On the structure of linear pseudo-BCK-algebras

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Joint work with Anatolij Dvurečenskij
Every linear hoop/BL-algebra is an ordinal sum of linear Wajsberg hoops [Agliano & Montagna]

The \( \{ \to, 1 \} \)-subreducts of hoops are BCK-algebras satisfying the identity

\[
(x \to y) \to (x \to z) = (y \to x) \to (y \to z)
\]

[Blok, Ferreirim]

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1. Every linear pseudo-hoop/pseudo-BL-algebra is an ordinal sum of linear Wajsberg pseudo-hoops [Dvurečenskij]

2. The \( \{\to, \sim \to, 1\} \)-subreducts of pseudo-hoops are pseudo-BCK-algebras satisfying

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A **porim** (＝ partially ordered residuated integral monoid) is a structure \((A, \leq, \cdot, \to, \rightsquigarrow, 1)\) where

- \((A, \leq)\) is a poset with greatest element 1,
- \((A, \cdot, 1)\) is a monoid,
- \(c \leq a \to b\) iff \(c \cdot a \leq b\), and \(c \leq a \rightsquigarrow b\) iff \(a \cdot c \leq b\).

A **pseudo-hoop** [Georgescu, Leuștean & Preoteasa] is a porim satisfying

\[(x \to y) \cdot x = y \cdot (y \rightsquigarrow x).\]

A **Wajsberg pseudo-hoop** [Georgescu, Leuștean & Preoteasa] is a pseudo-hoop satisfying

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A **porim** (= partially ordered residuated integral monoid) is a structure \((A, \leq, \cdot, \rightarrow, \leadsto, 1)\) where

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A **Wajsberg pseudo-hoop** [Georgescu, Leuştean & Preoteasa] is a pseudo-hoop satisfying

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A **pseudo-hoop** [Georgescu, Leuștean & Preoteasa] is a porim satisfying

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A **Wajsberg pseudo-hoop** [Georgescu, Leuștean & Preoteasa] is a pseudo-hoop satisfying

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**Pseudo-MV-algebras** = bounded Wajsberg pseudo-hoops

**Pseudo-BL-algebras** = bounded pseudo-hoops satisfying

\[(x \to y) \to z \leq ((y \to x) \to z) \to z\]

\[(x \bowtie y) \bowtie z \leq ((y \bowtie x) \bowtie z) \bowtie z\]
Pseudo-MV-algebras = bounded Wajsberg pseudo-hoops

Pseudo-BL-algebras = bounded pseudo-hoops satisfying

\[(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z\]
\[(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z\]
Porims = algebras \( (A, \cdot, \to, \multimap, 1) \) of type \((2, 2, 2, 0)\) that satisfy:

\[
(x \to y) \multimap ((y \to z) \multimap (x \to z)) = 1, \\
(x \multimap y) \to ((y \multimap z) \to (x \multimap z)) = 1,
\]

\(1 \to x = x,\)

\(1 \multimap x = x,\)

\(x \to 1 = 1,\)

\((x \cdot y) \to z = x \to (y \to z),\)

\(x \to y = 1 \ \& \ y \to x = 1 \ \Rightarrow \ x = y.\)  

A \textbf{pseudo-BCK-algebra} [Georgescu & Iorgulescu] is an algebra \( (A, \to, \multimap, 1) \) of type \((2, 2, 0)\) satisfying (1)–(5) and (7).

Pseudo-BCK-algebras are the \{\to, \multimap, 1\}-subreducts of porims.
Porims = algebras \((A, \cdot, \rightarrow, \leadsto, 1)\) of type \((2, 2, 2, 0)\) that satisfy:

