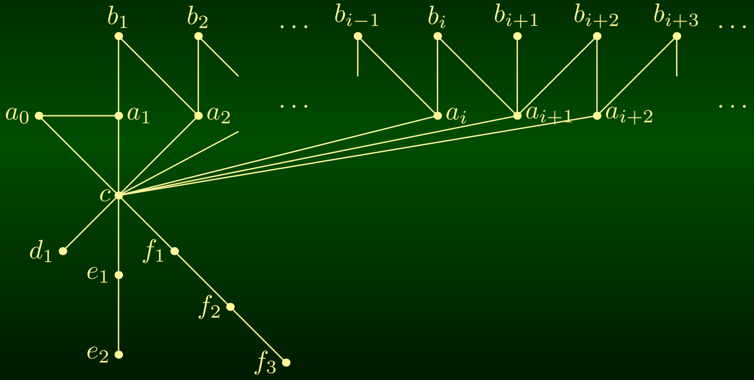


A better sort of infinite saws



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- $C = \{c, d_1, e_1, e_2, f_1, f_2.f_3\}$, $A = \{a_i \colon i \in \omega\}$, $B = \{b_i \colon b \in \omega \setminus \{0\}\}$.
- $d_1 E_Q c$, $e_1 E_Q c$, $f_1 E_Q c$.
- $e_1 E_Q e_2$, $f_1 E_Q f_2$, $f_2 E_Q f_3$.
- $a_0 E_Q a_1$, $cE_Q a_i$ for all $a_i \in A_Q$.

Uncountably many infinite saws

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- $a_{2k+1} E_Q b_{2k}$ iff $2k \notin Q$ and $a_{2k+1} E_Q b_{2k+2}$ iff $2k + 2 \in Q$.

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- $a_{2k} E_Q b_{2k-1}$ iff $2k \notin Q$ and $a_{2k} E_Q b_{2k+1}$ iff $2k + 2 \in Q$.

Uncountably many covers of $V(K_2)$

Let $\mathfrak{N}_Q = (N_Q, E_Q, \mathcal{I}_Q)$ be the frame on (N_Q, E_Q) with $\mathcal{I}_Q$ the modal algebra generated by $\{f_3\}$. It is easy to see that $\mathcal{I}_Q$ consists of precisely these subsets of N_Q whose intersection with A is either finite of cofinite in A and intersection with B is either finite of cofinite in B. Moreover, for distinct Q and Q', the dual algebras of $\mathfrak{N}_Q$ and $\mathfrak{N}_{Q'}$ are non-isomorphic.

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Theorem (T.K., Stevens)

Let $V(N_Q)$ be the variety generated by the dual algebra of $\mathfrak{N}_Q$. Then, $V(N_Q)$ is a cover of $V(K_2)$ in Λ^{KTB}. Thus, there are continuum covers of $V(K_2)$ in Λ^{KTB}.

An intimation of a proof

Proof.

Sketch: (1) show that any subset $X \subset N_Q$ such that $\Diamond X \setminus X \neq \neg X$, generates $\mathcal{I}_Q$. (2) show that any element x of any ultrapower of the dual algebra of $\mathfrak{N}_Q$ such that $\Diamond x \wedge \neg x \neq \neg x$, generates an algebra containing a subalgebra isomorphic to the dual algebra of $\mathfrak{N}_Q$. (3) show for distinct Q and Q', the varieties $V(N_Q)$ and $V(N_{Q'})$ are also distinct. From (1), (2) and some fiddling with Jónsson's Lemma conclude that $V(N_Q)$ covers $V(K_2)$. From (3) conclude that there are continuum such covers. □