# Semantical Aspects of a Logic for Pragmatics

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# Outline of talk

1. Logic for pragmatics
    - ◇ Key example
    - ◇ Formal system
2. Categorical semantics
    - ◇ Basic structure
    - ◇ Completeness
3. Degenerate models
    - ◇ Kripke semantics
    - ◇ Further issues

# Logic for pragmatics

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# Commonsense reasoning

1. Informal argument

   If a police man says to a taxi driver "Keep close to that red car!" and the red car speeds, then the taxi driver can interpret the command given to him as the command to speed

2. A possible formalization

$$\frac{K, S_r, (K \circ S_r) \supset S_t \Longrightarrow S_t}{(K)^{\circ}, S_r, (K \circ S_r) \supset S_t \Longrightarrow (S_t)^{\circ}}$$

- ◇ $K$ = *taxi keeps close to red car*
- ◇ $S_r$ = *red car speeds*, $S_t$ = *taxi speeds*

# Pragmatic language

1. Assertive formulae

$$\eta \ := \ \vdash p \mid \epsilon \mid \eta \circ \eta$$

2. Pure causal formulae

$$\xi \ := \ \eta \mid \eta \supset\!\!\!\!\!\!\text{c}\ \xi \mid \xi \circ \xi$$

3. Causal-deontic formulae

$$\gamma \ := \ \xi \mid \eta^{\circ} \mid \gamma \circ \gamma$$

4. Where we have that

$$(\eta_1 \circ \eta_2)^{\circ} \neq \ \eta_1^{\circ} \circ \eta_2^{\circ}$$

# Sequent calculus

$$\frac{}{\vdash p \Rightarrow_{\mathtt{a}} \vdash p} \qquad \frac{}{\Rightarrow_{\mathtt{a}} \epsilon} \qquad \frac{\vec{\eta} \Rightarrow_{\mathtt{a}} \eta}{\epsilon, \vec{\eta} \Rightarrow_{\mathtt{a}} \eta}$$

$$\frac{\Xi_1 \Rightarrow_{\mathtt{c}} \eta \quad \xi, \Xi_2 \Rightarrow_{\mathtt{c}} \xi}{\eta \supset\!\!\!\cdot\; \xi, \Xi_1, \Xi_2 \Rightarrow_{\mathtt{c}} \xi} \qquad \frac{\Xi, \eta \Rightarrow_{\mathtt{c}} \xi}{\Xi \Rightarrow_{\mathtt{c}} \eta \supset\!\!\!\cdot\; \xi}$$

$$\frac{\vec{\eta} \Rightarrow_{\mathtt{a}} \eta}{\vec{\eta} \Rightarrow_{\mathtt{c}} \eta} \qquad \frac{\Xi \Rightarrow_{\mathtt{c}} \xi}{\Xi \Rightarrow_{\mathtt{d}} \xi} \qquad \frac{\vec{\eta}, \Xi \Rightarrow_{\mathtt{c}} \eta}{\vec{\eta}^{\circ}, \Xi \Rightarrow_{\mathtt{d}} \eta^{\circ}}$$

$$\frac{\gamma', \gamma', \Gamma \Rightarrow_{\mathtt{x}} \gamma}{\gamma', \Gamma \Rightarrow_{\mathtt{x}} \gamma} \qquad \frac{\Gamma_1 \Rightarrow_{\mathtt{x}} \gamma' \quad \gamma', \Gamma_2 \Rightarrow_{\mathtt{x}} \gamma}{\Gamma_1, \Gamma_2 \Rightarrow_{\mathtt{x}} \gamma}$$

$$\frac{\gamma_1, \gamma_2, \Gamma \Rightarrow_{\mathtt{x}} \gamma}{\gamma_1 \circ \gamma_2, \Gamma \Rightarrow_{\mathtt{x}} \gamma} \qquad \frac{\Gamma_1 \Rightarrow_{\mathtt{x}} \gamma_1 \quad \Gamma_2 \Rightarrow_{\mathtt{x}} \gamma_2}{\Gamma_1, \Gamma_2 \Rightarrow_{\mathtt{x}} \gamma_1 \circ \gamma_2}$$