\[
(x \rightarrow y) \leadsto (((y \rightarrow z) \leadsto (x \rightarrow z))) = 1, \quad (1)
\]
\[
(x \leadsto y) \rightarrow (((y \leadsto z) \rightarrow (x \leadsto z))) = 1, \quad (2)
\]
\[
1 \rightarrow x = x, \quad (3)
\]
\[
1 \leadsto x = x, \quad (4)
\]
\[
x \rightarrow 1 = 1, \quad (5)
\]
\[
(x \cdot y) \rightarrow z = x \rightarrow (y \rightarrow z), \quad (6)
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\[
x \rightarrow y = 1 \quad \& \quad y \rightarrow x = 1 \quad \Rightarrow \quad x = y. \quad (7)
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Porims = algebras \((A, \cdot, \rightarrow, \simrightarrow, 1)\) of type \((2, 2, 2, 0)\) that satisfy:

\[
(x \rightarrow y) \simrightarrow ((y \rightarrow z) \simrightarrow (x \rightarrow z)) = 1,
\]

\[
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\]

\[
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\[
\begin{align*}
\text{1) } (x \rightarrow y) \rightsquigarrow ((y \rightarrow z) \rightsquigarrow (x \rightarrow z)) &= 1, \\
\text{2) } (x \rightsquigarrow y) \rightarrow ((y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)) &= 1, \\
\text{3) } 1 \rightarrow x &= x, \\
\text{4) } 1 \rightsquigarrow x &= x, \\
\text{5) } x \rightarrow 1 &= 1, \\
\text{6) } (x \cdot y) \rightarrow z &= x \rightarrow (y \rightarrow z), \\
\text{7) } x \rightarrow y = 1 &\quad \& \quad y \rightarrow x = 1 \quad \Rightarrow \quad x = y.
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Pseudo-BCK-algebras are the \(\{\rightarrow, \rightsquigarrow, 1\}\)-subreducts of porims.
Let $A = (A, \rightarrow, \leadsto, 1)$ be a pseudo-BCK-algebra. The relation $\leq$ given by

$$x \leq y \text{  iff  } x \rightarrow y = 1 \text{  (iff  } x \leadsto y = 1)$$

is a partial order on $A$; $1$ is the greatest element of $(A, \leq)$. If $(A, \leq)$ is a chain, then $A$ is a linear pseudo-BCK-algebra.
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If $(A, \leq)$ is a chain, then $A$ is a **linear** pseudo-BCK-algebra.
A pseudo-$\mathcal{L}$BCK-algebra [Dvurečenskij & Vetterlein] is a pseudo-BCK-algebra $A = (A, \rightarrow, \rightsquigarrow, 1)$ satisfying the identity
\[(x \rightarrow y) \rightsquigarrow y = (y \rightsquigarrow x) \rightarrow x,\]
and the following “relative cancellation” property:
\[x \geq y \land x \geq z \land x \rightarrow y = x \rightarrow z \Rightarrow y = z.\]

- RCP can be replaced by the identity
  \[(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z)\]
- pseudo-$\mathcal{L}$BCK-algebras = the $\{\rightarrow, \rightsquigarrow, 1\}$-subreducts of Wajsberg pseudo-hoops and pseudo-MV-algebras
- pseudo-$\mathcal{L}$BCK-algebras = Bosbach’s cone algebras
A *pseudo-$\ell$BCK-algebra* [Dvurečenskij & Vetterlein] is a pseudo-BCK-algebra $A = (A, \to, \leadsto, 1)$ satisfying the identity

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A **pseudo-ŁBCK-algebra** [Dvurečenskij & Vetterlein] is a pseudo-BCK-algebra \( A = (A, \rightarrow, \bowtie, 1) \) satisfying the identity

\[
(x \rightarrow y) \bowtie y = (y \bowtie x) \rightarrow x,
\]

and the following “relative cancellation” property:

\[
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- **pseudo-$\mathcal{L}$BCK-algebras** = Bosbach’s cone algebras
Let $(I, \leq)$ be a non-empty chain. The ordinal sum of linear pseudo-BCK-algebras $A_i$ ($i \in I$) such that $A_i \cap A_j = \{1\}$ for all $i \neq j \in I$ is a pseudo-BCK-algebra $\bigoplus_{i \in I} A_i = (\bigcup_{i \in I} A_i, \rightarrow, \leadsto, 1)$ where the operations $\rightarrow, \leadsto$ are defined as follows:

