# Boolean Algebras and Lambda Calculus

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- What is lambda calculus?
  - A theory of functions
  - The name of a function contains a description of the function as a program
  - Untyped world: every element in lambda calculus is contemporaneously
    * Function
    * A possible argument for a function
    * A possible result of the application of a function to an argument
  - No Partiality: every function can be applied to any other function including itself

# Lambda terms

- $\lambda$-notation:
  Expression: $a + 2$ Function: $f(a) = a + 2$ $\lambda_a(a + 2)$

- Algebraic similarity type $\Sigma$:
  - Nullary operators: $a, b, c, \dots \in A$ (formal parameters)
  - Binary operator: $\bullet$ (application)
  - Unary operators: $\lambda_a$ $(a \in A)$ ($\lambda$-abstractions)

- A $\lambda$-term is a ground $\Sigma$-term (no algebraic variable $x, y, z, \dots$)

  $\lambda_a(a)$ YES $\lambda_a(x)$ NO

  - $a$ = generic function
  - $M \cdot N$ = function $M$ applied to argument $N$
  - $\lambda_a(M)$ = function of $a$ whose body is expression $M$

# How to compute (informally)

- Bound and free parameters: $\lambda_a(a \cdot b)$
- $\alpha$-conversion: $\lambda_a(a \cdot b) = \lambda_c(c \cdot b)$
  The name of a bound parameter does not matter
- $\beta$-conversion: $\lambda_a(a) \cdot b = b$

$$\lambda_a(aa) \cdot \lambda_a(aa) = \lambda_a(aa) \cdot \lambda_a(aa) = \ldots$$

# The classic $\lambda$-calculus

- The $\lambda$-term algebra is the absolutely free $\Sigma$-algebra over an empty set of generators:

$$\mathbf{\Lambda} = (\Lambda, \cdot, \lambda_a, a)_{a \in A}$$

The object of study of $\lambda$-calculus is any congruence on $\mathbf{\Lambda}$ (called $\lambda$-theory) including $\alpha$- and $\beta$-conversion:

  - $\beta$-conversion:

$$\lambda_a(M) \cdot N = M[N/a]$$

    $M[N/a]$ is a "meta-operation" defined by induction over $M$.

  - $\alpha$-conversion:

$$\lambda_a(M) = \lambda_b(M[b/a]) \quad (b \text{ not free in } M)$$

- The lattice of $\lambda$-theories $\equiv$ The congruence lattice of $\mathbf{\Lambda}/\lambda\beta$
($\lambda\beta$ is the least congruence on $\mathbf{\Lambda}$ including $\alpha$- and $\beta$-conversion)

## Is the untyped $\lambda$-calculus algebraic? YES

- CA combinatory algebras (Curry-Schönfinkel)

- LAA lambda abstraction algebras (Pigozzi-S. 1993)

**Theorem 1** (S. 2000)

1. *Variety*($\mathbf{\Lambda}/\lambda\beta$) = LAA.
2. *Lattice of $\lambda$-theories = Lattice of eq. theories of* LAAs.
   *$\lambda$-theory $T \Leftrightarrow$ variety generated by the term algebra of $T$.*

Are CAs and LAAs good algebras?

The properties of a variety $\mathcal{V}$ of algebras are usually studied through the lattice identities satisfied by the congruence lattices of all algebras in $\mathcal{V}$

# Some negative algebraic results

**Theorem 2** (Lusin-S. 2004) *Every nontrivial lattice identity fails in the congruence lattice of a suitable* LAA *(CA).*

Conclusion: We cannot apply thirty years of Universal Algebra to LAA (CA)!

Lambda calculus was introduced around 1930 by Alonzo Church as part of a foundational formalism of mathematics and logic based on functions as primitive. After some years this formalism was shown inconsistent. Why?

**Theorem 3** *Classic logic is inconsistent with combinatory logic.*

Proof: The variety of Boolean algebras is congruence permutable. Plotkin and Simpson have shown that the Malcev conditions for congruence permutability are inconsistent with combinatory logic.

**Theorem 4** *The implication fragment of classic logic is inconsistent with combinatory logic.*

Proof: An implication algebra is 3-permutable. Plotkin and Selinger have shown that the Malcev conditions for congruence 3-permutability are inconsistent with combinatory logic.

We should be pessimistic!