# Categorical semantics

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# Causal-deontic frame

A causal-deontic frame consists of three categories

1. $\mathbb{A} = (\mathbb{A}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta)$
2. $\mathbb{C} = (\mathbb{C}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta)$
3. $\mathbb{D} = (\mathbb{D}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta)$

together with three structure-preserving functors

1. $\mathcal{J}_{\mathtt{ac}} = (\mathcal{J}_{\mathtt{ac}}, \iota_2, \iota_0)\colon \mathbb{A} \longrightarrow \mathbb{C}$
2. $\mathcal{J}_{\mathtt{cd}} = (\mathcal{J}_{\mathtt{cd}}, \iota_2, \iota_0)\colon \mathbb{C} \longrightarrow \mathbb{D}$
3. $\mathcal{O} = (\mathcal{O}, \theta_2, \theta_1)\colon \mathbb{A} \longrightarrow \mathbb{D}$

such that both $\mathcal{J}_{\mathtt{ac}}$ and $\mathcal{J}_{\mathtt{cd}}$ are strong, and the category $\mathbb{C}$ is closed with respect to the category $\mathcal{J}_{\mathtt{ac}}\mathbb{A}$

# The basic problem

1. Given a rule such as

$$\frac{\vec{\eta}, \Xi \Rightarrow_{\mathtt{c}} \eta}{\vec{\eta}^{\circ}, \Xi \Rightarrow_{\mathtt{d}} \eta^{\circ}}$$

2. How can we go from

$$\mathcal{J}_{\mathtt{ac}}\mathrm{A}_1 \otimes \mathrm{C} \longrightarrow \mathcal{J}_{\mathtt{ac}}\mathrm{A}_2 \text{ in } \mathbb{C}$$

to

$$\mathcal{O}\mathrm{A}_1 \otimes \mathcal{J}_{\mathtt{cd}}\mathrm{C} \longrightarrow \mathcal{O}\mathrm{A}_2 \text{ in } \mathbb{D}?$$

# Restriction functor

The functor $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}\colon \mathbb{C}^{op} \longrightarrow \mathbf{SCat}$ is defined as follows

1. $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C})$ is a semi-category that has
   (a) objects $\mathcal{J}_{\mathsf{ac}}\mathrm{A}$ of $\mathbb{C}$ as objects
   (b) morphisms $\mathcal{J}_{\mathsf{ac}}\mathrm{A}_1 \otimes \mathrm{C} \longrightarrow \mathcal{J}_{\mathsf{ac}}\mathrm{A}_2$ of $\mathbb{C}$ as morphisms
2. $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{f})\colon \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}_2) \longrightarrow \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}_1)$ is a semi-functor that maps
   (a) an object $\mathcal{J}_{\mathsf{ac}}\mathrm{A}$ to the object $\mathcal{J}_{\mathsf{ac}}\mathrm{A}$
   (b) a morphism $\mathrm{g}\colon \mathcal{J}_{\mathsf{ac}}\mathrm{A}_1 \otimes \mathrm{C}_2 \longrightarrow \mathcal{J}_{\mathsf{ac}}\mathrm{A}_2$ to the morphism

$$\mathcal{J}_{\mathsf{ac}}\mathrm{A}_1 \otimes \mathrm{C}_1 \xrightarrow{\mathrm{id} \otimes \mathrm{f}} \mathcal{J}_{\mathsf{ac}}\mathrm{A}_1 \otimes \mathrm{C}_2 \xrightarrow{\mathrm{g}} \mathcal{J}_{\mathsf{ac}}\mathrm{A}_2$$

# Expansion functor

The functor $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}} \colon \mathbb{C}^{\mathsf{op}} \longrightarrow \mathbf{SCat}$ is defined as follows

1. $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(\mathrm{C})$ is a semi-category that has
   (a) objects $\mathrm{D}$ of $\mathbb{D}$ as objects
   (b) morphisms $\mathrm{D}_1 \otimes \mathcal{J}_{\mathtt{cd}}\mathrm{C} \longrightarrow \mathrm{D}_2$ of $\mathbb{D}$ as morphisms
2. $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(\mathrm{f}) \colon \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(\mathrm{C}_2) \longrightarrow \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(\mathrm{C}_1)$ is a semi-functor that maps
   (a) an object $\mathrm{D}$ to the object $\mathrm{D}$
   (b) a morphism $\mathrm{g} \colon \mathrm{D}_1 \otimes \mathcal{J}_{\mathtt{cd}}\mathrm{C}_2 \longrightarrow \mathrm{D}_2$ to the morphism

$$\mathrm{D}_1 \otimes \mathcal{J}_{\mathtt{cd}}\mathrm{C}_1 \xrightarrow{\mathrm{id} \otimes \mathcal{J}_{\mathtt{cd}}\mathrm{f}} \mathrm{D}_1 \otimes \mathcal{J}_{\mathtt{cd}}\mathrm{C}_2 \xrightarrow{\mathrm{g}} \mathrm{D}_2$$