\[
x \rightarrow y = \begin{cases} 
x \rightarrow_i y & \text{if } x, y \in A_i, \\
1 & \text{if } x \in A_i \setminus \{1\}, y \in A_j, i < j, \\
y & \text{if } x \in A_i, y \in A_j, i > j,
\end{cases}
\]

\[
x \leadsto y = \begin{cases} 
x \leadsto_i y & \text{if } x, y \in A_i, \\
1 & \text{if } x \in A_i \setminus \{1\}, y \in A_j, i < j, \\
y & \text{if } x \in A_i, y \in A_j, i > j.
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\]
\[ y \in A_j \]
\[ i < j \]
\[ x \in A_i \]

\[ x \rightarrow y = 1 = x \bowtie y \]
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Which linear pseudo-BCK-algebras arise as ordinal sums of linear pseudo-ŁBCK-algebras?

A linear pseudo-BCK-algebra is an ordinal sum of linear pseudo-ŁBCK-algebras iff it satisfies the identities

\[(x \to y) \to (x \to z) = (y \to x) \to (y \to z), \quad (H)\]
\[(((x \to y) \multimap y) \to x) \multimap x = (((y \multimap x) \to x) \multimap y) \to y. \quad (J)\]

The identity (H), as well as

\[(x \multimap y) \multimap (x \multimap z) = (y \multimap x) \multimap (y \multimap z), \quad (H')\]

holds in all pseudo-hoops, but there exist pseudo-hoops that do not satisfy (J) (though it holds in hoops).
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Let $A$ be a linear pseudo-BCK-algebra. A cut of $A$ is $X \subseteq A \setminus \{1\}$ such that

- $x < y$ for all $x \in X$ and $y \in A \setminus X$,
- $A \setminus X$ is closed under $\to$, $\simto$,
- $y \to x = x = y \simto x$ for all $x \in X$ and $y \in A \setminus X$.

A cut is trivial if $X = \emptyset$ or $X = A \setminus \{1\}$.

1. If $A$ is the ordinal sum $A_1 \oplus A_2$ of linear pseudo-BCK-algebras $A_1$ and $A_2$, then $X = A_1 \setminus \{1\}$ is a cut of $A$. If $A_1$ and $A_2$ are non-trivial pseudo-BCK-algebras, then the cut is non-trivial.

2. Let $A$ be a linear pseudo-BCK-algebra and $X$ be a cut of $A$. Then $A_1 = (X \cup \{1\}, \to, \simto, 1)$ and $A_2 = (A \setminus X, \to, \simto, 1)$ are subalgebras of $A$, and $A = A_1 \oplus A_2$. If the cut $X$ is non-trivial, then $A_1$, $A_2$ are non-trivial pseudo-BCK-algebras.
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Let $A$ be a linear pseudo-BCK-algebra. For $a \in A \setminus \{1\}$ we put

$$X_a = \{ x \in A \setminus \{1\} \mid a \rightarrow x = x \}.$$  

We have

$$X_a = \{ x \in A \setminus \{1\} \mid a \bowtie x = x \}.$$

If $A$ satisfies the identities (H) and (J), then for every $a \in A \setminus \{1\}$, $X_a$ is a cut of $A$. The cut is non-trivial provided that $X_a \neq \emptyset$.

Let $A$ be a linear pseudo-BCK-algebra satisfying (H) and (J). The following statements are equivalent:

1. $A$ is sum irreducible.
2. For all $a, b \in A$, if $a \rightarrow b = b$ (or $a \bowtie b = b$), then $a = 1$ or $b = 1$.
3. $A$ is a pseudo-$Ł$BCK-algebra.
Let $A$ be a linear pseudo-BCK-algebra. For $a \in A \setminus \{1\}$ we put

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We have

$$X_a = \{ x \in A \setminus \{1\} \mid a \bowtie x = x \}.$$  

If $A$ satisfies the identities $(H)$ and $(J)$, then for every $a \in A \setminus \{1\}$, $X_a$ is a cut of $A$. The cut is non-trivial provided that $X_a \neq \emptyset$.