# Boolean algebras for $\lambda$-calculus

- Let $\mathbf{A}$ be any algebra. There exists a bijective correspondence between:
  - Pairs $(\rho, \rho')$ of complementary factor congruences: $\rho \cap \rho' = \Delta$; $\rho \circ \rho' = \nabla$
  - Factorizations $\mathbf{A} = \mathbf{A}/\rho \times \mathbf{A}/\rho'$.
  - Decomposition operations $f : A \times A \to A$ defined by

$$f(x, y) = u \text{ iff } x \rho u \rho' y.$$

- Let $\mathbf{t} \equiv \lambda_a(\lambda_b(a))$ and $\mathbf{f} \equiv \lambda_a(\lambda_b(b))$.

$$(\mathbf{t}x)y = x; \quad (\mathbf{f}x)y = y.$$

  (The least reflexive compatible relation on the term algebra $\mathbf{\Lambda}/\lambda\beta$ including $\mathbf{t} = \mathbf{f}$ is trivial)

- We have for a pair $(\rho, \rho')$ of complementary factor congruences:

$$\mathbf{t} \rho e \rho' \mathbf{f} \;\Rightarrow\; (\mathbf{t}x)y \;\rho\; (ex)y \;\rho'\; (\mathbf{f}x)y \;\Rightarrow\; x \rho \;(ex)y\; \rho' y.$$

$$f(x, y) = (ex)y$$

# The Boolean algebra of central elements

**Definition 1** *Let* $\mathbf{A}$ *be an* LAA *(*CA*). We say an element* $e \in A$ *is* central *when it satisfies the following equations, for all* $x, y, z, v \in A$*:*

*(i)* $(ex)x = x$.

*(ii)* $(e((ex)y))z = (ex)z = (ex)((ey)z)$.

*(iii)* $(e(xy))(zv) = ((ex)z)((ey)v)$.

*(iv)* $e = (e\mathbf{t})\mathbf{f}$.

- $e$ is central $\Leftrightarrow$ $\mathbf{A} = \mathbf{A}/\theta(\mathbf{t}, e) \times \mathbf{A}/\theta(\mathbf{f}, e)$
- $\mathbf{A}$ is directly indecomposable iff $\mathbf{t}, \mathbf{f}$ are the unique central elements.

**Theorem 5** *Let* $\mathbf{A}$ *be an* LAA *(*CA*). Then the algebra* $(C(\mathbf{A}), \wedge, ^-)$ *of central elements of* $\mathbf{A}$*, defined by*

$$e \wedge d = (e\mathbf{t})d; \quad e^- = (e\mathbf{f})\mathbf{t},$$

*is a Boolean algebra.*

Proof: LAAs have skew factor congruences $\Rightarrow$ Factor congruences are a Boolean sublattice of Con$(\mathbf{A})$.

# The Stone representation theorem

**Theorem 6** *Let* $\mathbf{A}$ *be an* LAA *(or a* CA*) and* $I$ *be the Boolean space of maximal ideals of the Boolean algebra of central elements. Then the map*

$$f : A \to \Pi_{i \in I}(A/ \cup i),$$

*defined by*

$$f(x) = (x/ \cup i : i \in I),$$

*gives a* **weak** *Boolean product representation of* $\mathbf{A}$*, where the quotient algebras* $\mathbf{A}/ \cup i$ *are directly indecomposable.*

Proof: From a theorem by Vaggione.

# Central elements at work

The directly indecomposable LAAs (CAs) (there exist a lot of them!) are the building blocks of LAA (CA).

How to use central elements and directly indecomposable LAAs (CAs) to get results on lambda calculus?

- Church (around 1930): Lambda calculus
- Scott (1969): First model
- Meyer-Scott (around 1980): There exists a first-order axiomatization of what is a model of $\lambda$-calculus as a particular class of CAs.

$$\mathcal{D} \text{ model} \Rightarrow \ \mathrm{Th}(\mathcal{D}) = \{M = N : M \text{ and } N \text{ have the same interpretation}\}$$

- Scott Semantics and its refinements (1969-2007) A Scott topological space $\mathcal{D}$ and two Scott continuous maps

$$i : \mathcal{D} \to [\mathcal{D} \to \mathcal{D}]; \quad j : [\mathcal{D} \to \mathcal{D}] \to \mathcal{D}; \quad i \circ j = id_{[\mathcal{D} \to \mathcal{D}]}$$

- A semantics $\mathcal{C}$ of lambda calculus is incomplete if there exists a consistent $\lambda$-theory T s.t.

$$T \neq \mathrm{Th}(\mathcal{D}), \text{for all models } \mathcal{D} \in \mathcal{C}.$$

# Central elements at work

**Theorem 7** *The semantics of lambda calculus given in terms of directly indecomposable models (this includes Scott Semantics and its refinements) is incomplete.*

Proof:

1. CA$_{di}$'s is a universal class $\Rightarrow$ CA$_{di}$ is closed under subalgebras $\Rightarrow$ the directly decomposable CAs are closed under expansion.