# Summary of results

1. Completeness

   A natural transformation $\vartheta_{\mathrm{C}}\colon \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}) \longrightarrow \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(\mathrm{C})$ that maps

   (a) an object $\mathcal{J}_{\mathtt{ac}}\mathrm{A}_i$ of $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C})$ to the object $\mathcal{O}\mathrm{A}_i$ of $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(\mathrm{C})$

   (b) a morphism $\mathcal{J}_{\mathtt{ac}}\mathrm{A}_1 \otimes \mathrm{C} \longrightarrow \mathcal{J}_{\mathtt{ac}}\mathrm{A}_2$ of $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C})$ to the morphism $\mathcal{O}\mathrm{A}_1 \otimes \mathcal{J}_{\mathtt{cd}}\mathrm{C} \longrightarrow \mathcal{O}\mathrm{A}_2$ of $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(\mathrm{C})$

2. Soundness

   A coherence condition on the natural transformation $\vartheta_{\mathrm{C}}\colon \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}) \longrightarrow \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(\mathrm{C})$

# Degenerate models

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# Kripke frame

1. Let $U = (U, \cdot, 1, \preceq)$ and $W = (W, \cdot, 1, \preceq)$ be preordered commutative monoids such that
   (a) $U$ is a proper (preord. comm.) submonoid of $W$
   (b) for all $w \in W$ (resp. $u \in U$), $w \preceq ww$ (resp. $u \preceq uu$)
2. A Kripke frame is a triple $(W, U, \lhd)$ where $\lhd \subseteq U \times W$ is a binary relation such that
   (a) $\forall u \in U. u \lhd u$
   (b) $\forall u, u' \in U. \forall w, w' \in W. u \lhd w \wedge u' \lhd w' \rightarrow uu' \lhd ww'$
   (c) $\forall u \in U. \forall w, w', w'' \in W. w \preceq w'w'' \wedge u \lhd w$
   $\rightarrow \exists u' \in U. u \preceq u'w'' \wedge u' \lhd w'$

# Kripke model

1. A Kripke model $(W, U, \lhd, \Vdash)$ is a Kripke frame $(W, U, \lhd)$ endowed with a forcing relation $\Vdash \subseteq W \times \Gamma$
2. Given a downward closed subset $(\vdash p)^\star$ of $U$ for each propositional atom $p$, the forcing relation is defined by the following clauses
   (a) $w \Vdash \vdash p \Longleftrightarrow w \in (\vdash p)^\star$
   (b) $w \Vdash \epsilon \Longleftrightarrow w \preceq 1$
   (c) $w \Vdash \eta^\circ \Longleftrightarrow \forall u \in U. u \lhd w \rightarrow u \Vdash \eta$
   (d) $w \Vdash \eta \supset\!\!\!\!\!\!\circ\ \xi \Longleftrightarrow \forall w' \in W. w' \Vdash \eta \rightarrow ww' \Vdash \xi$
   (e) $w \Vdash \gamma_1 \circ \gamma_2 \Longleftrightarrow \exists w_1, w_2 \in W. w \preceq w_1 w_2 \wedge w_1 \Vdash \gamma_1 \wedge w_2 \Vdash \gamma_2$

# Causal-deontic algebra

A causal-deontic algebra $(A, C, D, i, j, o)$ consists of three preordered commutative monoids

$$A = (A, \cdot, 1, \leq_{\mathtt{a}}) \qquad C = (C, \cdot, 1, \leq_{\mathtt{c}}) \qquad D = (D, \cdot, 1, \leq_{\mathtt{d}})$$

together with three monotone functions

$$i: A \longrightarrow C \qquad j: C \longrightarrow D \qquad o: A \longrightarrow D$$

such that the following conditions are satisfied

1. $i$ and $j$ are monoid homomorphisms
2. $C$ is residuated (closed) with respect to $i(A)$
3. $o$ is such that $1 \leq_{\mathtt{d}} o(1)$ and $o(a_1)o(a_2) \leq_{\mathtt{d}} o(a_1 a_2)$
4. if $i(a_1)c \leq_{\mathtt{c}} i(a_2)$ then $o(a_1)j(c) \leq_{\mathtt{d}} o(a_2)$

# Summary of results (con't)

1. The logic is sound and complete with respect to the Kripke semantics
2. A causal-deontic algebra can be recovered from the downward closed subsets of the set of possible worlds
3. Given an interpretation function $[-]$, the logic is sound and complete with respect to the algebraic semantics
4. Kripke models provide a degenerate example of the categorical model

# Future work

1. Coherence condition

2. Non-degenerate models

3. More general construction

4. Indexed categories vs. fibrations