Let $A$ be a linear pseudo-BCK-algebra satisfying $(H)$ and $(J)$. The following statements are equivalent:

1. $A$ is sum irreducible.
2. For all $a, b \in A$, if $a \rightarrow b = b$ (or $a \bowtie b = b$), then $a = 1$ or $b = 1$.
3. $A$ is a pseudo-$\check{L}$BCK-algebra.
Every non-trivial linear pseudo-BCK-algebra satisfying the equations

\[(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z) \quad (H)\]
\[((x \rightarrow y) \hyperlink{1}{\rightsquigarrow} y) \rightarrow x) \hyperlink{1}{\rightsquigarrow} x = (((y \hyperlink{1}{\rightsquigarrow} x) \rightarrow x) \hyperlink{1}{\rightsquigarrow} y) \hyperlink{1}{\rightsquigarrow} y) \quad (J)\]

can uniquely be represented as an ordinal sum of non-trivial linear pseudo-ŁBCK-algebras.

For every linear pseudo-BCK-algebra \(A\), the following statements are equivalent:

1. \(A\) satisfies the identities \((H)\) and \((J)\).
2. \(A\) is an ordinal sum of linear pseudo-ŁBCK-algebras.
3. \(A\) is a \({\rightarrow, \hyperlink{1}{\rightsquigarrow}, 1}\)-subreduct of a linear pseudo-hoop.
4. \(A\) is a \({\rightarrow, \hyperlink{1}{\rightsquigarrow}, 1}\)-subreduct of a linear pseudo-BL-algebra.
Every non-trivial linear pseudo-BCK-algebra satisfying the equations

\[(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z) \quad (H)\]
\[(((x \rightarrow y) \sim y) \rightarrow x) \sim x = (((y \sim x) \rightarrow x) \sim y) \sim y \quad (J)\]

can uniquely be represented as an ordinal sum of non-trivial linear pseudo-ŁBCK-algebras.

For every linear pseudo-BCK-algebra \( A \), the following statements are equivalent:

1. \( A \) satisfies the identities (H) and (J).
2. \( A \) is an ordinal sum of linear pseudo-ŁBCK-algebras.
3. \( A \) is a \( \{\rightarrow, \sim, 1\} \)-subreduct of a linear pseudo-hoop.
4. \( A \) is a \( \{\rightarrow, \sim, 1\} \)-subreduct of a linear pseudo-BL-algebra.
Every non-trivial linear pseudo-BCK-algebra satisfying the equations

\[(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z) \quad (H)\]
\[(((x \rightarrow y) \bowtie y) \rightarrow x) \bowtie x = (((y \bowtie x) \rightarrow x) \bowtie y) \bowtie y \quad (J)\]

can uniquely be represented as an ordinal sum of non-trivial linear pseudo-ŁBCK-algebras.

For every linear pseudo-BCK-algebra \(A\), the following statements are equivalent:

1. \(A\) satisfies the identities \((H)\) and \((J)\).
2. \(A\) is an ordinal sum of linear pseudo-ŁBCK-algebras.
3. \(A\) is a \{\(\rightarrow\), \(\bowtie\), \(1\)\}-subreduct of a linear pseudo-hoop.
4. \(A\) is a \{\(\rightarrow\), \(\bowtie\), \(1\)\}-subreduct of a linear pseudo-BL-algebra.
Every non-trivial linear pseudo-BCK-algebra satisfying the equations

\[(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z) \quad (H)\]
\[(((x \rightarrow y) \bowtie y) \rightarrow x) \bowtie x = (((y \bowtie x) \rightarrow x) \bowtie y) \bowtie y \quad (J)\]

can uniquely be represented as an ordinal sum of non-trivial linear pseudo-$\mathcal{L}$BCK-algebras.