2. The lambda theory $T$ generated by $\lambda_a(aa) \cdot \lambda_a(aa) = \mathbf{t}$ is consistent.

3. The lambda theory $S$ generated by $\lambda_a(aa) \cdot \lambda_a(aa) = \mathbf{f}$ is consistent.

4. $\lambda_a(aa) \cdot \lambda_a(aa)$ is a nontrivial central element in the term model of $T \cap S$

5. All the models of $T \cap S$ are directly decomposable.

# Central elements at work

**Theorem 8** *For every r.e. lambda theory $T$, the lattice interval $[T) = \{S : T \subseteq T\}$ contains a continuum of "decomposable" lambda theories.*

**Theorem 9** *The set of lambda theories representable in EACH of the following semantics is not closed under finite intersection, so that it does not constitute a sublattice of the lattice of lambda theories:*

- *Graph models*
- *Filter models*
- *Continuous models*
- *Stable models.*

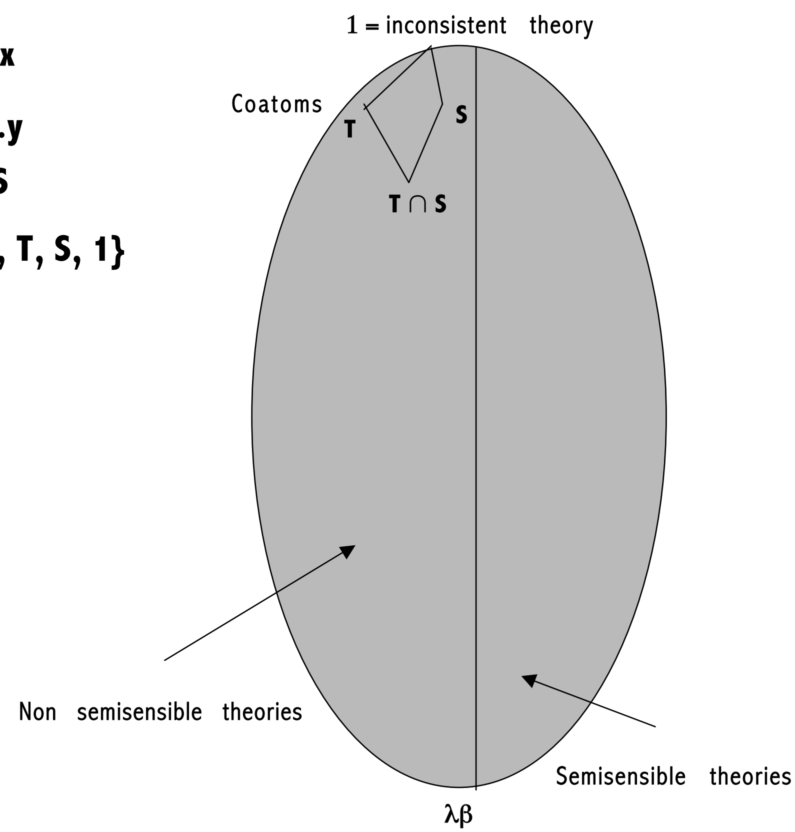
# Finite Boolean Sublattices

**T coatom containing $\Omega = \lambda xy.x$**

**S coatom containing $\Omega = \lambda xy.y$**

**$\Omega$ is nontrivial central in $T \cap S$**

**The interval $[T \cap S) = \{T \cap S, T, S, 1\}$**



# The lattice $\lambda T$ of $\lambda$-theories

Conjecture: Every nontrivial lattice identity fails in $\lambda T$

- (S. 2000) $\lambda T$ is isomorphic to the lattice of equational theories of LAA's.
- (Lampe 1986) $\lambda T$ satisfies the Zipper condition:

$$\vee\{b : a \wedge b = c\} = 1 \ \Rightarrow\ a = c.$$

- (S. 2001) $\lambda T$ is not modular.
- (Berline-S. 2006) ($\exists$ $\lambda$-theory $T$) the interval $[T, \nabla]$ is distributive.
- (Statman 2001) The meet of all coatoms is $\neq \lambda\beta$.
- (Visser 1980)
  - Every countable poset embeds into $\lambda T$ by an order-preserving map.
  - Every interval $[T, S]$ with $T, S$ r.e. has a continuum of elements.

- (S. 2006) $(\forall n)(\exists T_n)$ such that the interval sublattice $[T_n, \nabla]$ is isomorphic to the finite Boolean lattice with $2^n$ elements.

- (Diercks-Erné-Reinhold 1994) There exists no $\lambda$-theory $T$ such that the interval sublattice $[T, \nabla]$ is isomorphic to an infinite Boolean lattice.