For every linear pseudo-BCK-algebra $A$, the following statements are equivalent:

1. $A$ satisfies the identities (H) and (J).
2. $A$ is an ordinal sum of linear pseudo-$\mathcal{L}$BCK-algebras.
3. $A$ is a $\{\rightarrow, \bowtie, 1\}$-subreduct of a linear pseudo-hoop.
4. $A$ is a $\{\rightarrow, \bowtie, 1\}$-subreduct of a linear pseudo-BL-algebra.
Every non-trivial linear pseudo-BCK-algebra satisfying the equations

\[(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z) \quad (H)\]
\[(((x \rightarrow y) \bowtie y) \rightarrow x) \bowtie x = (((y \bowtie x) \rightarrow x) \bowtie y) \bowtie y \quad (J)\]

can uniquely be represented as an ordinal sum of non-trivial linear pseudo-ŁBCK-algebras.

For every linear pseudo-BCK-algebra \(A\), the following statements are equivalent:

1. \(A\) satisfies the identities \((H)\) and \((J)\).
2. \(A\) is an ordinal sum of linear pseudo-ŁBCK-algebras.
3. \(A\) is a \(\{\rightarrow, \bowtie, 1\}\)-subreduct of a linear pseudo-hoop.
4. \(A\) is a \(\{\rightarrow, \bowtie, 1\}\)-subreduct of a linear pseudo-BL-algebra.
Every non-trivial linear pseudo-BCK-algebra satisfying the equations

\[(x \to y) \to (x \to z) = (y \to x) \to (y \to z) \quad (H)\]
\[
(((x \to y) \bowtie y) \to x) \bowtie x = (((y \bowtie x) \to x) \bowtie y) \bowtie y \quad (J)
\]
can uniquely be represented as an ordinal sum of non-trivial linear pseudo-ŁBCK-algebras.

For every linear pseudo-BCK-algebra \(A\), the following statements are equivalent:

1. \(A\) satisfies the identities (H) and (J).
2. \(A\) is an ordinal sum of linear pseudo-ŁBCK-algebras.
3. \(A\) is a \(\{\to, \bowtie, 1\}\)-subreduct of a linear pseudo-hoop.
4. \(A\) is a \(\{\to, \bowtie, 1\}\)-subreduct of a linear pseudo-BL-algebra.
A pseudo-hoop/pseudo-BL-algebra/pseudo-BCK-algebra which is a subdirect product of ones with an underlying linear order is said to be **representable**.
Representable pseudo-BCK-algebras/pseudo-hoops/pseudo-BL-algebras are axiomatized by the identity

\[(x \rightarrow y) \rightarrow u \leq ((((((y \rightarrow x) \rightarrow z) \rightarrow z) \rightsquigarrow w) \rightsquigarrow w) \rightarrow u) \rightarrow u.\]

(R)

For every pseudo-BCK-algebra \(A\), the following are equivalent:

1. \(A\) is a \(\{\rightarrow, \rightsquigarrow, 1\}\)-subreduct of a representable pseudo-BL-algebra;
2. \(A\) is a \(\{\rightarrow, \rightsquigarrow, 1\}\)-subreduct of a representable pseudo-hoop;
3. \(A\) satisfies the equations (R), (H) and (J).
A pseudo-hoop/pseudo-BL-algebra/pseudo-BCK-algebra which is a subdirect product of ones with an underlying linear order is said to be **representable**.

Representable pseudo-BCK-algebras/pseudo-hoops/pseudo-BL-algebras are axiomatized by the identity

\[(x \rightarrow y) \rightarrow u \leq ((((((y \rightarrow x) \rightarrow z) \rightarrow z) \bowtie w) \bowtie w) \rightarrow u) \rightarrow u.\]

\[(R)\]

For every pseudo-BCK-algebra \(A\), the following are equivalent:

1. \(A\) is a \(\{\rightarrow, \bowtie, 1\}\)-subreduct of a representable pseudo-BL-algebra;

2. \(A\) is a \(\{\rightarrow, \bowtie, 1\}\)-subreduct of a representable pseudo-hoop;

3. \(A\) satisfies the equations \((R)\), \((H)\) and \((J)\).
A pseudo-hoop/pseudo-BL-algebra/pseudo-BCK-algebra which is a subdirect product of ones with an underlying linear order is said to be **representable**. Representable pseudo-BCK-algebras/pseudo-hoops/pseudo-BL-algebras are axiomatized by the identity

\[(x \rightarrow y) \rightarrow u \leq ((((((y \rightarrow x) \rightarrow z) \rightarrow z) \rightsquigarrow w) \rightsquigarrow w) \rightarrow u) \rightarrow u.\]

\((R)\)

For every pseudo-BCK-algebra \(A\), the following are equivalent:

1. \(A\) is a \(\{\rightarrow, \rightsquigarrow, 1\}\)-subreduct of a representable pseudo-BL-algebra;
2. \(A\) is a \(\{\rightarrow, \rightsquigarrow, 1\}\)-subreduct of a representable pseudo-hoop;
3. \(A\) satisfies the equations \((R)\), \((H)\) and \((J)\).
A pseudo-hoop/pseudo-BL-algebra/pseudo-BCK-algebra which is a subdirect product of ones with an underlying linear order is said to be **representable**.

Representable pseudo-BCK-algebras/pseudo-hoops/pseudo-BL-algebras are axiomatized by the identity

\[(x \rightarrow y) \rightarrow u \leq (((((((y \rightarrow x) \rightarrow z) \rightarrow z) \wedge w) \wedge w) \rightarrow u) \rightarrow u).\]

(R)

For every pseudo-BCK-algebra \(A\), the following are equivalent:

1. \(A\) is a \(\{\rightarrow, \wedge, 1\}\)-subreduct of a representable pseudo-BL-algebra;
2. \(A\) is a \(\{\rightarrow, \wedge, 1\}\)-subreduct of a representable pseudo-hoop;
3. \(A\) satisfies the equations (R), (H) and (J).
A pseudo-hoop/pseudo-BL-algebra/pseudo-BCK-algebra which is a subdirect product of ones with an underlying linear order is said to be **representable**.

Representable pseudo-BCK-algebras/pseudo-hoops/pseudo-BL-algebras are axiomatized by the identity

\[(x \to y) \to u \le ((((((y \to x) \to z) \to z) \sim w) \sim w) \to u) \to u.\]  

\((R)\)

For every pseudo-BCK-algebra \(A\), the following are equivalent:

1. \(A\) is a \(\{\to, \sim, 1\}\)-subreduct of a representable pseudo-BL-algebra;
2. \(A\) is a \(\{\to, \sim, 1\}\)-subreduct of a representable pseudo-hoop;
3. \(A\) satisfies the equations \((R)\), \((H)\) and \((J)\).
The class of all \( \{ \rightarrow, \rightsquigarrow, 1 \} \)-subreducts of representable pseudo-BL-algebras/pseudo-hoops is the variety of pseudo-BCK-algebras satisfying

\[
(x \rightarrow y) \rightarrow u \leq ((((((y \rightarrow x) \rightarrow z) \rightarrow z) \rightsquigarrow w) \rightsquigarrow w) \rightarrow u) \rightarrow u,
\]

\((R)\)

\[
(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z),
\]

\((H)\)

\[
(((x \rightarrow y) \rightsquigarrow y) \rightarrow x) \rightsquigarrow x = (((y \rightsquigarrow x) \rightarrow x) \rightsquigarrow y) \rightarrow y.
\]

\((J)\)
The class of all \(\{\rightarrow, \leadsto, 1\}\)-subreducts of representable pseudo-BL-algebras/pseudo-hoops is the variety of pseudo-BCK-algebras satisfying

\[
(x \rightarrow y) \rightarrow u \leq ((((((y \rightarrow x) \rightarrow z) \rightarrow z) \leadsto w) \leadsto w) \rightarrow u) \rightarrow u, \quad \text{(R)}
\]

\[
(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z), \quad \text{(H)}
\]

\[
(((x \rightarrow y) \leadsto y) \rightarrow x) \leadsto x = (((y \leadsto x) \rightarrow x) \leadsto y) \rightarrow y. \quad \text{(J)}
\]

THANK YOU